Linguistic group decision making with induced aggregation operators and probabilistic information

José M. Merigó, Montserrat Casanovas, Daniel Palacios-Marqués

A Manchester Business School, The University of Manchester, Booth Street West, M15 6PQ Manchester, United Kingdom
b Department of Business Administration, University of Barcelona, Av. Diagonal 690, 08034 Barcelona, Spain
c Department of Management Control and Information Systems, University of Chile, Av. Diagonal Paraguay 257, 8330015 Santiago, Chile
d Department of Business Organisation, Universitat Politècnica de València, Camino Vera s/n, 46022 València, Spain

Abstract

A new approach for linguistic group decision making by using probabilistic information and induced aggregation operators is presented. It is based on the induced linguistic probabilistic ordered weighted average (ILPOWA). It is an aggregation operator that uses probabilities and OWA operators in the same formulation considering the degree of importance that each concept has in the formulation. It uses complex attitudinal characters that can be assessed by using order inducing variables. Furthermore, it deals with an uncertain environment where the information cannot be studied in a numerical scale but it is possible to use linguistic variables. Several extensions to this approach are presented by using moving averages and Bonferroni means. The applicability of this approach is also studied with a focus on multi-criteria group decision making by using multi-person aggregation operators in order to deal with the opinion of several experts in the analysis. An illustrative example regarding importation strategies in the administration of a country is developed.

1. Introduction

The available information in an environment can be assessed in different ways. Usually, people try to represent the information by using a numerical scale. However, this framework cannot be used always since we may find uncertain environments where the information is very imprecise and it is not possible to represent it in a quantitative way. Therefore, it is necessary to use another framework such as the use of linguistic variables that represent the information in a qualitative way. By using linguistic information we can represent expressions such as high, low, very high and so on. The classic approach for representing linguistic information by using fuzzy sets [1] was introduced by Zadeh [2]. Since its introduction, it has been studied by a lot of authors [3–5]. A very useful approach has been introduced by Xu [5] where he extends the model into a continuous setting. Thus, it is easier to deal with the information without losing information, especially when using operations in the analysis.

Linguistic information has been used in a wide range of decision making problems. For example, Xu [5,6] and Wei [7] studied linguistic decision making problems by using group information. Merigó and Casanovas [8] analyzed a model by using distance measures that was later extended by Zeng and Su [9]. Merigó et al. [10] studied the use of the Dempster–Shafer belief structure. Herrera et al. analyzed a model that used unbalanced linguistic information [3]. Xu and Wang [11,12] introduced a model by using power averages. Xu et al. [13] developed a similar approach when dealing with the 2-tuple linguistic methodology. Xu et al. [14] have designed different scales for dealing with interactive approaches in linguistic decision making.

When dealing with decision making problems [15,16], it is necessary to aggregate the available information in order to make decisions. A very useful aggregation operator is the ordered weighted average (OWA) [17]. It provides a parameterized family of aggregation operators between the minimum and the maximum. The reordering process of the OWA operator can be generalized by using order inducing variables obtaining the induced OWA (IOWA) operator [18]. The OWA and the IOWA operator have also been studied under linguistic environments forming the linguistic OWA (LOWA) [19] and the induced LOWA (ILOWA) operator [6]. These
operators have been extended by using generalized and quasi-arithmetic means [9,20].

Recently, Mergí [21] has introduced the probabilistic OWA (POWA) operator. It unifies probabilistic aggregations and the OWA operator in the same formulation and considering the degree of importance that each concept has in the analysis. Thus, we are able to consider decision making problems with probabilities and with the attitudinal character of the decision maker. Note that some previous studies already studied the use of the OWA operator in the probability such as the concept of immediate probability [22–24]. Moreover, other authors have developed similar approaches by using the weighted average including the probabilistic weighted average [25], the weighted OWA (WOWA) operator [26], the hybrid average [27] and the importance OWA operator [28,29].

The aim of this paper is to analyze decision making problems under linguistic environments where it is possible to use probabilities and the attitudinal character of the decision maker. For doing so, it introduced the induced linguistic OWA (ILPOWA) operator. It is an aggregation operator that unifies the OWA operator and the probability in the same formulation under a linguistic environment. Its main advantage is that it can deal with probabilistic information and under or overestimate it according to the attitudinal character of the decision maker. This issue is important because the probabilities provide a neutral expectation of the future but cannot guarantee that it is the correct one. Thus, sometimes we may find situations where the results tend to be higher or lower than those provided by the probabilities. Some of its main properties and particular cases are studied. Several extensions are also developed by using moving averages forming the induced linguistic probabilistic ordered weighted moving average (ILPOMMA). Furthermore, it also presented the use of Bonferroni means in the ILPOWA operator. Note that the main contribution of this paper is the integration of OWA aggregation operators with probabilities under linguistic environments being the first POWA model that uses linguistic information in the aggregation process.

This approach is studied in a linguistic multi-criteria group decision making problem by using multi-person aggregation operators. Thus, it formed the multi-person ILPOWA (MP-ILPOWA) operator. It is an aggregation operator that deals with the opinion of several experts in the analysis when this information is provided in the form of linguistic variables. Several particular cases are studied including the multi-person linguistic probabilistic aggregation (MP-LPA), the multi-person LOWA (MP-LOWA) and the multi-person linguistic arithmetic mean (MP-LAM). An illustrative example is also presented focused on a linguistic decision making application regarding the importation strategy of a country.

The rest of the paper is organized as follows. Section 2 reviews some basic preliminaries. Section 3 presents the ILPOWA operator and some basic families. Section 4 develops several extensions by using moving averages and Bonferroni means. Section 5 analyzes the new linguistic group decision making approach and the MP-ILPOWA operator. Section 6 summarizes the main results of the paper.

2. Preliminaries

In this section, we briefly review the linguistic approach and some basic aggregation operators including the OWA, the linguistic OWA, the induced OWA and the probabilistic OWA operator.

2.1. The linguistic approach

Usually, people are used to work in a quantitative setting, where the information is expressed by means of numerical values. However, many aspects of the real world cannot be assessed in a quantitative form. Instead, it is possible to use a qualitative one, i.e., with vague or imprecise knowledge such as the use of linguistic assessments instead of numerical values. The linguistic approach represents qualitative aspects as by means of linguistic variables [2].

We have to select the appropriate linguistic descriptors for the term set and their semantics. One possibility for generating the linguistic term set consists in directly supplying the term set by considering all terms distributed on a scale on which a total order is defined. For example, a set of eleven terms $S$ could be given as follows:

$$S = \{S_0 = N, S_1 = EL, S_2 = VL, S_3 = L, S_4 = LM, S_5 = M, S_6 = MH, S_7 = H, S_8 = VH, S_9 = EH, S_{10} = P\}.$$

Note that $N = None$, $EL = Extremely$ $low$, $VL = Very$ $low$, $L = Low$, $LM = Low-Medium$, $M = Medium$, $MH = Medium-High$, $H = High$, $VH = Very$ $high$, $EH = Extremely$ $high$, $P = Perfect$. Usually, in these cases, it is required that in the linguistic term set there exists:

- A negation operator: $Neg(s_j) = s_j$ such that $j = g + 1 - i$.
- The set is ordered: $s_i \leq s_j$ if and only if $i \leq j$.
- Max operator: $Max(s_i, s_j) = s_i$ if $s_i \geq s_j$.
- Min operator: $Min(s_i, s_j) = s_j$ if $s_i \leq s_j$.

Different approaches have been developed for dealing with linguistic information such as [2–5]. In this paper, we follow the ideas of Xu [5,6]. Thus, in order to preserve all the given information, we extend the discrete linguistic term set $\bar{S}$ to a continuous linguistic term set $\bar{S} = \{\bar{s}_i | i \in [1, t] \}$, where, if $s_a \in \bar{S}$, we call $s_a$ the original linguistic term, otherwise, we call $s_a$ the virtual linguistic term.

Consider any two linguistic terms $s_a, s_{\beta} \in \bar{S}$, and $\mu, \mu_1, \mu_2 \in [0,1]$, we define some operational laws as follows [5,6]:

- $\mu s_{\alpha} = s_{\mu \alpha}$.
- $s_{\alpha} + s_{\beta} = s_{\alpha + \beta}$.
- $s_{\alpha} \cdot s_{\beta} = s_{\alpha \beta}$.
- $s_{\alpha} \cdot s_{\beta} = s_{\beta} \cdot s_{\alpha} = s_{\alpha \beta}$.

Note that this model is very useful for computing with words because it is very easy to use and it follows a similar methodology as the numerical information.

2.2. The OWA operator

The OWA operator [17] is an aggregation operator that provides a parameterized family of aggregation operators between the minimum and the maximum. It is very useful in decision making under uncertainty since it allows the use of decisions that considers the degree of optimism of the decision maker. It is defined as follows.

**Definition 1.** An OWA operator of dimension $n$ is a mapping OWA: $\mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weighting vector $W$ of dimension $n$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$OWA(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j,$$

where $b_j$ is the $j$th largest of the $a_i$.

Note that different properties can be studied such as the distinction between descending and ascending orders, different measures for characterizing the weighting vector and different families of OWA operators. Note that it is commutative, monotonic, bounded and idempotent. For further reading, refer, for example to [30–35].
2.3. The linguistic OWA operator

In the literature, we find a wide range of linguistic aggregation operators [3–6]. In this study, we consider the linguistic ordered weighted averaging (LOWA) operator [5,8]. It is an extension of the OWA operator for environments where the available information is imprecise and can be assessed with linguistic variables. It can be defined as follows.

Definition 2. A LOWA operator of dimension $n$ is a mapping $LOWA: \mathbb{S}^n \rightarrow \hat{S}$, which has an associated weighting vector $W$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$LOWA(s_1, s_2, \ldots, s_n) = \sum_{j=1}^{n} w_j s_j$$

where $s_j$ is the $j$th largest of the $s_i$.

Note that it is possible to distinguish between the descending LOWA (DLOWA) and the ascending LOWA (ALOWA) operator. The weights of these operators are related by $w_j = w_{n+1-j}$ where $w_j$ is the $j$th weight of the DLOWA (or LOWA) operator and $w_{n+1-j}$ the $j$th weight of the ALOWA operator.

The LOWA operator provides a parameterized family of aggregation operators that includes as special cases the LA and the linguistic weighted average (LWA). The LA is obtained when all the weights $w_j$ are equal for all $j$. The LWA is obtained if the ordered position of the $s_j$ is the same as the ordered position of the $s_i$.

2.4. The induced OWA operator

The IOWA operator [18] is an extension of the OWA operator. Its main difference is that the reordering step is not carried out with the values of the arguments $a_i$. In this case, the reordering step is developed with order-inducing variables that reflect a more complex reordering process. The IOWA operator also includes as particular cases the maximum, the minimum and the average criteria. It can be defined as follows.

Definition 3. An IOWA operator of dimension $n$ is a mapping $IOWA: \mathbb{R}^n \times \mathbb{R}^n \rightarrow R$ that has an associated weighting vector $W$ of dimension $n$ with $W = \sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, such that:

$$IOWA((a_1, 0), (a_2, 0), \ldots, (a_n, 0)) = \sum_{j=1}^{n} w_j b_j$$

where $b_j$ is the $j$th largest value of the $a_i$, $u_j$ is the $j$th largest $u_i$, $v_j$ is the order inducing variable and $a_j$ is the argument variable.

The IOWA operator is also monotonic, bounded, idempotent and commutative. It has been studied in a wide range of problems [36–40].

2.5. The probabilistic OWA operator

The probabilistic ordered weighted averaging (POWA) operator is an aggregation operator that unifies the probability and the OWA operator in the same formulation considering the degree of importance that each concept has in the analysis [18,41]. It is defined as follows.

Definition 4. A POWA operator of dimension $n$ is a mapping $POWA: \mathbb{R}^n \rightarrow R$ that has an associated weighting vector $W$ of dimension $n$ such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$POWA(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} \beta_j b_j$$

where $b_j$ is the $j$th largest of the $a_i$, each argument $a_i$ has an associated weight $\langle a_i \rangle$ with $\sum_{j=1}^{n} v_i = 1$ and $v_i \in [0,1]$, $\tilde{v}_j = \beta w_j + (1 - \beta) v_j$ with $\beta \in [0,1]$ and $v_j$ is the weight $\langle a_i \rangle$ ordered according to $b_j$, that is, according to the $j$th largest of the $a_i$.

By choosing a different manifestation in the weighting vector, we are able to obtain a wide range of particular types of POWA operators [18,41]. Especially, when $\beta = 0$, we get the probabilistic aggregation, and if $\beta = 1$, we get the OWA operator.

3. The induced linguistic probabilistic OWA operator

3.1. Main concepts

The induced linguistic probabilistic ordered weighted averaging (ILPOWA) operator is a new aggregation model that unifies the linguistic probabilistic aggregation (LPA) and the induced LOWA (ILOWA) operator in the same formulation considering the degree of importance that each concept has in the analysis. Thus, it is able to deal with objective information represented in the form of probabilities and with the attitudinal character of the decision maker. Moreover, it also assesses uncertain information that cannot be assessed with numerical values but it is possible to use linguistic variables. The ILPOWA operator can be defined as follows.

Definition 5. An ILPOWA operator of dimension $n$ is a mapping $ILPOWA: \mathbb{S}^n \times \mathbb{S}^n \rightarrow \hat{S}$ that has an associated weighting vector $W$ of dimension $n$ such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$ILPOWA((u_1, s_1), (u_2, s_2), \ldots, (u_n, s_n)) = \sum_{j=1}^{n} \tilde{v}_j s_j$$

where $s_j$ is the $s_a$ linguistic value having the $j$th largest $u_i$, $v_j$ is the order inducing variable, each argument $s_i$ has an associated probability $p_i$ with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0,1]$, $\tilde{v}_j = \beta w_j + (1 - \beta) p_j$ with $\beta \in [0,1]$ and $p_j$ is the probability $p_i$ ordered according to $s_j$, that is, according to the $j$th largest $u_i$.

The ILPOWA operator can also be formulated separating the part that affects the LPA and the IOWA operator in the following way:

$$ILPOWA((u_1, s_1), \ldots, (u_n, s_n)) = \sum_{j=1}^{n} \tilde{v}_j s_j$$

where $s_j$ is the $s_a$ linguistic value having the $j$th largest $u_i$ and $p_i \in [0,1]$.

Next, we present a simple example regarding the aggregation process used in the ILPOWA operator.

Example 1. Assume a linguistic term set of eleven elements as explained in the previous sections and the following linguistic arguments in an aggregation process: $\{s_4, s_5, s_6, s_7\}$. Assume the following weighting vector $W = \{0.2, 0.2, 0.3, 0.3\}$, the probabilistic weighting vector $V = \{0.3, 0.3, 0.2, 0.2\}$ and the order inducing variables $U = \{8, 4, 9, 6\}$. Note that the probabilistic information has a degree of importance of 60% while the weighting vector $W$ of the ILOWA a degree of 40%. If we want to aggregate this information by using the ILPOWA operator, we get the following. The aggregation can be solved either with (4) or (5). With (4) we get the following:

$$ILPOWA = 0.4 \times (0.2 \times s_8 + 0.2 \times s_4 + 0.3 \times s_4 + 0.3 \times s_2) + 0.6 \times (0.3 \times s_4 + 0.3 \times s_2 + 0.2 \times s_8 + 0.2 \times s_6) = s_{4.68}$$

Note that if we use Eq. (5) we should also get the same result.

We can distinguish between the descending ILPOWA (DILPOWA) and the ascending ILPOWA (AILPOWA) operator...
by using in the ILOWA \( w_j = w_{n-j+1} \), where \( w_j \) is the \( j \)th weight of the \( \text{DILOWA} \) and \( w_{n-j+1} \) the \( j \)th weight of the \( \text{ILPOWA} \) operator.

Note that if the weighting vector is not normalized, the \( \text{ILPOWA} \) operator can be expressed as:

\[
\text{ILPOWA}((u_1, s_{a_1}), \ldots, (u_n, s_{a_n})) = \frac{1}{\beta} \sum_{j=1}^{n} \beta_j s_{j}.
\] 

(7)

The \( \text{ILPOWA} \) is monotonic, bounded and idempotent. It is monotonic because if \( s_{a_i} \geq s_{a_j} \), for all \( i \), then, \( \text{ILPOWA} ((u_1, s_{a_1}), \ldots, (u_n, s_{a_n})) \geq \text{ILPOWA} ((u_1, s_{a_1}), \ldots, (u_n, s_{a_n})) \). It is bounded because the \( \text{ILPOWA} \) aggregation is delimited by the minimum and the maximum. That is, \( \text{Min}(s_{a_i}) \leq \text{ILPOWA} ((u_1, s_{a_1}), \ldots, (u_n, s_{a_n})) \leq \text{Max}(s_{a_i}) \). It is idempotent because if \( s_{a_i} = s_{a_j} \), for all \( i \), then, \( \text{ILPOWA} ((u_1, s_{a_1}), \ldots, (u_n, s_{a_n})) = s_{a}. \) A further interesting property that appears with the \( \text{ILPOWA} \) operator is a semi boundary condition based on the probabilistic information of the problem.

**Theorem 1 (Semi boundary conditions).** Assume \( f \) is the \( \text{ILPOWA} \) operator, then:

\[
\beta \times \text{Min}(s_{a_i}) + (1 - \beta) \leq \beta \times \text{Max}(s_{a_i}) + (1 - \beta) \leq \beta \times \text{Max}(s_{a_i}) + (1 - \beta) \times \sum_{i=1}^{n} p_i s_{a_i}.
\]

(8)

**Proof.** Let \( \text{max}(s_{a_i}) = c \), and \( \text{min}(s_{a_i}) = d \), then

\[
f((u_1, s_{a_1}), \ldots, (u_n, s_{a_n})) = \beta \sum_{j=1}^{n} w_j s_{j} + (1 - \beta) \sum_{i=1}^{n} p_i s_{a_i} \leq \beta \sum_{j=1}^{n} w_j c + (1 - \beta) \sum_{i=1}^{n} p_i s_{a_i},
\]

(9)

\[
f((u_1, s_{a_1}), \ldots, (u_n, s_{a_n})) = \beta \sum_{j=1}^{n} w_j s_{j} + (1 - \beta) \sum_{i=1}^{n} p_i s_{a_i} \geq \beta \sum_{j=1}^{n} w_j d + (1 - \beta) \sum_{i=1}^{n} p_i s_{a_i},
\]

(10)

Since \( \sum_{j=1}^{n} w_j = 1 \), we get

\[
f((u_1, s_{a_1}), \ldots, (u_n, s_{a_n})) \leq \beta c + (1 - \beta) \sum_{i=1}^{n} p_i s_{a_i},
\]

(11)

\[
f((u_1, s_{a_1}), \ldots, (u_n, s_{a_n})) \geq \beta d + (1 - \beta) \sum_{i=1}^{n} p_i s_{a_i},
\]

(12)

Therefore,

\[
\beta \times \text{Min}(s_{a_i}) + (1 - \beta) \leq \beta \times \text{Max}(s_{a_i}) + (1 - \beta) \leq \beta \times \text{Max}(s_{a_i}) + (1 - \beta) \times \sum_{i=1}^{n} p_i s_{a_i}.
\]

Note that if \( \beta = 1 \), they become the common boundary conditions. This is important because sometimes the maximum and minimum may form an interval that is excessively broad and it is necessary to obtain more specific information. Therefore, the formation of semi bounds based on the probabilistic information seems to be a good alternative.

A further interesting issue is the analysis of measures for characterizing the weighting vector \( W \). Following a similar methodology as it has been developed for the \( \text{IOWA} \) weighted average \( \text{(IOWA/WA)} \) operator [42] we can formulate the degree of orness, the entropy of dispersion, the divergence of \( W \) and the balance operator. For example, the entropy of dispersion can be formulated as follows:

\[
H(\hat{W}) = -\left( \beta \sum_{j=1}^{n} w_j \ln(w_j) + (1 - \beta) \sum_{i=1}^{n} w_i \ln(w_i) \right).
\]

(13)

Note that if \( \beta = 0 \), it becomes the classical Shannon entropy [43] while if \( \beta = 1 \), the Yager entropy of dispersion used in the OWA aggregation.

It is worth noting that other linguistic approaches could be used when dealing with the \( \text{ILPOWA} \) operator such as the use of the 2-tuple linguistic approach [4] and the unbalanced linguistic model [3]. Additionally, we can use different techniques for representing the linguistic variables. If we look to the internal part of the linguistic label, we can use interval numbers, fuzzy numbers, probabilistic sets and more complex structures. Following [29,44], we can also study the external perspective of the linguistic label by using interval numbers or fuzzy numbers.

Note that in the case of ties in the reordering process of the inducing variables, the final aggregated results may be different depending on the linguistic argument selected first. This problem appears because the ordering does not depend on the arguments as in the \( \text{OWA} \) operator. In order to solve this problem, we recommend to use the policy explained by Yager and Filev [18] that consists in replacing each tied argument by the average. However, note that it is possible to use any aggregation operator from the minimum to the maximum.

A further interesting issue is that some previous models already considered the possibility of using \( \text{OWA} \) operators and probabilities in the same formulation such as the concept of immediate probabilities [22–24]. Using this concept, the \( \text{ILPOWA} \) would become the induced linguistic immediate probability as follows:

\[
\text{ILPOWA}((u_1, s_{a_1}), (u_2, s_{a_2}), \ldots, (u_n, s_{a_n})) = \sum_{j=1}^{n} \beta_j s_{j}.
\]

(14)

where each argument \( s_{a_j} \) has an associated probability \( p_j \) with \( \sum_{j=1}^{n} p_j = 1 \), then \( \beta_j = \frac{w_j p_j}{\sum_{j=1}^{n} w_j p_j} \) and \( p_j \) is the probability \( p_j \) ordered according to \( s_{j} \) that is, according to the \( j \)th largest \( u_j \).

Moreover, it is possible to consider other approaches that unify the weighted average with the \( \text{OWA} \) operator because the weighted average can also be seen as a probability. For example, it is possible to extend the \( \text{ILPOWA} \) operator by using hybrid averages [27] and \( \text{WOWA} \) operators [26]. The induced linguistic hybrid probabilistic average is defined as follows:

\[
\text{ILHPA}((u_1, s_{a_1}), \ldots, (u_n, s_{a_n})) = \sum_{j=1}^{n} w_j s_{j} \text{, where } s_{j} \text{ is the } j \text{th largest } u_j, \text{ } u_j \text{ is the order-inducing variable, } \omega = (\omega_1, \omega_2, \ldots, \omega_n) \text{ is the weighting vector of the } s_{a_j}, \text{ with } \omega_i \in [0,1] \text{ and } \sum_{i=1}^{n} \omega_i = 1.
\]
The induced linguistic WOAWA (ILWOWA) operator is defined as follows:

$$ILWOWA((u_1, s_{a_1}), \ldots, (u_n, s_{a_n})) = \frac{1}{n} \sum_{i=1}^{n} \alpha_i s_{a_i}$$

where $\alpha_i = \sigma(1), \ldots, \sigma(n)$ is a permutation of $\{1, \ldots, n\}$ such that $u_{\sigma(i-1)} \geq u_{\sigma(i)}$ for all $i = 2, \ldots, n$, and the weight $\alpha_i$ is defined as:

$$\alpha_i = \frac{w^*}{\sum_{j=1}^{n} p_{\sigma(j)}} - \frac{w^*}{\sum_{j=1}^{n} p_{\sigma(j)}}$$

with $w^*$ a monotonically increasing function that interpolates the points $(i/n, \sum_{j=1}^{i} w_j)$ together with the point $(0, 0)$. $w^*$ is required to be a straight line when the points can be interpolated in this way.

Finally, note that these approaches have been developed by using the continuous linguistic model of Xu [5]. However, it is possible to use other linguistic representation approaches including the 2-tuple linguistic approach [4], the unbalanced linguistic framework [3] and the use of uncertain linguistic variables [6,7]. For the last case, it is possible to consider linguistic expressions that are also imprecise. This issue becomes useful when the environment is very uncertain and with a lot of complexities. In this situation, a decision maker can provide a linguistic expression such as [very low, low], where he indicates that the result tends to be low but he is not sure how low is the result. Under this framework, it is possible to introduce the uncertain ILPOWA (ULPOWA) operator. It is very similar to the ILPOWA operator with the difference that now the linguistic variables are intervals instead of crisp values. For further information regarding the use of uncertain linguistic variables with aggregation operators, see [6,7].

3.2. Families of ILPOWA operators

Next, we are going to analyze a wide range of particular cases of ILPOWA operators following the methodology commonly used in the OWA literature [45,46]. Thus, we can see different scenarios that may occur in the aggregation process from the minimum to the maximum. The main advantage of this issue is that it shows that the ILPOWA is very flexible because it can produce different results depending on the attitudinal character of the decision maker in the specific problem considered.

First, let us consider the two main cases of the ILPOWA operator found by analyzing the coefficient $\beta$. If $\beta = 0$, we get the linguistic probabilistic aggregation (LPA), also known as the linguistic expected value, and if $\beta = 1$, the ILPOWA operator. Note that when $\beta$ increases, we give more importance to the ILPOWA operator and vice versa. Some other interesting examples are the maximum-LPA, the minimum-LPA and the step-ILPOWA operator.

Remark 1. The maximum-LPA is found when $w_1 = 1$ and $w_j = 0$ for all $j \neq 1$. The minimum-LPA is formed when $w_n = 1$ and $w_j = 0$ for all $j \neq n$. More generally, the step-LPA is formed when $w_k = 1$ and $w_j = 0$ for all $j \neq k$. Note that if $k = 1$, the step-ILPOWA is transformed into the maximum-LPA, and if $k = n$, it becomes the minimum-LPA operator.

Remark 2. The linguistic arithmetic-LPA is obtained when $w_j = 1/n$ for all $j$. It can be formulated as follows:

$$LA-LPA((u_1, s_{a_1}), \ldots, (u_n, s_{a_n})) = \frac{1}{n} \sum_{i=1}^{n} v_i s_{a_i} + (1 - \beta) \sum_{i=1}^{n} v_i s_{a_i}. \quad (18)$$

This is the linguistic arithmetic-ILPOWA (LA-ILPOWA). The LA-ILPOWA operator can be formulated as follows:

$$LA-ILPOWA((u_1, s_{a_1}), \ldots, (u_n, s_{a_n})) = \beta \sum_{j=1}^{n} w_j s_{a_i} + (1 - \beta) \frac{1}{n} \sum_{i=1}^{n} v_i s_{a_i}. \quad (19)$$

Remark 3. Some other interesting families are the following:

- The median-ILPOWA: if $n$ is odd we assign $w_{(n+1)/2} = 1$ and $w_j = 0$ for all others. If $n$ is even we assign for example, $w_{n/2} = w_{(n+1)/2} = 0.5$ and $w_j = 0$ for all others.
- The weighted median-ILPOWA: we select the argument $b_k$ that has the $k$th largest argument such that the sum of the weights from 1 to $k$ is equal or higher than 0.5 and the sum of the weights from 1 to $k - 1$ is less than 0.5.
- The S-ILPOWA: $w_1 = (1/n)(1 - (\alpha + \beta) + \alpha)$, $w_2 = (1/n)(1 - (\alpha + \beta) + \beta)$, and $w_j = (1/n)(1 - (\alpha + \beta))$ for $j = 2$ to $n - 1$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$.
- The Olympic-ILPOWA: when $w_1 = w_n = 0$, and for all others $w_j = 1/(n - 2)$.
- The General Olympic-ILPOWA: if $w_j = w_{n-1-j}$ is symmetric, strongly decaying and inclusive. It is symmetric if $w_j = w_{n-1-j}$. It is strongly decaying when $i < j \leq (n+1)/2$ then $w_i < w_j$ and when $i > j \geq (n+1)/2$ then $w_i > w_j$. And it is inclusive if $w_j > 0$.

Remark 4. Other families of ILPOWA operators could be studied by using them both in the probabilities and in the OWA operator. And many other types following the OWA literature regarding the formation of the OWA weights [21,34,47].

4. Moving averages and Bonferroni means in the ILPOWA operator

Further extensions to the ILPOWA operator can be developed by using other well-known aggregation operators [30]. For example, it is possible to extend it by using moving averages [48,49] and Bonferroni means [50].

Moving averages are very useful for representing dynamic information, for example, when considering several periods of time in the same problem. By using OWA operators in the moving average [48,49] it is possible to represent dynamic information from the minimum to the maximum potential result. In this case, it is formed the ordered weighted moving average (OWMA) that can be defined as follows:

$$OWMA(a_{1-t}, a_{2-t}, \ldots, a_{m-t}) = \sum_{j=1-t}^{m} w_j b_j, \quad (20)$$

where $b_j$ is the $j$th largest argument of the $a$, $m$ is the total number of arguments considered from the whole sample and $t$ indicates the movement done in the average from the initial analysis.

More generally, it is possible to extend the moving average to the whole OWA literature. Focussing on the ILPOWA operator, the combination of both concepts introduces the linguistic moving aggregation operators. That is, aggregation operators that moves toward a set of imprecise arguments that can be represented with linguistic variables. In this case, it is formed the induced linguistic probabilistic ordered weighted moving average (ILPOMWA) and it is formulated in the following way:

$$ILPOMWA((u_{1-t}, s_{a_{1-t}}), \ldots, (u_{n-t}, s_{a_{n-t}})) = \sum_{j=1-t}^{m-t} b_j s_{a_j}. \quad (21)$$
where \( s_{rj} \) is the \( s_{ur} \) linguistic value of the ILPOWMA pair \( (u_j, s_{ur}) \) having the \( j \)th smallest \( u_i \), \( u_i \) is the order inducing variable, each argument \( s_{ur} \) has an associated probability \( p_i \) with \( \sum_{i=1}^{n} p_i = 1 \) and \( p_i \in [0,1] \). \( F = \beta w_j + (1-\beta) p_j \) with \( \beta \in [0,1] \), \( w_j \) is the OWMA weight with \( \sum_{j=1}^{n} w_j = 1 \) and \( w_j \in [0,1] \), \( p_j \) is the probability \( p_j \) ordered according to \( s_{rj} \), that is, according to the \( j \)th smallest \( u_i \), \( m \) is the number of arguments considered and \( t \) indicates the movement from the initial analysis.

Observe that if \( \beta = 1 \), the ILPOWMA operator becomes the OWMA operator and if \( \beta = 0 \), the linguistic probabilistic moving average. Furthermore, the ILPOWMA accomplishes similar properties than the ILPO including monotonicity, idempotency and the boundary conditions.

Next, let us look into the use of the OWA operator in the Bonferroni mean. It was introduced by Yager [50]. Following this methodology, it is also possible to extend it by using ILPOWMA operators. Recall that the Bonferroni mean is defined as follows:

\[
B(a_1, a_2, \ldots, a_n) = \frac{1}{n(n-1)} \sum_{i \neq j} a_i a_j \tag{22}
\]

By rearranging the terms [51], it can be expressed as follows:

\[
B(a_1, a_2, \ldots, a_n) = \left( \frac{1}{n} \sum_{i=1}^{n} a_i \right) \left( \frac{1}{n(n-1)} \sum_{j=1}^{n} a_j \right)^{1/(1+q)} \tag{23}
\]

In this paper, instead of introducing the OWA operators in the Bonferroni mean as it was done by Yager [50], let us introduce it as follows:

\[
BON-OWA(a_1, a_2, \ldots, a_n) = \left( \frac{1}{n} \sum_{i=1}^{n} w_i a_i \right) \left( \frac{1}{n-1} \sum_{j=1}^{n} a_j \right)^{1/(1+q)} \tag{24}
\]

where \( a_k \) is the \( k \)th smallest of the \( a_i \).

Following this methodology, it is possible to extend it to a linguistic context forming the BON-LOWA as follows:

\[
BON-LOWA(s_{a_1}, s_{a_2}, \ldots, s_{a_n}) = \left( \frac{1}{n} \sum_{k=1}^{n} w_k s_{a_k} \right) \left( \frac{1}{n-1} \sum_{j=1}^{n} s_{a_j} \right)^{1/(1+q)} \tag{25}
\]

where \( s_{rj} \) is the \( k \)th smallest of the \( s_{ur} \).

More generally, it can be generalized by using the ILPOWMA operator becoming the BON-ILPOWMA operator as follows:

\[
BON-ILPOWMA((u_1, s_{ur_1}), \ldots, (u_n, s_{ur_n})) = \lambda \left( \sum_{k=1}^{n} w_k s_{r_k} \right) \left( \frac{1}{n-1} \sum_{j=1}^{n} s_{r_j} \right)^{1/(1+q)} + (1-\lambda) \left( \sum_{i=1}^{n} p_j s_{u_i} \right) \left( \frac{1}{n-1} \sum_{j=1}^{n} s_{u_j} \right)^{1/(1+q)} \tag{26}
\]

\section{5. Group decision making with the ILPOWMA operator}

\subsection{5.1. Theoretical framework}

The ILPOWMA operator can be used in a wide range of applications. In this paper, we focus on a multi-criteria group decision making problem regarding importation management in the government of a country. The main advantage of using the ILPOWMA operator is that we can use probabilities and the attitudinal character of the decision maker in the analysis and under an imprecise environment that can be assessed with linguistic information. Thus, in linguistic decision making it permits to deal with uncertain and risk environments in the same formulation and considering their relevance in the specific problem considered. In other words, it is possible to form linguistic decision making problems under risk and uncertainty. Note that in the literature there are many other methodologies for decision making [15,52,53].

The process to follow in the selection of importation strategies with the ILPOWMA operator can be summarized as follows.

Step 1: Let \( A = \{a_1, a_2, \ldots, a_n\} \) be a set of finite alternatives, \( T = \{T_1, T_2, \ldots, T_k\} \) a set of finite states of nature (or attributes), forming the payoff matrix \( (s_{gi})_{p \times n} \). This matrix is formed for each criteria \( CR=(C_1, C_2, \ldots, C_k) \). All these criterion are aggregated through the weighting vector \( Y = (y_1, y_2, \ldots, y_k) \) with \( \sum_{j=1}^{n} y_j = 1 \) and \( y_j \in [0,1] \).

Step 2: Let \( E = \{e_1, e_2, \ldots, e_p\} \) be a finite set of decision makers. Let \( Z = \{z_1, z_2, \ldots, z_p\} \) be the weighting vector of the decision makers such that \( \sum_{k=1}^{p} z_k = 1 \) and \( z_k \in [0,1] \). Each decision maker provides his own payoff matrix \((s_{gi})_{m \times n}^{(E)}\). Use the weighted average, to aggregate the information of the decision makers \( E \) by using the weighting vector \( Z \). The result is the collective payoff matrix \((s_{gi})_{m \times n}^{(Z)}\).

Step 3: Calculate the order inducing variables \((u_{gi})_{m \times n}^{(Z)}\) to be used in the payoff matrix for each alternative \( g \) and state of nature \( i \). Moreover, calculate also the weighting vector \( W \) to be used in the ILPOWMA aggregation and the parameter \( \beta \). Note that \( W = (w_1, w_2, \ldots, w_n) \) such that \( \sum_{j=1}^{n} w_j = 1 \) and \( w_j \in [0,1] \) and \( \beta = (p_1, p_2, \ldots, p_n) \) such that \( \sum_{i=1}^{n} p_i = 1 \) and \( p_i \in [0,1] \).

Step 4: Calculate the aggregated results by using the ILPOWMA operator explained in Eq. (5). Consider different particular manifestations of the ILPOWMA by using different expressions as has been explained in Section 3.2.

Step 5: Adoption of decisions according to the results found in the previous steps. Select the alternative(s) that provides the best result/s. Moreover, establish a ranking of the alternatives from the most to the least preferred alternative in order to be able to consider more than one selection.

Note that this aggregation process can be summarized using the following aggregation operator that we call the multi-criteria multi-person – ILPOWMA (MC-MP-ILPOWMA) operator.
Definition 6. Let $S$ be the set of linguistic variables. An MC-MP-ILPOWA operator of dimension $n$ is a mapping $MC$-$MP$-$ILPOWA$:
$n^*: S^n \times S^n \times S^n \rightarrow S$ that has a weighting vector $z$ of dimension $p$ with $\sum_{k=1}^{p} z_k = 1$ and $z_k \in [0,1]$ and a weighting vector $W$ of dimension $n$ with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, such that:

$$f(\langle u_1, (s^1_{u_1}, \ldots, s^n_{u_1}) \rangle, \ldots, \langle u_n, (s^1_{u_n}, \ldots, s^n_{u_n}) \rangle) = \sum_{j=1}^{n} b_j s_j,$$

where $s_j$ is the $s_{ui}$ linguistic value having the $j$th largest $u_i$, and is the order inducing variable, each argument $s_{ui}$ has an associated probability $p_i$ with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0,1]$, $\beta$ is the probability $p_i$ ordered according to $s_j$, that is, according to the $j$th largest $u_i$, and $s_{ui} = \sum_{k=1}^{p} z_k s^k_{ui}$, being $s^k_{ui}$ the argument variable provided by each expert represented with linguistic variables after having considered the different criterion, $s^h_{ui} = \sum_{k=1}^{n} y_{k} s^k_{ui}$, $s_j$ is the linguistic argument provided by each expert for each criteria and $Y$ is a weighting vector of dimension $h$ with $\sum_{i=1}^{h} y_{i} = 1$ and $y_{i} \in [0,1]$. The MC-MP-ILPOWA operator has properties similar to those explained in Section 3. The MC-MP-ILPOWA operator includes a wide range of particular cases following the methodology explained in Section 4. Thus, we can find as special cases:

- The multi-criteria-multi-person-LPA (MC-MP-LPA) operator.
- The multi-criteria-multi-person-linguistic arithmetic mean (MC-MP-LAM).
- The multi-criteria-multi-person-ILOWA (MC-MP-ILOWA) operator.
- The multi-criteria-multi-person-LOWA (MC-MP-LOWA) operator.
- The multi-criteria-multi-person-LPOWA (MC-MP-LPOWA) operator.

More complex situations could be considered by using different types of aggregation operators in the aggregation of the experts opinion instead of using the WA operator. For example, we can use the ILPOWA operator, among others.

5.2. Illustrative example

In the following, we develop a brief illustrative example of the new approach in a linguistic multi-criteria multi-person decision-making problem concerning the selection of importation strategies. Although we do not use real data, note that the methodology of the example is adapted to the structure of a real problem.

Step 1: Assume a government that plans its importation strategy for the next year and they find that an important product for their economy is not produced in the country. Therefore, they have to import it from another country. After analyzing the market, the government considers six possible countries where the product is found. The decision problem consists in finding the country where it is optimal to buy this product according to the standards of the national government:

1. $A_1$: Import the product from China.
2. $A_2$: Import from India.
3. $A_3$: Import from Brazil.
5. $A_5$: Import from Russia.
6. $A_6$: Import from Turkey.

The government uses a group of experts to assess the problem. After careful review of the information, the group of experts establishes that two criteria determine the optimal strategy:

1. Economic environment of the country (internal).
2. World economic situation for the next year (external).

For both criteria they consider six possible scenarios that may occur in the future. They do not know which scenario will occur in the future. The objective is to consider all of them in order to do not lose any information. With this information, they want to search for the optimal alternative according to the interests of the government.

- $S_1$: Very good economic situation.
- $S_2$: Good economic situation.
- $S_3$: Regular – good economic situation.
- $S_4$: Regular economic situation.
- $S_5$: Bad economic situation.
- $S_6$: Very bad economic situation.

The group of experts is constituted by three persons, each offering their own opinion regarding the results found with each strategy and according to the scenario that occurs in the future. Since the information is very imprecise, the experts analyze the information providing linguistic evaluations of the results. They use a linguistic term set of eleven elements as shown in Section 2.1. The results given by the first expert for each criterion are shown in Tables 1 and 2.

With this linguistic information, the first expert forms his general opinion. In this paper we assume that the three experts give the same degree of importance to both criteria. That is: $Y = (0.5, 0.5)$. Thus, a simple arithmetic mean is used for aggregating the results between the two criteria. The results are shown in Table 3.

Next, a similar methodology is developed for the other two experts. Tables 4 and 5 present the results given by expert 2 for

Table 1

<table>
<thead>
<tr>
<th>Expert 1 – criteria 1.</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_4$</th>
<th>$S_5$</th>
<th>$S_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$s_1$</td>
<td>$s_4$</td>
<td>$s_2$</td>
<td>$s_1$</td>
<td>$s_7$</td>
<td>$s_7$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$s_3$</td>
<td>$s_6$</td>
<td>$s_4$</td>
<td>$s_5$</td>
<td>$s_4$</td>
<td>$s_1$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$s_4$</td>
<td>$s_6$</td>
<td>$s_4$</td>
<td>$s_5$</td>
<td>$s_4$</td>
<td>$s_5$</td>
</tr>
<tr>
<td>$A_4$</td>
<td>$s_4$</td>
<td>$s_7$</td>
<td>$s_5$</td>
<td>$s_6$</td>
<td>$s_6$</td>
<td>$s_5$</td>
</tr>
<tr>
<td>$A_5$</td>
<td>$s_7$</td>
<td>$s_8$</td>
<td>$s_7$</td>
<td>$s_8$</td>
<td>$s_6$</td>
<td>$s_6$</td>
</tr>
<tr>
<td>$A_6$</td>
<td>$s_4$</td>
<td>$s_5$</td>
<td>$s_2$</td>
<td>$s_7$</td>
<td>$s_2$</td>
<td>$s_7$</td>
</tr>
</tbody>
</table>

Table 2

<table>
<thead>
<tr>
<th>Expert 1 – criteria 2.</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_4$</th>
<th>$S_5$</th>
<th>$S_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$s_7$</td>
<td>$s_5$</td>
<td>$s_4$</td>
<td>$s_3$</td>
<td>$s_5$</td>
<td>$s_5$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$s_6$</td>
<td>$s_6$</td>
<td>$s_6$</td>
<td>$s_5$</td>
<td>$s_5$</td>
<td>$s_5$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$s_6$</td>
<td>$s_8$</td>
<td>$s_4$</td>
<td>$s_5$</td>
<td>$s_5$</td>
<td>$s_5$</td>
</tr>
<tr>
<td>$A_4$</td>
<td>$s_5$</td>
<td>$s_5$</td>
<td>$s_2$</td>
<td>$s_4$</td>
<td>$s_4$</td>
<td>$s_4$</td>
</tr>
<tr>
<td>$A_5$</td>
<td>$s_5$</td>
<td>$s_5$</td>
<td>$s_2$</td>
<td>$s_4$</td>
<td>$s_4$</td>
<td>$s_4$</td>
</tr>
<tr>
<td>$A_6$</td>
<td>$s_5$</td>
<td>$s_4$</td>
<td>$s_2$</td>
<td>$s_4$</td>
<td>$s_4$</td>
<td>$s_4$</td>
</tr>
</tbody>
</table>

Table 3

<table>
<thead>
<tr>
<th>Expert 1 – general expected result.</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_4$</th>
<th>$S_5$</th>
<th>$S_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$s_6$</td>
<td>$s_6$</td>
<td>$s_5$</td>
<td>$s_2$</td>
<td>$s_5$</td>
<td>$s_8$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$s_7$</td>
<td>$s_6$</td>
<td>$s_4$</td>
<td>$s_3$</td>
<td>$s_5$</td>
<td>$s_2$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$s_6$</td>
<td>$s_5$</td>
<td>$s_3$</td>
<td>$s_4$</td>
<td>$s_5$</td>
<td>$s_5$</td>
</tr>
<tr>
<td>$A_4$</td>
<td>$s_7$</td>
<td>$s_5$</td>
<td>$s_4$</td>
<td>$s_4$</td>
<td>$s_6$</td>
<td>$s_6$</td>
</tr>
<tr>
<td>$A_5$</td>
<td>$s_6$</td>
<td>$s_5$</td>
<td>$s_2$</td>
<td>$s_4$</td>
<td>$s_5$</td>
<td>$s_5$</td>
</tr>
<tr>
<td>$A_6$</td>
<td>$s_5$</td>
<td>$s_4$</td>
<td>$s_2$</td>
<td>$s_4$</td>
<td>$s_4$</td>
<td>$s_4$</td>
</tr>
</tbody>
</table>

The group of experts is constituted by three persons, each offering their own opinion regarding the results found with each strategy and according to the scenario that occurs in the future. Since the information is very imprecise, the experts analyze the information providing linguistic evaluations of the results. They use a linguistic term set of eleven elements as shown in Section 2.1. The results given by the first expert for each criterion are shown in Tables 1 and 2.
Table 4
Expert 2 – criteria 1.

<table>
<thead>
<tr>
<th></th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
<th>S4</th>
<th>S5</th>
<th>S6</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>s6</td>
<td>s8</td>
<td>s4</td>
<td>s5</td>
<td>s7</td>
<td>s6</td>
</tr>
<tr>
<td>A2</td>
<td>s5</td>
<td>s7</td>
<td>s6</td>
<td>s6</td>
<td>s1</td>
<td>s4</td>
</tr>
<tr>
<td>A3</td>
<td>s9</td>
<td>s5</td>
<td>s6</td>
<td>s2</td>
<td>s3</td>
<td>s5</td>
</tr>
<tr>
<td>A4</td>
<td>s7</td>
<td>s6</td>
<td>s5</td>
<td>s4</td>
<td>s3</td>
<td>s6</td>
</tr>
<tr>
<td>A5</td>
<td>s6</td>
<td>s6</td>
<td>s7</td>
<td>s3</td>
<td>s6</td>
<td>s6</td>
</tr>
<tr>
<td>A6</td>
<td>s5</td>
<td>s4</td>
<td>s6</td>
<td>s7</td>
<td>s4</td>
<td>s4</td>
</tr>
</tbody>
</table>

Table 5
Expert 2 – criteria 2.

<table>
<thead>
<tr>
<th></th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
<th>S4</th>
<th>S5</th>
<th>S6</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>s6</td>
<td>s6</td>
<td>s4</td>
<td>s5</td>
<td>s3</td>
<td>s4</td>
</tr>
<tr>
<td>A2</td>
<td>s7</td>
<td>s7</td>
<td>s6</td>
<td>s7</td>
<td>s4</td>
<td>s3</td>
</tr>
<tr>
<td>A3</td>
<td>s9</td>
<td>s9</td>
<td>s5</td>
<td>s3</td>
<td>s5</td>
<td>s3</td>
</tr>
<tr>
<td>A4</td>
<td>s5</td>
<td>s5</td>
<td>s5</td>
<td>s4</td>
<td>s4</td>
<td>s5</td>
</tr>
<tr>
<td>A5</td>
<td>s6</td>
<td>s6</td>
<td>s7</td>
<td>s3</td>
<td>s6</td>
<td>s6</td>
</tr>
<tr>
<td>A6</td>
<td>s5</td>
<td>s4</td>
<td>s4</td>
<td>s5</td>
<td>s3</td>
<td>s5</td>
</tr>
</tbody>
</table>

Table 6
Expert 2 – general expected result.

<table>
<thead>
<tr>
<th></th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
<th>S4</th>
<th>S5</th>
<th>S6</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>s7</td>
<td>s7</td>
<td>s4</td>
<td>s5</td>
<td>s5</td>
<td>s5</td>
</tr>
<tr>
<td>A2</td>
<td>s6</td>
<td>s6</td>
<td>s7</td>
<td>s6</td>
<td>s4</td>
<td>s5</td>
</tr>
<tr>
<td>A3</td>
<td>s9</td>
<td>s8</td>
<td>s5</td>
<td>s3</td>
<td>s5</td>
<td>s3</td>
</tr>
<tr>
<td>A4</td>
<td>s5</td>
<td>s5</td>
<td>s5</td>
<td>s4</td>
<td>s4</td>
<td>s5</td>
</tr>
<tr>
<td>A5</td>
<td>s6</td>
<td>s6</td>
<td>s7</td>
<td>s3</td>
<td>s6</td>
<td>s6</td>
</tr>
<tr>
<td>A6</td>
<td>s5</td>
<td>s4</td>
<td>s5</td>
<td>s5</td>
<td>s5</td>
<td>s6</td>
</tr>
</tbody>
</table>

Table 7
Expert 3 – criteria 1.

<table>
<thead>
<tr>
<th></th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
<th>S4</th>
<th>S5</th>
<th>S6</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>s6</td>
<td>s5</td>
<td>s4</td>
<td>s5</td>
<td>s7</td>
<td>s6</td>
</tr>
<tr>
<td>A2</td>
<td>s5</td>
<td>s7</td>
<td>s6</td>
<td>s5</td>
<td>s1</td>
<td>s4</td>
</tr>
<tr>
<td>A3</td>
<td>s9</td>
<td>s6</td>
<td>s4</td>
<td>s5</td>
<td>s5</td>
<td>s3</td>
</tr>
<tr>
<td>A4</td>
<td>s4</td>
<td>s7</td>
<td>s6</td>
<td>s5</td>
<td>s5</td>
<td>s8</td>
</tr>
<tr>
<td>A5</td>
<td>s6</td>
<td>s3</td>
<td>s8</td>
<td>s6</td>
<td>s5</td>
<td>s9</td>
</tr>
<tr>
<td>A6</td>
<td>s7</td>
<td>s4</td>
<td>s7</td>
<td>s4</td>
<td>s3</td>
<td>s6</td>
</tr>
</tbody>
</table>

Table 8
Expert 3 – criteria 2.

<table>
<thead>
<tr>
<th></th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
<th>S4</th>
<th>S5</th>
<th>S6</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>s6</td>
<td>s5</td>
<td>s4</td>
<td>s5</td>
<td>s5</td>
<td>s7</td>
</tr>
<tr>
<td>A2</td>
<td>s9</td>
<td>s5</td>
<td>s4</td>
<td>s5</td>
<td>s1</td>
<td>s4</td>
</tr>
<tr>
<td>A3</td>
<td>s9</td>
<td>s7</td>
<td>s4</td>
<td>s5</td>
<td>s1</td>
<td>s4</td>
</tr>
<tr>
<td>A4</td>
<td>s6</td>
<td>s5</td>
<td>s4</td>
<td>s5</td>
<td>s1</td>
<td>s4</td>
</tr>
<tr>
<td>A5</td>
<td>s2</td>
<td>s8</td>
<td>s5</td>
<td>s3</td>
<td>s5</td>
<td>s7</td>
</tr>
<tr>
<td>A6</td>
<td>s3</td>
<td>s9</td>
<td>s4</td>
<td>s5</td>
<td>s5</td>
<td>s8</td>
</tr>
</tbody>
</table>

Table 10
Collective results.

<table>
<thead>
<tr>
<th></th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
<th>S4</th>
<th>S5</th>
<th>S6</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>s7</td>
<td>s7</td>
<td>s4</td>
<td>s5</td>
<td>s4</td>
<td>s6</td>
</tr>
<tr>
<td>A2</td>
<td>s6</td>
<td>s6</td>
<td>s3</td>
<td>s4</td>
<td>s4</td>
<td>s5</td>
</tr>
<tr>
<td>A3</td>
<td>s6</td>
<td>s7</td>
<td>s7</td>
<td>s5</td>
<td>s4</td>
<td>s7</td>
</tr>
<tr>
<td>A4</td>
<td>s5</td>
<td>s7</td>
<td>s4</td>
<td>s4</td>
<td>s4</td>
<td>s6</td>
</tr>
<tr>
<td>A5</td>
<td>s4</td>
<td>s4</td>
<td>s5</td>
<td>s7</td>
<td>s6</td>
<td>s7</td>
</tr>
<tr>
<td>A6</td>
<td>s5</td>
<td>s5</td>
<td>s6</td>
<td>s4</td>
<td>s4</td>
<td>s6</td>
</tr>
</tbody>
</table>

Step 2: With this information provided by the experts, it is possible to aggregate the opinions in order to reach a collective result. We use the WA to obtain this matrix assuming that $Z = (0.3, 0.3, 0.4)$. The results are shown in Table 10.

Step 3: To analyze the attitudinal character of the group of experts, we consider that they use order-inducing variables which represent the complex attitudinal character in the decision process: $U = (14, 18, 10, 8, 15, 17)$. The main advantage of using order-inducing variables is that we can represent complex decision processes that include psychological factors such as time pressure, personal characteristics to each alternative and other related aspects.

Next, it is also necessary to determine the parameter $\beta$ and the weighting vector $W$ and $P$. We assume that $\beta = 0.5$ and the following weighting vectors: $W = (0.1, 0.1, 0.1, 0.2, 0.2, 0.3)$ and $P = (0.1, 0.1, 0.2, 0.2, 0.2, 0.1)$. Observe that the weighting vector $W$ represents the attitudinal character of the government in this problem. On the other hand, the weighting vector $P$ expresses the available probabilistic information regarding the probability that each scenario will occur. Since the probability cannot guarantee which scenario will occur and there are a lot of uncertainties in the problem, the OWA aggregation permits to under or overestimate the probabilistic results according to the attitude of the decision maker.

Step 4: Once the previous information is available, we can develop different methods based on the ILPOWA operator for the selection of an importation strategy. The use of the ILPOWA provides a more complete picture to the decision maker because it is possible to consider different future scenarios. Due to our uncertainty, we do not know which scenario is the correct one. Therefore, the representation of different particular cases that could happen (from the minimum to the maximum) seems to be useful for gaining a complete picture of the future different situations. Thus, the decision maker knows the results that could be obtained with each alternative and select the one that seems to be in closest accordance with his interests.

In this example, we consider the linguistic maximum, the linguistic minimum, the linguistic maximum probabilistic aggregation (Max-LPA), the linguistic minimum probabilistic aggregation (Min-LPA), the linguistic arithmetic mean (LAM), the LPA, the LOWA, the ILLOWA and the ILPOWA operator. Observe that the ILPOWA operator together with the Min-LPA and the Max-LPA are the contributions of the paper. The other aggregation methods are the classical ones used in the literature. We present all of them because it is useful in order to get a complete picture of the problem without losing information in the analysis. Moreover, it permits us to compare the results of each aggregation so we can see which method provides the highest results, the lowest ones, and so on. The results are shown in Table 11.

Step 5: Each method may give a different result and decision although in most of the cases $A_3$ seems to be the optimal choice excepting for the Max-LPA and the maximum where $A_3$ is better. Thus, in this example, the solution would be to import the product from Russia. Note that sometimes it is interesting to establish a ranking of the importation strategies in order to consider more than one alternative. The results are shown in Table 12.

Depending on the particular type of aggregation operator used, the results may lead to different decisions. The main advantage
of this methodology is that the decision maker knows all the different scenarios that may occur and select the one in closest accordance to his interests. Technically, these interests should be represented by using the ILPOWA operator because its weighting vector has been specifically designed in order to achieve this issue.

### 6. Conclusions

A new approach for linguistic group decision making by using the ILPOWA operator has been developed. Its main advantage is that it can deal with risk and uncertain environments in the same formulation considering imprecise information that can be assessed with linguistic variables. Moreover, this approach uses induced aggregation operators in order to represent complex attitudinal characters. A key issue of the ILPOWA operator is that it includes a wide range of particular cases including the linguistic average, the LOWA operator and the linguistic expected value. Further extensions have been introduced by using Bonferroni means and moving averages.

This new model can be implemented in a wide range of fields because all the previous studies that use the probability or the OWA operator in a linguistic framework can be revised and extended with this approach. The paper has focussed on a linguistic multi-criteria multi-person decision making process in political management regarding the selection of the optimal importation strategies. For doing so, it has been constructed the MC-MP-ILPOWA operator that permits to deal with the opinion of several experts and criteria in the analysis. We have seen that one of the main features of this approach is that it permits to assess linguistic decision making problems under risk and uncertainty in the same formulation.

In future research, we expect to design further developments by using other characteristics including the use of weighted averages and generalized aggregation operators. Furthermore, other applications will be studied in other areas including decision theory, engineering and economics [54].

### Acknowledgements

We would like to thank the anonymous reviewers for valuable comments that have improved the quality of the paper. Support from the MAPFRE Foundation, project 099311 from the University of Barcelona and PIEF-GA-2011-300062 from the European Commission, is gratefully acknowledged.

### References