On a conjecture of Shanks

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Abstract
The conjecture in question concerns the function $\phi_n$ related to the distribution of the zeroes of the Riemann zeta-function, $\gamma_n$, over the Gram points $g_n$. It is the purpose of this article to show that for any $\alpha > 0$ the sum

$$\sum_{n=1}^{K} \phi_n \alpha \to 0,$$

and this was conjectured, on numerical evidence, by Shanks (1961) [7] to be true for $\alpha = \frac{1}{2}$.

1. Introduction

One of the first large scale numerical calculations relating to the Riemann zeta-function was conducted by Haselgrove [3] in 1960. Included in these tables are calculations of $\Re \zeta\left(\frac{1}{2} + it\right)$, $\Im \zeta\left(\frac{1}{2} + it\right)$ and of the functions $Z(t)$ and $\theta(t)$ defined by

$$Z(t) = e^{\i \theta(t)} \zeta\left(\frac{1}{2} + it\right).$$

It can be shown (see, e.g. [9, §4] or [1, §§6–8]) that $Z(t)$ is real for real-valued $t$ and that $\theta(t)$ is ultimately increasing. Therefore when $n \geq -1$, one defines the Gram points $g_n$, to be such that $\theta(g_n) = n\pi$. Of interest in the location of the zeroes of the zeta-function is Gram’s Law, which states that each Gram interval $^1 (g_{n-1}, g_n)$ contains exactly one zero of $\zeta\left(\frac{1}{2} + it\right)$. Titchmarsh showed [8]
that Gram's Law fails infinitely often and results of the author [10] show, \textit{inter alia} that a positive proportion of Gram intervals do not contain a zero.

Table III in [3] includes values of \(g_n\), \(\gamma_n\) and \(\phi_n\), where \(\gamma_n\) denotes the ordinate of the \(n\)th zero on the critical line (counted with multiplicity) — in particular, the Riemann Hypothesis is not assumed. The numbers \(\phi_n\) are defined by

\[
\phi_n = n - \frac{3}{2} - \pi^{-1}\theta(\gamma_n),
\]

(1)

and Shanks [7] states that Gram's Law fails whenever \(|\phi_n| > \frac{1}{2}\). Care needs to be taken, since Shanks writes

\[
\phi_n = \pi^{-1}\arg\zeta'(\frac{1}{2} + i\gamma_n).
\]

as a definition for \(\phi_n\), and clearly the argument needs to be specified up to a multiple of \(2\pi\). But Shanks’s statement easily follows from (1), since, if Gram’s Law is true for all \(m < n\), then \(n - 2 < \pi^{-1}\theta(\gamma_n) < n - 1\).

Shanks gave some numerical data concerning the average of the sum \(\sum_{n=1}^{K} \phi_n\). He conjectured that \((1/K)\sum_{n=1}^{K} \phi_n \to 0\), and, in a note added in proof correction, that the stronger estimate \((1/K)\sum_{n=1}^{K} \phi_n \to 0\) may hold. His paper contains no \textit{prima facie} reason, other than the numerical evidence, to suggest why, if the latter conjecture were true, a similar result of the type \((1/K^\alpha)\sum_{n=1}^{K} \phi_n \to 0\) might not also be true, for some \(\alpha < \frac{1}{2}\). It is the object of this paper to answer both of Shanks's conjectures in the affirmative by proving a

**Theorem.** If \(\alpha > 0\) then

\[
\sum_{n=1}^{K} \phi_n \quad \rightarrow \quad 0.
\]

This conjecture of Shanks is not well known: there is a reference contained in [4, pp. 86–90]; the conjecture that \((1/K)\sum_{n=1}^{K} \phi_n \to 0\) is proved in [2]. Nevertheless the function \(\phi_n\) is closely related to the function \(\Delta(n)\) which has been studied by Titchmarsh [8] and by Selberg [6]. Define \(\Delta(n) = n - m\), where \(\gamma_n\) lies in the \(m\)th Gram interval \((g_{m-2}, g_{m-1})\), whence it follows from (1) that \(|\phi_n - \Delta(n)| \leq \frac{1}{2}\) whenever Gram’s Law holds up to \(n\). Thus \(\Delta(n)\) is a measure of how far the zeroes are ‘out of sync’ with the Gram points; indeed if Gram’s Law were to hold universally, then \(\Delta_n \equiv 0\). The function \(\Delta(n)\) is very similar to the argument function \(S(t)\) (properties of which can be found in [9, §9]) and so the sum considered by Shanks in (2) can be compared with \(\int_{0}^{T} S(t)\ dt\); the proof is achieved using estimates of this integral.

2. Proof of the theorem

Let \(N(T)\) denote, as usual, the number of non-trivial zeroes of \(\zeta(\sigma + it)\) with \(0 \leq t \leq T\). Working directly from (1) it follows that

\[
\sum_{n=1}^{K} \phi_n = \sum_{n=1}^{K} \left( n - \frac{3}{2} \right) - \pi^{-1} \sum_{n=1}^{K} \theta(\gamma_n).
\]

Since it can be verified\(^2\) that \(\theta(c) = 0\) for \(c \approx 17.3\ldots\), write

\(^2\) Indeed, \(\theta(0) = 0\); \(\theta(t)\) is decreasing for \(0 < t < c\) whereafter \(\theta(t)\) is monotonically increasing.
\[ \theta(\gamma_n) = \int_c^{\gamma_n} \theta'(t) \, dt. \]

The range of integration is taken beyond \( t = 0 \) to avoid future difficulties with the evaluation of logarithmic terms in the integrand. Then

\[ \sum_{n=1}^{K} \phi_n = \frac{K(K - 2)}{2} - \pi^{-1} \sum_{n=1}^{K} \int_c^{\gamma_n} \theta'(t) \, dt, \]

and the order of summation and integration can be inverted leading to

\[
\begin{align*}
\sum_{n=1}^{K} \phi_n &= \frac{K(K - 2)}{2} - \pi^{-1} \int_c^{\gamma_K} \theta'(t) \left( \sum_{n=1, \gamma_n \geq t}^{K} 1 \right) \, dt \\
&= \frac{K(K - 2)}{2} - \pi^{-1} \int_c^{\gamma_K} \theta'(t) \{ N(\gamma_K) - N(t) \} \, dt \\
&= \frac{K(K - 2)}{2} - \pi^{-1} K \theta(\gamma_K) + \pi^{-1} \int_c^{\gamma_K} \theta'(t) N(t) \, dt. \quad (3)
\end{align*}
\]

Now using (see, e.g. [1, p. 173])

\[ N(t) = S(t) + \pi^{-1} \theta(t) + 1, \quad (4) \]

one can rewrite the integral in (3) as

\[
\pi^{-1} \int_c^{\gamma_K} \theta'(t) N(t) \, dt = \pi^{-1} \int_c^{\gamma_K} \theta'(t) S(t) \, dt + \frac{\{ \theta(\gamma_K) \}^2}{2\pi^2} + \theta(\gamma_K),
\]

after integrating termwise and using \( \theta(c) = 0. \) Thus

\[
\sum_{n=1}^{K} \phi_n = \frac{K(K - 2)}{2} - \frac{\theta(\gamma_K)(K - 1)}{\pi} + \frac{\{ \theta(\gamma_K) \}^2}{2\pi^2} + \pi^{-1} \int_c^{\gamma_K} \theta'(t) S(t) \, dt.
\]

Applying (4) once more with \( t = \gamma_K \) one finds that

\[
\sum_{n=1}^{K} \phi_n = -\frac{1}{2} + \frac{1}{2} \{ S(\gamma_K) \}^2 + \pi^{-1} \int_c^{\gamma_K} \theta'(t) S(t) \, dt. \quad (5)
\]

For \( t > 0 \) one has the estimate

\[ \theta'(t) = \frac{1}{2} \log \frac{t}{2\pi} + O(t^{-2}), \quad (6) \]
and indeed sharp estimates of the implicit constant can be found in [5]. By the second mean-value theorem for integrals, or by integrating by parts, it follows from (6) that

$$\int_{c}^{\gamma K} \theta'(t)S(t)\,dt = O\left(\log \gamma K \int_{c}^{\gamma K} S(t)\,dt\right) + O\left(\max_{\varepsilon \leq t \leq \gamma K} \left| \int_{c}^{t} S(t)\,dt \right| \right)$$

$$= O\left(\log^{2} \gamma K\right), \quad (7)$$

by using the well-known result of Littlewood on the function $S(t)$, viz. $\int_{0}^{T} S(t)\,dt = O(\log T)$. The confluence of Eqs. (7) and (5) and the estimate $S(T) = O(\log T)$ is

$$\sum_{n=1}^{K} \phi_{n} = -\frac{1}{2} + O\left(\log^{2} \gamma K\right).$$

Since the Riemann–von-Mangoldt formula gives $N(\gamma K) = K \sim \frac{\gamma K}{2\pi} \log \frac{\gamma K}{2\pi}$, it follows that $\log K \sim \log \gamma K$, and hence that

$$\sum_{n=1}^{K} \phi_{n} \ll \log^{2} K, \quad (8)$$

whence the result in the theorem.

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**References**