DECISION MAKING WITH THE INDUCED GENERALIZED ADEQUACY COEFFICIENT

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Abstract. We introduce the induced generalized ordered weighted averaging adequacy coefficient (IGOWAAC) operator. The main advantage is that it provides a more complete generalization of the aggregation operators that includes a wide range of situations. We apply the new approach in a decision making problem.

Keywords: Decision Making, Aggregation Operator, OWA Operator, Adequacy Coefficient.

AMS Subject Classification: 62C86.

1. Introduction

The adequacy coefficient [8-10, 13, 16] is a method for calculating the differences between two sets, fuzzy sets [1, 15, 62] and interval-valued fuzzy sets. It is very similar to the Hamming distance with the difference that it establishes a threshold from which the results are always the same. In [29], Merigó and A.M. Gil-Lafuente suggested a generalization of the adequacy coefficient by using generalized means and ordered weighted averaging (OWA) operators [2, 37, 50-51, 61]. Thus, they provided a more general form of the adequacy coefficient that included a wide range of particular cases. Since its introduction, the adequacy coefficient has been analyzed in a lot of studies [12, 26].

An interesting extension of the OWA operator is the induced OWA (IOWA) operator [60]. It is an extension that uses order inducing variables in the reordering process of the aggregation. Thus, it is able to deal with complex environments where it is not easy to establish the attitudinal character of the decision maker. In [30] a generalization of the IOWA operator that includes a wide range of particular cases by using generalized means (induced generalized OWA (IGOWA) operator) and quasi-arithmetic means (Quasi-IOWA operator) has been suggested. Note that in the literature, we find a lot of papers dealing with induced [18, 19, 21, 23-24, 42-44] and generalized [2, 7, 14, 20, 32-33, 40, 53, 64] aggregation operators.

The aim of this paper is to present a generalization of the adequacy coefficient by using the IGOWA operator in the aggregation process. We present the IGOWA adequacy coefficient (IGOWAAC) operator. It gives a very general formulation that includes a wide range of aggregation operators including the adequacy coefficient, the OWA operator, the IGOWA operator and the Minkowski distance. We study some of its main properties and we see different families of IGOWAAC operators. We further generalize the IGOWAAC by using quasi-arithmetic means (Quasi-IOWAAC).

We also develop other extensions by using different types of aggregation operators. First, we consider the use of the hybrid average [41, 47, 63] in the IGOWAAC (or Quasi-IOWAAC) operator obtaining the induced generalized hybrid averaging adequacy coefficient (IGHAAC).

Second, we analyze the use of immediate probabilities [6, 17, 59] with the IGOWAAC operator

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Manuscript received 11 October 2010.
forming the immediate probabilistic IGOWAAC (IP-IGOWAAC) operator. Third, we extend the IGOWAAC to situations where we use Choquet integrals \([4, 34, 37, 45]\) obtaining the induced generalized Choquet integral adequacy coefficient (IGCIAC). Finally, we study the IGOWAAC operator with moving averages \([54]\). Thus, we get the induced generalized ordered weighted moving averaging adequacy coefficient (IGOWMAAC). We see that these extensions can be further extended by mixing them and by using other types of aggregations. We analyze a wide range of particular cases found in these aggregation operators.

We also see that the applicability of these new aggregation operators is very similar to the previous models developed in this direction \([8-13, 16, 22, 26-30, 46]\). We focus on a multi-person decision making problem regarding the selection of investment strategies. The main advantage is that we can provide a deep representation of the decision making problem by using the opinion of several experts. We analyze the results and we see that the IGOWAAC operator provides a robust representation of the problem that can consider a wide range of particular cases that may lead to different results and decisions.

This paper is organized as follows. In Section 2, we briefly review some basic concepts about the induced aggregation operators and the adequacy coefficient. In Section 3, we present the IGOWAAC operator and several particular cases. Section 4 introduces the Quasi-IOWAAC operator and Section 5 gives other extensions based on the use of hybrid aggregations, immediate probabilities, Choquet integrals and moving averages. Section 6 and 7 develop an illustrative example in multi-person decision making and Section 8 summarizes the main results of the paper.

2. Preliminaries

In this Section we briefly review the OWA operator, the GOWA operator, the adequacy coefficient and the IOWA operator.

2.1. The OWA operator. The OWA operator was introduced by Yager \([50]\) and it provides a parameterized family of aggregation operators that include the arithmetic mean, the maximum and the minimum. It can be defined as follows.

**Definition 1.** An OWA operator of dimension \(n\) is a mapping \(\text{OWA}: \mathbb{R}^n \rightarrow \mathbb{R}\) that has an associated weighting \(W = (w_1, w_2, ..., w_n)\) of dimension \(n\) with \(\sum_{j=1}^{n} w_j = 1\) and \(w_j \in [0, 1]\), such that:

\[
\text{OWA}(a_1, ..., a_n) = \sum_{j=1}^{n} w_j b_j,
\]

where \(b_j\) is the \(j^{th}\) largest of the \(a_i\).

From a generalized perspective of the reordering step, we can distinguish between the descending OWA (DOWA) operator and the ascending OWA (AOWA) operator. The OWA operator is commutative, monotonic, bounded and idempotent. Since its appearance, the OWA operator has been studied by a lot of authors \([3, 5, 25, 51, 55]\).

2.2. The GOWA operator. The generalized OWA (GOWA) operator \([14, 53]\) generalizes a wide range of aggregation operators that includes the OWA operator with its particular cases, the OWG operator, the ordered weighted harmonic averaging (OWHA) operator and the OWQA operator. It can be defined as follows.

**Definition 2.** A GOWA operator of dimension \(n\) is a mapping \(\text{GOWA}: \mathbb{R}^n \rightarrow \mathbb{R}\) that has an associated weighting \(W\) of dimension \(n\) with \(\sum_{j=1}^{n} w_j = 1\) and \(w_j \in [0, 1]\), such that:

\[
\text{GOWA}(a_1, a_2, ..., a_n) = \left( \sum_{j=1}^{n} w_j b_j^\lambda \right)^{1/\lambda},
\]
where \( b_j \) is the \( j \)th largest of the \( a_i \) and \( \lambda \) is a parameter such that \( \lambda \in (-\infty, \infty) \).

Note that it is possible to distinguish between descending (DGOWA) and ascending (AGOWA) orders. The GOWA operator is commutative, monotonic, bounded and idempotent. It can also be demonstrated that it has as special cases the maximum, the minimum and the generalized mean. If we look to different values of the parameter \( \lambda \), we can also obtain other special cases such as the usual OWA operator, the geometric OWA (OWGA) operator and the quadratic OWA (OWQA) operator.

### 2.3. The adequacy coefficient.

The normalized adequacy coefficient [16] is an index used for calculating the differences between two elements, two sets, etc. In fuzzy set theory, it can be useful, for example, for the calculation of distances between fuzzy sets and interval-valued fuzzy sets. It is very similar to the Hamming distance with the difference that it neutralizes the result when the comparison shows that the real element is higher than the ideal one. The weighted adequacy coefficient is an adequacy coefficient that normalizes the information using the weighted average. It can be defined as follows for two sets \( X = \{x_1, ..., x_n\} \) and \( Y = \{y_1, ..., y_n\} \).

**Definition 3.** A weighted adequacy coefficient (WAC) of dimension \( n \) is a mapping \( WAC: [0,1]^n \times [0,1]^n \rightarrow [0,1] \) that has an associated weighting \( W \) of dimension \( n \) with \( \sum_{i=1}^n w_i = 1 \) and \( w_i \in [0,1] \), such that:

\[
WAC(X,Y) = \sum_{i=1}^n w_i [1 \land (1 - x_i + y_i)], \tag{3}
\]

where \( x_i \) and \( y_i \) is the \( i \)th arguments of the sets \( X \) and \( Y \) respectively.

In [29,31], they proposed a new version of the adequacy coefficient that uses the OWA operator in the aggregation. They called it the OWAAC operator. It can be defined as follows for two sets \( X = \{x_1, ..., x_n\} \) and \( Y = \{y_1, ..., y_n\} \).

**Definition 4.** An OWAAC operator of dimension \( n \) is a mapping \( OWAAC: [0,1]^n \times [0,1]^n \rightarrow [0,1] \) that has an associated weighting \( W \) of dimension \( n \) with \( \sum_{j=1}^n w_j = 1 \) and \( w_j \in [0,1] \), such that:

\[
OWAAC(\langle x_1, y_1 \rangle, ..., \langle x_n, y_n \rangle) = \sum_{j=1}^n w_j K_j, \tag{4}
\]

where \( K_j \) is the \( j \)th largest of the \([1 \land (1 - x_i + y_i)]\), and \( x_i, y_i \in [0,1] \).

The OWAAC operator can be generalized by using generalized and quasi-arithmetic means. The result is the generalized OWAAC (GOWAAC) and the Quasi-OWAAC operator [29]. The GOWAAC operator can be defined as follows.

**Definition 5.** A GOWAAC operator of dimension \( n \) is a mapping \( GOWAAC: [0,1]^n \times [0,1]^n \rightarrow [0,1] \) that has an associated weighting \( W \) of dimension \( n \) with \( \sum_{j=1}^n w_j = 1 \) and \( w_j \in [0,1] \), such that:

\[
GOWAAC(\langle x_1, y_1 \rangle, ..., \langle x_n, y_n \rangle) = \left( \sum_{j=1}^n w_j K_j^\lambda \right)^{1/\lambda}, \tag{5}
\]

where \( K_j \) is the \( j \)th largest of the \([1 \land (1 - x_i + y_i)]\), \( x_i, y_i \in [0,1] \), \( \lambda \) is a parameter such that \( \lambda \in (-\infty, \infty) \).

### 2.4. The IOWA operator.

The IOWA operator was introduced by Yager and Filev [60] and it represents an extension of the OWA operator. The main difference is that the reordering step of the IOWA is carried out with order-inducing variables, rather than depending on the values of the arguments \( a_i \). The IOWA operator also includes the maximum, the minimum and the average operator, as special cases. It can be defined as follows.
Definition 6. An IOWA operator of dimension $n$ is a mapping $\text{IOWA}: R^n \times R^n \rightarrow R$ that has an associated weighting $W$ of dimension $n$ with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, and a set of order-inducing variables $u_i$, by a formula of the following form:

$$\text{IOWA}((u_1, a_1), (u_2, a_2), \ldots, (u_n, a_n)) = \sum_{j=1}^{n} w_j b_j,$$

where $(b_1, \ldots, b_n)$ is simply $(a_1, a_2, \ldots, a_n)$ reordered in decreasing order of the values of the $u_i$, $u_i$ is the order-inducing variable and $a_i$ is the argument variable.

The IOWA operator can be generalized by using generalized and quasi-arithmetic means. The result is the IGOWA and the Quasi-IOWA operator. For example, the Quasi-IOWA operator can be defined as follows.

Definition 7. A Quasi-IOWA operator of dimension $n$ is a mapping $\text{QIOWA}: R^n \times R^n \rightarrow R$ that has an associated weighting $W$ of dimension $n$ with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$\text{QIOWA}((u_1, a_1), (u_2, a_2), \ldots, (u_n, a_n)) = g^{-1}\left(\sum_{j=1}^{n} w_j g(b_{(j)})\right),$$

where $(b_1, \ldots, b_n)$ is simply $(a_1, a_2, \ldots, a_n)$ reordered in decreasing order of the values of the $u_i$, $u_i$ is the argument variable, $a_i$ is the argument variable and $g$ is a strictly continuous monotonic function.

3. The Induced Generalized Ordered Weighted Averaging Adequacy Coefficient

In this Section we analyze the IGOWAAC and study several of its main properties including a wide range of particular cases.

3.1. Introduction. The IGOWAAC operator is a new aggregation operator that uses induced aggregation operators, generalized means and the adequacy coefficient in the OWA operator. Thus, it uses complex reordering processes assessed with order-inducing variables in the adequacy coefficient representing different attitudinal characters of the decision maker. The main advantage is that it provides a complete generalization of the adequacy coefficient that includes a wide range of particular cases. It can be defined as follows.

Definition 8. A IGOWAAC operator of dimension $n$ is a mapping $\text{IGOWAAC}: [0, 1]^n \times [0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ that has an associated weighting $W$ of dimension $n$ with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$\text{IGOWAAC}((u_1, x_1, y_1), \ldots, (u_n, x_n, y_n)) = \left(\sum_{j=1}^{n} w_j K_j^\lambda\right)^{1/\lambda},$$

where $K_j$ is the $[1 \wedge (1 - x_i + y_i)]$ value of the IGOWAAC pair $(u_i, a_i)$ having the $j$th largest $u_i$, $u_i$ is the order inducing variable, $x_i, y_i \in [0, 1]$ and $\lambda$ is a parameter such that $\lambda \in (-\infty, \infty)$.

Example 1. Assume the following arguments in an aggregation process: $X = (0.4, 0.8, 0.6, 0.7)$, $Y = (0.2, 0.9, 0.4, 0.6)$. Assume the following weighting vector $W = (0.2, 0.2, 0.3, 0.3)$, the order inducing variables $U = (23, 17, 13, 28)$ and the parameter $\lambda = 1$. If we calculate the similarity between $X$ and $Y$ using the IGOWAAC operator, we get the following:

$$\text{IGOWAAC}(X, Y) = 0.2 \times [1 \wedge (1 - 0.7 + 0.6)] + 0.2 \times [1 \wedge (1 - 0.4 + 0.2)]$$
$$+ 0.3 \times [1 \wedge (1 - 0.8 + 0.9)] + 0.3 \times [1 \wedge (1 - 0.6 + 0.4)] = 0.88.$$
Note that it is possible to distinguish between the descending induced generalized OWAAC (DIGOWAAC) operator and the ascending induced generalized OWAAC (AIGOWAAC) operator by using $w_j = w_{n+1-j}$, where $w_j$ is the $j$th weight of the DIGOWAAC operator and $w_{n+1-j}$ the $j$th weight of the AIGOWAAC operator.

Another interesting transformation can be developed [51] by using $w_i^* = (1 + w_i)/(n - 1)$. Furthermore, we can also analyze situations with buoyancy measures [51]. In this case, we assume that $w_i \geq w_j$, for $i < j$. Note that it is also possible to consider a stronger case known as extensive buoyancy measure where $w_i > w_j$, for $i < j$. Additionally, we can also consider the contrary case, that is, $w_i \leq w_j$, for $i < j$, and the contrary case of the extensive measure $w_i < w_j$, for $i < j$.

If $B$ is the vector consisting of the ordered arguments $K_j^\lambda$, and $W^T$ is the transpose of the weighting vector, then the IGOWAAC operator can be expressed as

$$IGOWAAC \left(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle\right) = (W^T B)^{1/\lambda}. \quad (9)$$

Note that if the weighting vector is not normalized, i.e., $W = \sum_{j=1}^{n} w_j \neq 1$, then, the IGOWAAC operator can be expressed as

$$IGOWAAC \left(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle\right) = \left(\frac{1}{W} \sum_{j=1}^{n} w_j K_j^\lambda\right)^{1/\lambda}. \quad (10)$$

Analogously to the IGOWAAC operator, we can suggest a removal index that is the dual of the IGOWAAC operator, because $Q \left(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle\right) = 1 - K \left(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle\right)$. We will call it the IGOWADAC operator.

The IGOWAAC operator is a mean or averaging operator. This is a reflection of the fact that the operator is commutative, monotonic, bounded and idempotent. The IGOWAAC is also nonnegative and reflexive. These properties can be proved with the following theorems.

**Theorem 1** (Monotonicity). Assume $f \left(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle\right)$ and $g \left(\langle u_1, s_1, t_1 \rangle, \ldots, \langle u_n, s_n, t_n \rangle\right)$ are two examples of IGOWAAC operators, if $[1 \land (1 - x_i + y_i)] \geq [1 \land (1 - s_i + t_i)]$, for all $i$, then:

$$f \left(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle\right) \geq g \left(\langle u_1, s_1, t_1 \rangle, \ldots, \langle u_n, s_n, t_n \rangle\right). \quad (11)$$

**Proof.** Let

$$f \left(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle\right) = \sum_{j=1}^{n} w_j K_j, \quad (12)$$

$$g \left(\langle u_1, s_1, t_1 \rangle, \ldots, \langle u_n, s_n, t_n \rangle\right) = \sum_{j=1}^{n} w_j Q_j. \quad (13)$$

Since $[1 \land (1 - x_i + y_i)] \geq [1 \land (1 - s_i + t_i)]$, for all $i$, it follows that $K_j \geq Q_j$ and then

$$f \left(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle\right) \geq f \left(\langle u_1, s_1, t_1 \rangle, \ldots, \langle u_n, s_n, t_n \rangle\right).$$

**Theorem 2** (Commutativity). Assume $f$ is the IGOWAAC operator, then:

$$f \left(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle\right) = g \left(\langle u_1, s_1, t_1 \rangle, \ldots, \langle u_n, s_n, t_n \rangle\right). \quad (14)$$

where $(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle)$ is any permutation of the arguments $(\langle u_1, s_1, t_1 \rangle, \ldots, \langle u_n, s_n, t_n \rangle)$.

**Proof.** Let

$$f \left(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle\right) = \sum_{j=1}^{n} w_j K_j, \quad (15)$$
\[ g(\langle u_1, s_1, t_1 \rangle, \ldots, \langle u_n, s_n, t_n \rangle) = \sum_{j=1}^{n} w_j Q_j. \]  

(16)

Since \((\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle)\) is a permutation of \((\langle u_1, s_1, t_1 \rangle, \ldots, \langle u_n, s_n, t_n \rangle)\), we have \(K_j = Q_j\) for all \(j\) and then

\[ f(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle) = g(\langle u_1, s_1, t_1 \rangle, \ldots, \langle u_n, s_n, t_n \rangle). \]

**Theorem 3** (Idempotency). Assume \(f\) is the IGOWAAC operator, if \([1 \land (1 - x_i + y_i)] = [1 \land (1 - x + y)]\), for all \(i\), then:

\[ f(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle) = [1 \land (1 - x + y)]. \]  

(17)

**Proof.** Since \([1 \land (1 - x_i + y_i)] = [1 \land (1 - x + y)]\), for all \(i\), we have

\[ f(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle) = \sum_{j=1}^{n} w_j K_j = \sum_{j=1}^{n} w_j [1 \land (1 - x + y)] \]

\[ = [1 \land (1 - x + y)] \sum_{j=1}^{n} w_j. \]  

(18)

Since \(\sum_{j=1}^{n} w_j = 1\), we get

\[ f(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle) = [1 \land (1 - x + y)]. \]  

**Theorem 4** (Bounded). Assume \(f\) is the IGOWAAC operator, then

\[ \text{Min}\{[1 \land (1 - x_i + y_i)]\} \leq f(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle) \leq \text{Max}\{[1 \land (1 - x_i + y_i)]\}. \]  

(19)

**Proof.** Let \(\text{Max}\{[1 \land (1 - x_i + y_i)]\} = b\), and \(\text{Min}\{[1 \land (1 - x_i + y_i)]\} = a\), then

\[ f(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle) = \sum_{j=1}^{n} w_j K_j \leq \sum_{j=1}^{n} w_j b = b \sum_{j=1}^{n} w_j, \]  

(20)

\[ f(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle) = \sum_{j=1}^{n} w_j K_j \geq \sum_{j=1}^{n} w_j a = a \sum_{j=1}^{n} w_j. \]  

(21)

Since \(\sum_{j=1}^{n} w_j = 1\), we get

\[ f(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle) \leq b, \]  

(22)

\[ f(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle) \geq a. \]  

(23)

Therefore,

\[ \text{Min}\{[1 \land (1 - x_i + y_i)]\} \leq f(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle) \leq \text{Max}\{[1 \land (1 - x_i + y_i)]\}. \]

**Theorem 5** (Nonnegativity). Assume \(f\) is the IGOWAAC operator, then

\[ f(\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle) \geq 0. \]  

(24)

**Proof.** Let
\[ f(\langle u_1, x_1, y_1 \rangle, ..., \langle u_n, x_n, y_n \rangle) = \sum_{j=1}^{n} w_j K_j. \]  

Since \([1 \wedge (1 - x_i + y_i)] \geq 0\), for all \(i\), we obtain
\[ f(\langle u_1, x_1, y_1 \rangle, ..., \langle u_n, x_n, y_n \rangle) \geq 0. \]

**Theorem 6** (Nonnegativity). Assume \(f\) is the IGOWAAC operator, then
\[ f(\langle u_1, x_1, x_1 \rangle, ..., \langle u_n, x_n, x_n \rangle) = \sum_{j=1}^{n} w_j K_j. \]

Since \(x_i = x_i, [1 \wedge (1 - x_i + x_i)] = 1\), for all \(i\), we obtain
\[ f(\langle u_1, x_1, x_1 \rangle, ..., \langle u_n, x_n, x_n \rangle) = 1. \]

Another interesting issue to consider is that the IGOWAAC operator becomes the induced Minkowski OWA distance (IMOWAD) operator under certain conditions [26,29]. As it is explained in [26], the adequacy coefficient and the Hamming distance (and also further generalizations by using generalized and quasi-arithmetic means) become the same measure when the adequacy coefficient fulfills the following theorem.

**Theorem 7.** Assume \(IMOWAD(\langle u_1, x_1, x_1 \rangle, ..., \langle u_n, x_n, x_n \rangle)\) is the IMOWAD operator, and \(IGOWADAC(\langle u_1, x_1, x_1 \rangle, ..., \langle u_n, x_n, x_n \rangle)\) is the IGOWADAC operator. If \(x_i \geq y_i\) for all \(i\), then:
\[ IMOWAD(\langle u_1, x_1, y_1 \rangle, ..., \langle u_n, x_n, y_n \rangle) = IGOWADAC(\langle u_1, x_1, x_1 \rangle, ..., \langle u_n, x_n, x_n \rangle). \]

**Proof.** Let
\[ IGOWADAC(\langle u_1, x_1, x_1 \rangle, ..., \langle u_n, x_n, x_n \rangle) = \left( \sum_{j=1}^{n} w_j [0 \vee (x_i - y_i)]^{1/\lambda} \right). \]
\[ IMOWAD(\langle u_1, x_1, y_1 \rangle, ..., \langle u_n, x_n, y_n \rangle) = \left( \sum_{j=1}^{n} w_j |x_i - y_i|^{1/\lambda} \right). \]

Since \(x_i \geq y_i\) for all \(i\), \([0 \vee (x_i - y_i)] = (x_i - y_i)\) for all \(i\), then
\[ IMOWAD(\langle u_1, x_1, y_1 \rangle, ..., \langle u_n, x_n, y_n \rangle) = IGOWADAC(\langle u_1, x_1, x_1 \rangle, ..., \langle u_n, x_n, x_n \rangle). \]

Another interesting extension can be developed by using infinitary aggregation operators [35]. In this case, we can represent an aggregation process where there is an unlimited number of arguments that appear in the aggregation process. Note that \(\sum_{j=1}^{\infty} w_j = 1\). By using the IGOWAAC operator we get the infinitary IGOWAAC (\(\infty\)-IGOWAAC) operator as follows:
\[ \infty - IGOWAAC(\langle u_1, x_1, y_1 \rangle, ..., \langle u_n, x_n, y_n \rangle) = \sum_{j=1}^{\infty} w_j K_j. \]
order inducing variables such that \( r : I \to R \), being that \( I \subset R \) is a closed interval \( I = [a, b] \). In this paper, we use a more general representation by using a generating function for the arguments such that \( s : R^m \to R \) where \( m \) is the number of previous arguments considered for obtaining the final arguments to be aggregated. Moreover, we use a weighting function \( f \) for the weighting vector. Thus, we present a general type of function induced mixture generalized adequacy coefficient (IMGAC). Note that we present this formulation by using quasi-arithmetic means [2, 7, 30]. In this definition, we refer to the arguments as two sets \( X = \{x_1, x_2, ..., x_n\} \) and \( Y = \{y_1, y_2, ..., y_n\} \).

**Definition 9.** An IMGAC operator of dimension \( n \) is a mapping IMGAC: \([0, 1]^n \times [0, 1]^n \times R^n \to [0, 1]\) that has associated two vectors of weighting functions \( f, r : I \to [0, \infty] \) and \( s : R^m \to R \), such that:

\[
IMGAC ((r_0(u_1), s_p(x_1), s_q(y_1)), ..., (r_0(u_n), s_p(x_n), s_q(y_n))) = g^{-1} \left( \frac{\sum_{j=1}^{n} f_j(s_p(K_j))g(s_q(K_j))}{\sum_{j=1}^{n} f_j(s_q(K_j))} \right),
\]

(31)

where \( g \) is a strictly continuous monotonic function, \( s_q(K_j) \) is the \([1 \wedge (1 - s_p(x_i) + s_q(y_i))] \) value of the IMGAC triplet \((r_0(u_i), s_p(x_i), s_q(y_i))\) having the \( j \)th largest \( r_0(u_i) \): \( u_i \) is the order-inducing variable, \([1 \wedge (1 - s_p(x_i) + s_q(y_i))] \) is the argument variable represented in the form of individual similarities, and \( 0, p \) and \( q \) indicates that each order-inducing variable and each argument is formed by using a different function.

Another interesting issue to analyze is the different measures used to characterize the weighting vector of the IGOWAAC operator. For example, we could consider the entropy of dispersion and the divergence of \( W \). The dispersion is a measure that provides knowledge on the type of information being used. It can be defined as follows.

\[
H(W) = -\sum_{j=1}^{n} w_j \ln(w_j).
\]

(32)

For example, if \( w_j = 1 \) for some \( j \) then \( H(W) = 0 \) and the least amount of information is used. If \( w_j = 1/n \) for all \( j \), then the amount of information used is maximum.

The divergence can be defined as follows.

\[
Div(W) = \sum_{j=1}^{n} w_j \left( \frac{n - j}{n - 1} - \alpha(W) \right)^2.
\]

(33)

A further interesting issue is the problem of ties in the reordering step of the aggregation with order-inducing variables. To solve this problem, we recommend following the method developed by Yager and Filev [60] where they replace each argument of the tied IGOWAAC pair by its average. For the IGOWAAC operator, instead of using the arithmetic mean, we replace each argument of the tied IGOWAAC pair by the generalized adequacy coefficient depending on the parameter of \( \lambda \).

### 3.2. Families of IGOWAAC operators.

Different types of IGOWAAC operators may be used in the aggregation process. Mainly, we can distinguish between those families found in the weighting vector \( W \) and those found in the parameter \( \lambda \).

**Remark 1.** When \( \lambda \to \infty \), we obtain the maximum AC.

\[
IGOWAAC ((u_1, x_1, y_1), ..., (u_n, x_n, y_n)) = \max \{[1 \wedge (1 - x_i + y_i)]\}.
\]

(34)

**Remark 2.** When \( \lambda \to -\infty \), we obtain the minimum AC.

\[
IGOWAAC ((u_1, x_1, y_1), ..., (u_n, x_n, y_n)) = \min \{[1 \wedge (1 - x_i + y_i)]\}.
\]

(35)
Remark 3. When $\lambda = 1$, the IGOWAAC operator becomes the IOWAAC operator.

$$IGOWAAC ((u_1, x_1, y_1), \ldots, (u_n, x_n, y_n)) = \sum_{j=1}^{n} w_j K_j.$$  \hspace{1cm} (36)

Remark 4. When $\lambda \to 0$, we form the IOWGAAC operator.

$$IGOWAAC ((u_1, x_1, y_1), \ldots, (u_n, x_n, y_n)) = \prod_{j=1}^{n} K_j^{w_j}.$$  \hspace{1cm} (37)

Remark 5. When $\lambda = -1$, we get the IOWHAAC operator.

$$IGOWAAC ((u_1, x_1, y_1), \ldots, (u_n, x_n, y_n)) = \frac{1}{\sum_{j=1}^{n} w_j K_j^\lambda}.$$  \hspace{1cm} (38)

Remark 6. When $\lambda = 2$, we obtain the IOWQAAC operator.

$$IGOWAAC ((u_1, x_1, y_1), \ldots, (u_n, x_n, y_n)) = \left( \sum_{j=1}^{n} w_j K_j^\lambda \right)^{1/2}.$$  \hspace{1cm} (39)

Remark 7. If we analyse the weighting vector $W$, then, we find the following cases:

- The maximum ($w_1 = 1$ and $w_j = 0$, for all $j \neq 1$).
- The minimum ($w_n = 1$ and $w_j = 0$, for all $j \neq n$).
- The generalized adequacy coefficient ($w_j = 1/n$, for all $i$).
- The weighted generalized adequacy coefficient (the ordered position of $i$ is the same as the ordered position of $u_i$).
- The generalized Hurwicz adequacy coefficient criteria ($w_p = \alpha$, $w_q = 1 - \alpha$, $w_j = 0$, for all $j \neq p, q$, and $u_p = \text{Max} \{a_i\}$ and $u_q = \text{Min} \{a_i\}$).
- The GOWAAC operator (the ordered position of $j$ is the same as the ordered position of $u_i$).
- The step-IGOWAAC ($w_k = 1$ and $w_j = 0$, for all $k \neq j$).
- The S-IGOWAAC ($w_p = (1/n)(1 - (\alpha + \beta)) + \alpha$, $w_q = (1/n)(1 - (\alpha + \beta))$ + $\beta$, $u_p = \text{Max} \{a_i\}$ and $u_q = \text{Min} \{a_i\}$, and $w_j = (1/n)(1 - (\alpha + \beta))$ for $j \neq p, q$, where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$).
- The centered-IGOWAAC (if it is symmetric, strongly decaying from the center to the maximum and the minimum, and inclusive).
- The olympic-IGOWAAC operator ($w_1 = w_n = 0$, and $w_j = 1/(n - 2)$ for all others).
- Etc.

Remark 8. We could develop a lot of other families of IGOWAAC weights in a similar way as it has been developed in a lot of studies [5, 23, 31, 51].

4. **The Quasi-IOWAAC Operator**

Note that it is possible to further generalize the IGOWAAC operator by using quasi-arithmetic means. The result is the Quasi-IOWAAC operator.

**Definition 10.** A Quasi-IOWAAC operator of dimension $n$ is a mapping $QIOWAAC: [0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ that has an associated weighting $W$ of dimension $n$ with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$QIOWAAC ((u_1, x_1, y_1), \ldots, (u_n, x_n, y_n)) = g^{-1} \left( \sum_{j=1}^{n} w_j g (b_{(j)}) \right),$$  \hspace{1cm} (40)
Thus, the radical IOWAAC operator is:

\[ \text{QIOWAAC} (\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle) = \frac{2}{\pi} \arctan \left( \frac{\sum_{j=1}^{n} w_j \tan \left( \frac{\pi}{2} K_j \right)}{\sum_{j=1}^{n} w_j \sec \left( \frac{\pi}{2} K_j \right)} \right) \]

The exponential IOWAAC is found when \( g(t) = r^t \) if \( r \neq 1 \) and \( g(t) = t \) if \( r = 1 \). Then, the exponential IOWAAC operator is:

\[ \log_r \left( \frac{\sum_{j=1}^{n} w_j r^{b_j}}{\sum_{j=1}^{n} w_j} \right) \]

The radical IOWAAC is found if \( r > 0, r \neq 1 \), and the generating function is \( g(t) = r^{1/t} \). Thus, the radical IOWAAC operator is:

\[ \text{QIOWAAC} (\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle) = \left( \log_r \left( \frac{\sum_{j=1}^{n} w_j r^{1/b_j}}{\sum_{j=1}^{n} w_j} \right) \right)^{-1} \]

5. OTHER EXTENSIONS BY USING THE IGOWAAC OPERATOR

In this Section, we analyze several extensions of the IGOWAAC operator by using other techniques such as the use of hybrid aggregations [19, 47, 63], immediate probabilities [6, 17, 59], Choquet integrals [4, 23-24, 34, 37, 45, 53] and moving averages [54]. First, we consider the use of hybrid aggregations. Thus, we get the induced generalized hybrid averaging adequacy coefficient (IGHAAC). This aggregation operator uses induced and generalized aggregation operators in the adequacy coefficient in a unified framework between the weighted average and the OWA operator. Furthermore, we can generalize this aggregation operator by using quasi-arithmetic means obtaining a deeper generalization. We call it the induced quasi-arithmetic hybrid averaging adequacy coefficient (Quasi-IHAAC). It is defined as follows.

**Definition 11.** A Quasi-IHAAC operator of dimension \( n \) is a mapping \( \text{QIHAAC}: [0,1]^n \times [0,1]^n \rightarrow [0,1] \) that has an associated weighting \( W \) of dimension \( n \) with \( \sum_{j=1}^{n} w_j = 1 \) and

\[ v_i = \sum_{j=1}^{n} w_j f_{ij} \]

where \( b_j \) is the \( [1 \wedge (1 - x_i + y_i)] \) value of the Quasi-IOWAAC pair \( (u_i, x_i, y_i) \) having the \( j \)th largest \( u_i, u_i \) is the order inducing variable, \( x_i, y_i \in [0,1] \) and \( g \) is a strictly continuous monotonic function.

Note that it is also possible to suggest an equivalent removal index that it is a dual of the Quasi-IOWAAC because \( Q(X,Y) = 1 - K(X,Y) \). We call it the Quasi-IOWADAC. Note also that all the properties and particular cases commented in the IGOWAAC operator are also applicable in the Quasi-IOWAAC operator. For example, if \( w_j = 1/n \), for all \( a_i \), then, we get the Quasi-NAC operator and if the ordered position of \( i \) is the same as the ordered position of \( u_i \), then, we get the Quasi-WAC.

A further interesting issue is that the Quasi-IOWAAC operator includes a lot of other particular cases that are not included in the IGOWAAC operator. For example, we could mention the trigonometric IOWAAC operator, the exponential IOWAAC operator and the radical IOWAAC operator.

The trigonometric IOWAAC is found when \( g_1(t) = \sin((\pi/2)t) \), \( g_2(t) = \cos((\pi/2)t) \) and \( g_3(t) = \tan((\pi/2)t) \) are the generating functions. Thus, the trigonometric IOWAAC functions are:

\[ \text{QIOWAAC} (\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle) = \frac{2}{\pi} \arcsin \left( \sum_{j=1}^{n} w_j \sin \left( \frac{\pi}{2} K_j \right) \right) \]

\[ \text{QIOWAAC} (\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle) = \frac{2}{\pi} \arccos \left( \sum_{j=1}^{n} w_j \cos \left( \frac{\pi}{2} K_j \right) \right) \]

\[ \text{QIOWAAC} (\langle u_1, x_1, y_1 \rangle, \ldots, \langle u_n, x_n, y_n \rangle) = \frac{2}{\pi} \arctan \left( \sum_{j=1}^{n} w_j \tan \left( \frac{\pi}{2} K_j \right) \right) \]
\[ QIHAA \left( \left< u_1, x_1, y_1 \right>, \ldots, \left< u_n, x_n, y_n \right> \right) = g^{-1} \left( \sum_{j=1}^{n} w_j g \left( K_j \left< K_{(j)} \right> \right) \right), \tag{45} \]

where \( K_j \) is the \([1 \wedge (1 - x_i + y_i)]^* = n \omega_i[1 \wedge (1 - x_i + y_i)]\) values reordered in decreasing order of the values of the \( u_i \), \( u_i \) is the order inducing variable, \( x_i \) and \( y_i \) are the argument variables and \( g \) is a strictly continuous monotonic function.

A further generalization can also be extended by using quasi-arithmetic means, obtaining the immediate probabilistic - Quasi-IOWAAC (IP-Quasi-IOWAAC) operator. It can be defined as follows.

**Definition 12.** An IP-Quasi-IOWAAC operator of dimension \( n \) is a mapping \( IP\text{-QIOWAAC}: [0, 1]^n \times [0, 1]^n \times [0, 1]^n \rightarrow [0, 1] \) that has an associated weighting \( W \) of dimension \( n \) with \( \sum_{j=1}^{n} w_j = 1 \) and \( w_j \in [0, 1] \), such that:

\[ IP - QIOWAAC \left( \left< u_1, x_1, y_1 \right>, \ldots, \left< u_n, x_n, y_n \right> \right) = g^{-1} \left( \sum_{j=1}^{n} v_j g \left( K_j \left< K_{(j)} \right> \right) \right), \tag{46} \]

where \( K_j \) is the \([1 \wedge (1 - x_i + y_i)]^* \) values reordered in decreasing order of the values of the \( u_i \), \( u_i \) is the order inducing variable, \( x_i, y_i \in [0, 1] \), each \([1 \wedge (1 - x_i + y_i)]^* \) has associated a \( u_i \), \( v_j \) is the associated probability of \( K_j \), \( v_j = (w_j v_j / \sum_{j=1}^{n} w_j v_j) \) and \( g \) is a strictly continuous monotonic function.

Note that this model can also be extended using quasi-arithmetic Choquet integral [4]. In this case, we obtain the induced quasi-arithmetic Choquet interval adequacy coefficient (Quasi-ICDIAC). It can be defined as follows.

**Definition 13.** Let \( m \) be a fuzzy measure on \( X \). An induced quasi-arithmetic Choquet integral adequacy coefficient (Quasi-ICIAC) operator of dimension \( n \) is a function \( QICIAC: [0, 1]^n \times [0, 1]^n \times [0, 1]^n \rightarrow [0, 1] \), such that:

\[ QICIAC \left( \left< u_1, x_1, y_1 \right>, \ldots, \left< u_n, x_n, y_n \right> \right) = g^{-1} \left( \sum_{j=1}^{n} g \left( K_j \left( m \left( A(j) \right) - m \left( A(i-1) \right) \right) \right) \right), \tag{47} \]

where \( g \) is strictly continuous monotonic function; \( K_j \) is the \([1 \wedge (1 - x_i + y_i)]^* \) value of the triplet \( \left< u_i, x_i, y_i \right> \) having the \( j \)th largest \( u_i \), \( u_i \) is the order inducing variable, \( [1 \wedge (1 - x_i + y_i)]^* \) is the argument variable represented in the form of individual similarities; \( A(i) = \{ x(i), \ldots, x_{(i)} \}, i \geq 1 \) and \( A(0) = \emptyset \).

Furthermore, we can extend the IGOWAAC and the Quasi-IOWAAC operators by using moving averages. Thus, we get the induced generalized ordered weighted moving average adequacy coefficient (IGOWMAAC) and the induced quasi-arithmetic ordered weighted moving average adequacy coefficient (Quasi-IGOWMAAC), respectively. For two sets, \( X = \{ x_{1+t}, x_{2+t}, \ldots, x_{m+t} \} \) and \( Y = \{ y_{1+t}, y_{2+t}, \ldots, y_{m+t} \} \), we can define it as follows.

**Definition 14.** A Quasi-IGOWMAAC operator of dimension \( m \) is a mapping \( QIOWMAAC: [0, 1]^m \times [0, 1]^m \rightarrow [0, 1] \) that has an associated weighting \( W \) of dimension \( n \) with
\[ \sum_{j=1+t}^{m+t} w_j = 1 \text{ and } w_j \in [0, 1] \], according to the following formula:

\[
QIOW MAAC((u_{1+t}, x_{1+t}, y_{1+t}), \ldots, (u_{m+t}, x_{m+t}, y_{m+t})) = g^{-1} \left( \sum_{j=1+t}^{m+t} g(K_j) \right), \tag{48} \]

where \( K_j \) is the \([1 \wedge (1 - x_i + y_i)]\) value of the QIOWMAAC triplet \((u_i, x_i, y_i)\) having the \(j\)th largest \( u_i \), \( u_i \) is the order inducing variable, \([1 \wedge (1 - x_i + y_i)]\) is the argument variable represented in the form of individual similarities, \( m \) is the total number of arguments considered from the whole sample, \( t \) indicates the movement done in the average from the initial analysis and \( g \) is a strictly continuous monotonic function.

Note that the Quasi-IOWMAAC operator can also be extended by using hybrid aggregations, immediate probabilities and Choquet integrals. Additionally, it is also possible to formulate these new approaches in a more complete way by using a function for the formation of the order-inducing variables and the arguments as explained in Definition 9, that is, by using mixture operators and related techniques. Thus, we get the quasi-arithmetic induced mixture hybrid averaging adequacy coefficient (Quasi-IMHAAC), the immediate probabilistic quasi-arithmetic induced mixture adequacy coefficient (IP-Quasi-IMAC), the quasi-arithmetic induced mixture Choquet adequacy coefficient (Quasi-IMCAC), the quasi-arithmetic induced mixture moving adequacy coefficient (Quasi-IMMAC) and so on. Moreover, we can also consider the dual of all these extensions by using \( Q(X, Y) = 1 - K(X, Y) \).

In the following, we analyze several particular cases of the four previous definitions given in this section. We use the methodology explained in Section 3.2 where we are able to analyze a wide range of particular types of IGOWAAC operators. In this case, we analyze several families of the Quasi-IHAAC, IP-Quasi-IOWAAC, Quasi-ICDIAC and Quasi-IOWMAAC operators. The results are given in Table 1.

<table>
<thead>
<tr>
<th>Particular case</th>
<th>Quasi-IHAAC</th>
<th>IP-QIOWAAC</th>
<th>Quasi-ICIAC</th>
<th>QIOWMAAC</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_j = 1/n ), for all ( j )</td>
<td>Quasi-WAC</td>
<td>Quasi-PAC</td>
<td>Quasi-NAC</td>
<td>Quasi-NMAC</td>
</tr>
<tr>
<td>( g(a) = a^\lambda )</td>
<td>IGHAAAC</td>
<td>IP-IGOWAAC</td>
<td>IGGIAC</td>
<td>IGOWMAAC</td>
</tr>
<tr>
<td>( g(a) = a )</td>
<td>IHAAAC</td>
<td>IP-IOWAAC</td>
<td>ICIAC</td>
<td>IOWMAAC</td>
</tr>
<tr>
<td>( g(a) \rightarrow a^\lambda, \lambda \rightarrow 0 )</td>
<td>IHQAAC</td>
<td>IP-IOWQAAC</td>
<td>ICQIAC</td>
<td>IOWQMAAC</td>
</tr>
<tr>
<td>( g(a) \rightarrow a^\lambda, \lambda \rightarrow -\infty )</td>
<td>IHCAAC</td>
<td>IP-IOWCAAC</td>
<td>ICCIAC</td>
<td>IOWCMAAC</td>
</tr>
</tbody>
</table>
Note that it is also possible to use other techniques such as the use of Bonferroni means \[56\], different types of norms \[58\], prioritized aggregations \[57\] and a lot of other methods.

6. Financial decision making with the IGOWAAC operator

The IGOWAAC operator is applicable in a wide range of situations such as in decision making, statistics, engineering and economics. In this paper, we consider a multi-person decision making application in investment selection. The use of the IGOWAAC operator can be useful in a lot of situations, for example, when the board of directors of a company wants to take a decision. Obviously, the attitudinal character of the board of directors is very complex because it involves the decision of different persons with different interests.

The process to follow in the selection of investments with the IGOWAAC operator is similar to the process developed in \[8, 16\], with the difference that now we are considering a financial management problem. The 5 steps of the decision process can be summarized as follows:

Step 1: Analysis and determination of the significant characteristics of the available investment strategies for the company. Theoretically, it is represented as:

\[ C = \{C_1, C_2, ..., C_i, ..., C_n\} \]

where \( C_i \) is the \( i \)th characteristic of the investment strategy and we suppose a limited number \( n \) of characteristics.

Step 2: Fixation of the ideal levels of each characteristic in order to form the ideal investment strategy.

<table>
<thead>
<tr>
<th>( P ) = ( x_1 \ x_2 \ ... \ x_i \ ... \ x_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 ) ( C_2 ) \ ... ( C_i ) \ ... ( C_n )</td>
</tr>
</tbody>
</table>

where \( P \) is the ideal investment strategy expressed by a fuzzy subset, \( C_i \) is the \( i \)th characteristic to consider and \( x_i \in [0, 1] \); \( i = 1, 2, ..., n \), is a number between 0 and 1 for the \( i \)th characteristic.

Step 3: Fixation of the real level of each characteristic for all the investment strategies considered.

<table>
<thead>
<tr>
<th>( P_k ) = ( y_1 \ y_2 \ ... \ y_i \ ... \ y_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 ) ( C_2 ) \ ... ( C_i ) \ ... ( C_n )</td>
</tr>
</tbody>
</table>

with \( k = 1, 2, ..., m \); where \( P_k \) is the \( k \)th investment strategy expressed by a fuzzy subset, \( C_i \) is the \( i \)th characteristic to consider and \( y_i \in [0, 1] \); \( i = 1, 2, ..., n \), is a number between 0 and 1 for the \( i \)th characteristic of the \( k \)th investment strategy. Note that in this step we may have to consider the opinion of several experts in the analysis. Thus, we have to develop a multi-person aggregation process. In this case, we get the multi-person IGOWAC (MP-IGOWAAC) operator. It can be defined as follows.

Definition 15. An MP-IGOWAAC operator is an aggregation operator that has a weighting vector \( V \) of dimension \( p \) with and \( v_k \in [0, 1] \), and a weighting vector \( W \) of dimension \( n \) with \( m + t \sum_{j=1}^{m+t} w_j = 1 \) and \( w_j \in [0, 1] \), such that:

\[
f(\langle u_1, (x_1^1, ..., x_1^p), (y_1^1, ..., y_1^p) \rangle, ..., \langle u_n, (x_n^1, ..., x_n^p), (y_n^1, ..., y_n^p) \rangle) = \left( \sum_{j=1}^{n} w_j K_j^\lambda \right)^{1/\lambda}
\]

where \( K_j \) is the \([1 \land (1 - x_i + y_i)]\) value of the MP-IGOWAAC triplet \( \langle u_i, x_i, y_i \rangle \) having the \( j \)th
largest \( u_i \), \( u_i \) is the order inducing variable, 
\[
[1 \land (1 - x_i + y_i)] = \left( \sum_{k=1}^{p} v_k [1 \land (1 - x_{ik}^k + y_{ik}^k)] \right).
\]

\([1 \land (1 - x_{ik}^k + y_{ik}^k)]\) is the argument variable provided by each person represented in the form of individual similarities.

**Step 4:** Comparison between the ideal investment strategy and the different alternatives considered using the IGOWAAC operator. In this step, the objective is to express numerically the difference between the ideal investment strategy and the different alternatives considered. Note that it is possible to consider a wide range of IGOWAAC operators such as those described in Section 3, 4 and 5.

**Step 5:** Adoption of decisions according to the results found in the previous steps. Finally, we should take the decision about which investment strategy to select. Obviously, our decision is to select the investment strategy with the best results according to the type of IGOWAAC operator used.

### 7. Illustrative Example

In the following, we present a numerical example of the new approach in a decision making problem. We study a problem of investment selection where a decision maker is looking for the optimal strategy. Note that other decision-making applications could be developed such as in financial decision making [11, 26, 48], human resource management [8] and strategic decision making [29].

We analyze different particular cases of the IGOWAAC operator such as the NAC, the WAC, the OWAAC and the IOWAAC. Note that with this analysis, we obtain “optimal” choices that depend on the aggregation operator used in each particular case. Then, we see that each aggregation operator may lead to different results and decisions. The main advantage of the IGOWAAC is that it includes a wide range of particular cases, reflecting different potential factors to be considered in the decision-making problem depending on the situation found in the analysis. Thus, the decision maker is able to consider a lot of possibilities and select the aggregation operator that is in closest accordance with his interests.

Assume that a company wants to invest some money in a region. Initially, they consider five possible investment alternatives.

- \( A_1 \) = Invest in the European market.
- \( A_2 \) = Invest in the American market.
- \( A_3 \) = Invest in the Asian market.
- \( A_4 \) = Invest in the African market.
- \( A_5 \) = Do not invest money.

In order to evaluate these investments, the investor has brought together a group of experts. This group considers that each investment alternative can be described with the following characteristics:

- \( C_1 \) = Benefits in the short term.
- \( C_2 \) = Benefits in the mid term.
- \( C_3 \) = Benefits in the long term.
- \( C_4 \) = Risk of the investment.
- \( C_5 \) = Other variables.

The experts establish the values of an ideal investment as it is shown in Table 4. This ideal investment represents the optimal results for the company.

<table>
<thead>
<tr>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
<th>( C_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>0.9</td>
<td>0.8</td>
<td>0.9</td>
<td>1</td>
</tr>
</tbody>
</table>
The results of the available investment strategies, depending on the characteristic $C_i$ and the alternative $A_k$ that each of the 3 experts uses, is shown in Tables 5, 6 and 7.

**Table 5. Available investment strategies - Expert 1.**

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.7</td>
<td>0.6</td>
<td>0.7</td>
<td>0.6</td>
<td>0.9</td>
</tr>
<tr>
<td>$A_2$</td>
<td>0.8</td>
<td>1</td>
<td>0.2</td>
<td>1</td>
<td>0.6</td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.6</td>
<td>0.7</td>
<td>0.6</td>
<td>0.6</td>
<td>0.5</td>
</tr>
<tr>
<td>$A_4$</td>
<td>0.9</td>
<td>0.6</td>
<td>0.8</td>
<td>1</td>
<td>0.9</td>
</tr>
<tr>
<td>$A_5$</td>
<td>0.3</td>
<td>0.7</td>
<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
</tr>
</tbody>
</table>

**Table 6. Available investment strategies - Expert 2.**

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.6</td>
<td>0.9</td>
<td>1</td>
<td>0.9</td>
<td>0.9</td>
</tr>
<tr>
<td>$A_2$</td>
<td>1</td>
<td>0.7</td>
<td>0.1</td>
<td>1</td>
<td>0.8</td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.4</td>
<td>0.9</td>
<td>0.8</td>
<td>0.7</td>
<td>0.6</td>
</tr>
<tr>
<td>$A_4$</td>
<td>0.9</td>
<td>0.5</td>
<td>0.7</td>
<td>1</td>
<td>0.9</td>
</tr>
<tr>
<td>$A_5$</td>
<td>0.7</td>
<td>0.7</td>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
</tr>
</tbody>
</table>

**Table 7. Available investment strategies - Expert 3.**

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.5</td>
<td>0.7</td>
<td>0.9</td>
<td>0.8</td>
<td>0.9</td>
</tr>
<tr>
<td>$A_2$</td>
<td>0.9</td>
<td>0.9</td>
<td>0.2</td>
<td>1</td>
<td>0.7</td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.8</td>
<td>0.8</td>
<td>0.7</td>
<td>0.7</td>
<td>0.6</td>
</tr>
<tr>
<td>$A_4$</td>
<td>0.9</td>
<td>0.5</td>
<td>0.8</td>
<td>1</td>
<td>0.7</td>
</tr>
<tr>
<td>$A_5$</td>
<td>0.8</td>
<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Each expert has the same degree of importance, that is, $1/3$. With this information, we can aggregate the individual opinion of the experts in order to obtain a collective result that will be compared with the ideal strategy. Note that we assume that the ideal strategy is given in a single result that represents the consensus result between the three experts. Thus, the similarity analysis will be developed between these results and the collective results of the experts. The collective results given by the experts are presented in Table 8.

**Table 8. Available investment strategies - Collective results.**

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.6</td>
<td>0.7</td>
<td>0.9</td>
<td>0.8</td>
<td>0.9</td>
</tr>
<tr>
<td>$A_2$</td>
<td>0.9</td>
<td>0.9</td>
<td>0.2</td>
<td>1</td>
<td>0.7</td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.6</td>
<td>0.8</td>
<td>0.7</td>
<td>0.7</td>
<td>0.6</td>
</tr>
<tr>
<td>$A_4$</td>
<td>0.9</td>
<td>0.5</td>
<td>0.8</td>
<td>1</td>
<td>0.7</td>
</tr>
<tr>
<td>$A_5$</td>
<td>0.6</td>
<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
<td>0.8</td>
</tr>
</tbody>
</table>

In this problem, the experts assume the following weighting vector: $W = (0.3, 0.2, 0.2, 0.2, 0.1)$. Due to the fact that the attitudinal character is very complex because it involves the opinion of different members of the board of directors, the experts use order-inducing variables to represent it. The results are shown in Table 9.

With this information, we can aggregate the expected results for each state of nature in order to make a decision. In Table 10, we present different results obtained by using different types of IGOWAAC operators such as the NAC, the WAC, the OWAAC and the IOWAAC operator.

If we establish an ordering of the alternatives, a typical situation if we want to consider more than one alternative, then we get the results shown in Table 11. Note that the first alternative in each ordering is the optimal choice.

As we can see, depending on the aggregation operator used, the ordering of the investment strategies may be different. Therefore, the decision concerning which investment strategy select may be also different.
Table 9. Order inducing variables.

<table>
<thead>
<tr>
<th></th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>C5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>15</td>
<td>12</td>
<td>17</td>
<td>13</td>
<td>10</td>
</tr>
<tr>
<td>A2</td>
<td>17</td>
<td>20</td>
<td>15</td>
<td>14</td>
<td>16</td>
</tr>
<tr>
<td>A3</td>
<td>11</td>
<td>14</td>
<td>12</td>
<td>18</td>
<td>13</td>
</tr>
<tr>
<td>A4</td>
<td>10</td>
<td>19</td>
<td>17</td>
<td>15</td>
<td>13</td>
</tr>
<tr>
<td>A5</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>17</td>
<td>11</td>
</tr>
</tbody>
</table>

Table 10. Aggregated results.

<table>
<thead>
<tr>
<th></th>
<th>NAC</th>
<th>WAC</th>
<th>OWAAC</th>
<th>IOWAAC</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>0.9</td>
<td>0.9</td>
<td>0.92</td>
<td>0.91</td>
</tr>
<tr>
<td>A2</td>
<td>0.82</td>
<td>0.85</td>
<td>0.88</td>
<td>0.82</td>
</tr>
<tr>
<td>A3</td>
<td>0.82</td>
<td>0.85</td>
<td>0.85</td>
<td>0.81</td>
</tr>
<tr>
<td>A4</td>
<td>0.86</td>
<td>0.89</td>
<td>0.90</td>
<td>0.82</td>
</tr>
<tr>
<td>A5</td>
<td>0.9</td>
<td>0.91</td>
<td>0.92</td>
<td>0.92</td>
</tr>
</tbody>
</table>

Table 11. Ordering of the investment strategies.

<table>
<thead>
<tr>
<th></th>
<th>NAC</th>
<th>WAC</th>
<th>OWAAC</th>
<th>IOWAAC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ordering</td>
<td>A1 = A5 ≻ A4 ≻ A2 = A3</td>
<td>A5 ≻ A1 ≻ A4 ≻ A2 = A3</td>
<td>A1 = A5 ≻ A4 ≻ A2 ≻ A3</td>
<td>A5 ≻ A1 ≻ A2 ≻ A4 ≻ A3</td>
</tr>
</tbody>
</table>

8. Conclusions

We have presented the IGOWAAC operator. It is a new aggregation operator that generalizes a wide range of aggregation operators by using order inducing variables, generalized means, OWA operators and the adequacy coefficient. We have studied some of its main properties and we have seen that it is an extension of the Minkowski distance. We have analyzed a wide range of families of IGOWAAC operators such as the OWAAC, the OWQAAC, the step-IGOWAAC, the centered-IGOWAAC, etc. We have also presented a further generalization of the IGOWAAC operator by using quasi-arithmetic means (Quasi-IGOWAAC operator).

Moreover, we have also considered other extensions by using hybrid aggregations, probabilities, Choquet integrals and moving averages. We have presented the Quasi-IHAAC, the IP-Quasi-IOWAAC, the Quasi-ICIAC and the Quasi-IOWMAAC operators. We have also considered several particular cases of these approaches such as the use of generalized, arithmetic, geometric, quadratic, harmonic and cubic aggregation operators.

We have also developed an illustrative example in a multi-person decision making problem regarding the selection of investment strategies. We have introduced the MP-IGOWAAC operator that it is able to assess the opinion of several experts by using the IGOWAAC operator.

In future research, we expect to develop further extensions of this approach by using other extensions such as the use of unified aggregation operators, more complex structures and applying it to different decision making problems.

Acknowledgements

We would like to thank the anonymous referees for valuable comments that have improved the quality of the paper. Support from the project JC2009-00189 from the Spanish Ministry of Education and A/023879/09 from the Spanish Ministry of Foreign Affairs is gratefully acknowledged.

References


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