On a well-balanced high-order finite volume scheme for the shallow water equations with bottom topography and dry areas

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Eleventh International Conference on Hyperbolic Problems, Lyon, 2006
Plan

- Shallow water equations: nonconservative framework
- High-order Roe schemes based on reconstruction of states
- Treatment of wet/dry fronts: the MRoe scheme
- High-order extension: the HMRoe scheme
- Numerical results
Shallow water equations: nonconservative framework

Shallow water equations modelling the flow of a shallow layer of fluid through a straight channel with constant rectangular cross-section and bottom topography:

\[
\begin{align*}
\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} &= 0 \\
\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{q^2}{h} + \frac{g}{2} h^2 \right) &= gh \frac{dH}{dx}
\end{align*}
\]

- \( h(x, t) \): thickness of the fluid layer
- \( q(x, t) \): discharge
- \( H(x) \): depth function, measured from a fixed level of reference
- \( g \): gravity

- \( u = q/h \)
- \( c = \sqrt{gh} \)
Shallow water equations: nonconservative framework

Balance law formulation:
\[
\frac{\partial U}{\partial t} + \frac{\partial}{\partial x} F(U) = S(U) \frac{dH}{dx}
\]

\[
U = \begin{bmatrix} h \\ q \end{bmatrix}, \quad F(U) = \begin{bmatrix} q \\ \frac{q^2}{h} + \frac{g}{2} h^2 \end{bmatrix}, \quad S(U) = \begin{bmatrix} 0 \\ gh \end{bmatrix}
\]

\[
\frac{\partial H}{\partial t} = 0
\]

Nonconservative formulation:
\[
\frac{\partial W}{\partial t} + A(W) \frac{\partial W}{\partial x} = 0
\]

\[
W = \begin{bmatrix} h \\ q \\ H \end{bmatrix}, \quad A(W) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + c^2 & 2u & -c^2 \\ 0 & 0 & 0 \end{bmatrix}
\]

Generalized Roe schemes

Generalized Roe schemes for solving nonconservative hyperbolic systems [Parés, Castro, *M2AN*, 04]:

\[ W_{i}^{n+1} = W_{i}^{n} - \frac{\Delta t}{\Delta x} \left( \mathcal{A}_{i-1/2}^{+} (W_{i}^{n} - W_{i-1}^{n}) + \mathcal{A}_{i+1/2}^{-} (W_{i+1}^{n} - W_{i}^{n}) \right) \]

\( \mathcal{A}_{i+1/2} \): Roe linearization [Toumi, *J. Comput. Phys.*, 92]

Well-balancing:

– general stationary solutions are approximated with order two;

– water at rest solutions are exactly computed.
High-order generalized Roe schemes

- High-order Roe schemes: [Castro, Gallardo, Parés, Math. Comp., 06].
- The generalization is based on reconstruction of the states $W$:

\[ \begin{align*}
W_{i+1/2}^+ & \quad W_{i+1/2}^- \\
p_{i+1/2}^+(x) & \quad p_{i+1/2}^-(x)
\end{align*} \]

- The scheme is well-balanced, at least with the same order of the reconstruction operator.
**High-order generalized Roe schemes**

Semi-discrete scheme:

\[ W_i'(t) = -\frac{1}{\Delta x} \left( \mathcal{A}_{i-1/2}^+ (W_{i-1/2}^+ (t) - W_{i-1/2}^- (t)) 
+ \mathcal{A}_{i+1/2}^- (W_{i+1/2}^- (t) - W_{i+1/2}^+ (t)) 
+ \int_{x_{i-1/2}}^{x_i} \mathcal{A}(p_{i-1/2}^+ (x)) \frac{d}{dx} p_{i-1/2}^+(x) \, dx 
+ \int_{x_i}^{x_{i+1/2}} \mathcal{A}(p_{i+1/2}^- (x)) \frac{d}{dx} p_{i+1/2}^-(x) \, dx \right) \]

- \( \mathcal{A}_{i+1/2} \): Roe matrix associated to \( W_{i+1/2}^- (t) \) and \( W_{i+1/2}^+ (t) \)
- \( p_{i+1/2}^{t, \pm} (x) \): reconstructing functions at time \( t \)

Time discretization: Runge-Kutta TVD scheme

Very good results when applied to the shallow water equations in the wet bed case, with fifth-order WENO reconstructions and third-order RK TVD.
Treatment of wet/dry fronts: the MRoe scheme

**MRoe scheme:** modification of the Roe scheme that allows a treatment of wet/dry situations [Castro, González-Vida, Parés, *M3AS*, 06].

Remarks:

- It is based on the resolution, at each intercell where a wet/dry transition has been detected, of an appropriate *nonlinear* Riemann problem.
- In most cases, the nonlinear Riemann problems can be easily solved.
- The nature of the nonlinear Riemann problems depends on the kind of wet/dry situation found. The numerical fluxes are modified accordingly.

The MRoe scheme is well-balanced and preserves the positivity of the height $h$. 
**Treatment of wet/dry fronts: the MRoe scheme**

A case example: $H_L > H_R$ and the step acts as an obstacle for the fluid

Partial Riemann problem:

\[
\begin{align*}
\frac{\partial U}{\partial t} + \frac{\partial}{\partial x} F(U) &= 0 \quad \text{for } x < 0, \; t > 0 \\
U(x, 0) &= W_L \quad \text{if } x < 0 \\
U(x, 0) &\in \mathcal{V}_R \quad \text{if } x > 0
\end{align*}
\]

$W_L = [h_L, q_L]^T$, $\mathcal{V}_R = \{[h, 0]^T : h \geq 0\}$
High-order extension: the HMRoe scheme

**HMRoe scheme**: high-order Roe scheme + MRoe scheme

Remarks:

- The numerical fluxes must be modified accordingly to the kind of wet/dry transition found.
- The variables to be reconstructed are: the height $h$, the speed $u$ and the elevation $\eta = h - H$; the reconstructed bottom is then defined as $H = h - \eta$ and the discharge as $q = hu$. The reason of this choice is twofold: to maintain the well-balancing property and, at the same time, to avoid cancellation problems when the speed of the fluid $u$ is computed.
- An adequate reconstruction operator must be chosen: polynomial reconstructions introduce spurious oscillations that may lead to negative values of the height $h$. 
**High-order extension: the HMRoe scheme**

Hyperbolic reconstructions [Marquina, *SIAM J. Sci. Comput.*, 94] have been chosen for several reasons:

- Monotonicity: it avoids the appearance of negative values of the height $h$.
- Third-order of accuracy on the whole cell: the scheme results to be third-order accurate on smooth wet regions and well-balanced with the same order.
- Compactness of the stencil: robustness of the method, easier analysis of wet/dry situations and lower computational cost.
- Hyperbolas have lower total variation than parabolas, thus reducing oscillating behaviour near shocks.
Numerical experiment: steady flow over a bump

Depth function: \( H(x) = 2 - 0.2 \exp(-0.16(x - 10)^2), \ x \in [0, 20] \).

Initial conditions: \( q(x, 0) = 0, \ h(x, 0) = 0.13 + H(x) \).

Boundary conditions: \( h = 0.33 \) downstream; \( q = 0.18 \) upstream.

Data: CFL=0.9; \( \Delta x = 0.1 \); \( T = 200 \).

Solution (surface elevation):
Numerical experiment: small perturbation of steady state water

Depth function: \( H(x) = \begin{cases} 
-0.25(\cos(10\pi(x - 0.5)) + 1) & \text{if } 1.4 \leq x \leq 1.6, \\
0 & \text{otherwise.} 
\end{cases} \)

Initial data: small perturbation of height \( \varepsilon = 10^{-3} \) of the stationary solution.
Data: CFL=0.9; \( \Delta x = 0.01 \); \( T = 0.2 \).
Numerical experiment: dambreak over a plane

Depth function: \( H(x) = 1 - x \tan(\alpha), \, x \in [-15, 15]. \)

Initial conditions: \( q(x, 0) = 0, \, h(x, 0) = \begin{cases} H(x) & \text{if } x < 0, \\ 0 & \text{otherwise}. \end{cases} \)

Boundary condition: \( q = 0 \) at \( x = -15. \)

Data: CFL=0.9; \( \Delta x = 0.05; \, T = 2; \, \varepsilon_h = 10^{-6}. \)

We consider three different cases:

- \( \alpha = 0 \): flat bottom.
- \( \alpha = \pi/60 \): ascending bottom.
- \( \alpha = -\pi/60 \): descending bottom.
Numerical experiment: dambreak over a plane ($\alpha = 0$)

- Initial condition
- Solution at $t = 2$
- Wet/dry front position
- Wet/dry front velocity
Numerical experiment: dambreak over a plane ($\alpha = \pi/60$)

Initial condition

Solution at $t = 2$

Wet/dry front position

Wet/dry front velocity
Numerical experiment: dambreak over a plane ($\alpha = -\pi/60$)

Initial condition

Solution at $t = 2$

Wet/dry front position

Wet/dry front velocity

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Numerical experiment: dry bed generation on a flat bottom

Initial conditions: $h(x, 0) = 0$, $q(x, 0) = \begin{cases} -0.3 & \text{if } x \geq 0, \\ 0.3 & \text{otherwise.} \end{cases}$

Data: $\text{CFL}=0.8; \Delta x = 0.05; T = 1; \varepsilon_h = 10^{-6}$. 

Free surface elevation

Discharge
Numerical experiment: dry bed generation on a non-flat bottom

Depth function: \( H(x) = \begin{cases} 13 & \text{if } 25/3 < x < 12.5, \\ 14 & \text{otherwise.} \end{cases} \)

Initial conditions: \( h(x, 0) = 10, q(x, 0) = \begin{cases} -350 & \text{if } x < 50/3, \\ 350 & \text{otherwise.} \end{cases} \)

Data: CFL=0.9; \( \Delta x = 0.05; T = 1; \varepsilon_h = 10^{-6}. \)
Numerical experiment: drain on a non-flat bottom

Depth function: \( H(x) = \begin{cases} 
0.05(x - 10)^2 & \text{if } 8 \leq x \leq 12, \\
0.2 & \text{otherwise.}
\end{cases} \)

Initial conditions: \( h(x, 0) = 0.3 + H(x) \), \( q(x, 0) = 0 \).

Boundary conditions: mirror state (left) and outlet (right).

Data: CFL=0.4; 300 nodes; \( T = 1000 \); \( \varepsilon_h = 10^{-6} \).

Free surface elevation

Discharge