On the $R$-order of convergence of Newton’s method under mild differentiability conditions

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Abstract

A new technique is used instead of the classical majorant principle to analyze the $R$-order of convergence of the Newton process when more general conditions than the Kantorovich ones are considered.

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1. Introduction

The problem of approximating a solution of a nonlinear equation $F(x) = 0$ is very interesting, since we can then solve a large number of different types of problems. So, if $F$ is a nonlinear operator defined on a non-empty open convex subset $Ω$ of a Banach space $X$ with values in a Banach space $Y$, the equation $F(x) = 0$ can represent a differential equation, a boundary value problem, an integral equation, etc. The normal way to approximate a solution of $F(x) = 0$ is by means of iterative processes. An iterative process is defined by an algorithm such that, from an initial approximation $x_0$, a sequence $\{x_n\}$ is constructed satisfying $\lim_{n} x_n = x^*$, where $F(x^*) = 0$.

In the study of iterative methods there are two especially important sides: the convergence of the sequence $\{x_n\}$ to a solution $x^*$ of $F(x) = 0$ and the speed of this convergence. So, we can do then different analysis of convergence: local, semilocal or global (depending on the required conditions).

The best known iteration to solve nonlinear equations is the Newton method

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n), \quad n \geq 0 \quad \text{given } x_0,$$

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provided that \( F'(x_n)^{-1} \) exists for all \( n \geq 0 \). Results concerning convergence have been published under assumptions of Newton–Kantorovich type. Initially, see [7], the required assumptions to study the convergence of Newton’s method were

\[
\text{(A1)} \quad \Gamma_0 = F'(x_0)^{-1} \text{ exists at some } x_0 \in \Omega, \text{ is a continuous linear operator and } \| \Gamma_0 \| \leq \beta, \\
\text{(A2)} \quad \| \Gamma_0 F(x_0) \| \leq \eta, \\
\text{(A3)} \quad \| F''(x) \| \leq M (x \in \Omega).
\]

Kantorovich obtained the convergence of the Newton iteration under assumptions (A1)–(A3). Next, this study can be modified by replacing condition (A3) for

\[
\| F'(x) - F'(y) \| \leq L \| x - y \|, \quad L \geq 0 \quad (x, y \in \Omega). \tag{1}
\]

(See [1,9,12]). Others ([2,6,8,11]) consider a generalization of the last conditions, which is given by

\[
\| F'(x) - F'(y) \| \leq K \| x - y \|^p, \quad K \geq 0, \quad p \in [0, 1] \quad (x, y \in \Omega). \tag{2}
\]

(Notice that conditions (1) and (2) mean, respectively, that \( F' \) is Lipschitz continuous and \( F' \) is \((K, p)\)-Hölder continuous in the domain \( \Omega \).

Under conditions (1) and (2) the number of equations that can be solved by Newton’s process is limited. For instance, we cannot analyze the convergence of the method to a solution of equations where additions of operators, which satisfy (1) or (2), appear. We then consider the following nonlinear integral equation of mixed Hammerstein type [5]:

\[
x(s) + \sum_{i=1}^{m} \int_{a}^{b} g_i(s, t) \ell_i(x(t)) \, dt = u(s), \quad s \in [a, b],
\]

where \(-\infty < a < b < \infty, u, \ell_i, \) and \( g_i, \) for \( i = 1, 2, \ldots, m, \) are known functions and \( x \) is a solution to be determined. If \( \ell_i'(x(t)) = (K_i, p_i)\)-Hölder continuous in \( \Omega, \) for \( i = 1, 2, \ldots, m, \) the corresponding operator \( F : \Omega \subseteq C[0, 1] \rightarrow C[0, 1], \)

\[
[F(x)](s) = x(s) + \sum_{i=1}^{m} \int_{a}^{b} g_i(s, t) \ell_i(x(t)) \, dt - u(s), \quad s \in [a, b], \tag{3}
\]

does not satisfy (1) neither (2) when, for instance, the max-norm is considered. In this case,

\[
\| F'(x) - F'(y) \| \leq \sum_{i=1}^{m} K_i \| x - y \|^p_i, \quad K_i \geq 0, \quad p_i \in [0, 1] \quad (x, y \in \Omega).
\]

To solve this type of equations and relax conditions (1) and (2) we can consider

\[
\| F'(x) - F'(y) \| \leq \omega(\| x - y \|) \quad (x, y \in \Omega), \tag{4}
\]

where \( \omega(z) = \sum_{i=1}^{m} K_i z^{p_i}. \) We then require that \( \omega(z) \) is a non-decreasing continuous real function for \( z > 0, \) such that \( \omega(0) = 0 \) (see [2,3]).

Obviously conditions (A3), (1) and (2) are relaxed by condition (4). Besides the former ones are generalized by the latter one if \( \omega(z) = M, \omega(z) = Lz \) and \( \omega(z) = Kz^p \) respectively, where \( K = \max\{K_1, K_2, \ldots, K_m\} \) and \( p = \max\{p_1, p_2, \ldots, p_m\}. \)

Another generalization, see [4], that we can consider to relax the usual conditions to facilitate the convergence of Newton’s method, is given by the following condition:

\[
\| F''(x) \| \leq \tilde{\omega}(\| x \|) \quad (x \in \Omega), \tag{5}
\]

where \( \tilde{\omega} : \mathbb{R}_+ \cup \{0\} \rightarrow \mathbb{R}_+ \cup \{0\} \) is a continuous real function such that \( \tilde{\omega}(0) \geq 0 \) and \( \tilde{\omega} \) is monotonous.

On the other hand, the convergence properties depend on the choice of the distance \( \| \cdot \|, \) but for a given distance, the speed of convergence of a sequence \( \{x_n\} \) is characterized by the speed of convergence of the sequence of non-negative numbers \( \| x^n - x_n \|. \) An important measure of the speed of convergence is the \( R \)-order of convergence (see [10]). It is
known that a sequence \( \{x_n\} \) converges to \( x^* \) with \( R \)-order at least \( \tau > 1 \) if there are constants \( C \in (0, \infty) \) and \( \gamma \in (0, 1) \) such that \( \|x^n - x_n\| \leq C\gamma^n, n = 0, 1, \ldots \). If \( F'' \) is continuous and bounded in \( \Omega \) or \( F' \) is Lipschitz continuous in \( \Omega \), the convergence of the Newton iteration is \( R \)-quadratic (see [10,12]).

If \( F' \) is \((K, p)\)-Hölder continuous in \( \Omega \), the \( R \)-order of convergence is \( 1 + p \) at least (see [6]). Real majorizing sequences are usually used to prove the convergence of Newton’s method and the \( R \)-order of convergence two is analyzed, but in [3] and [4] the difficulty in using majorizing sequences, when the required conditions are generalized, is laid out. An alternative technique, where a particular real sequence is involved, was provided. The application of this technique is very simple and allows us to generalize the results obtained under Newton–Kantorovich type conditions. Moreover, by using the technique considered, we analyze

Finally, we apply the results mentioned above to a nonlinear Hammerstein integral equation of the second kind.

Throughout the paper we denote

\[
B(x, r) = \{ y \in X; \|y - x\| \leq r \} \quad \text{and} \quad B(x, r) = \{ y \in X; \|y - x\| < r \}.
\]

2. Semilocal convergence of the Newton method

To establish a semilocal convergence result for the Newton method, certain conditions for the operator \( F \) and the initial approximation \( x_0 \) are required. Conclusions about the existence and uniqueness of solutions of the equation \( F(x) = 0 \) are also obtained. We provide the regions of existence and uniqueness of solutions from the theoretical significance of the method, without finding the solutions themselves. This is sometimes more important than the actual knowledge of a solution.

A new technique was developed in [3] and [4] to prove the semilocal convergence of Newton’s sequence, where we constructed, from some scalar parameters, systems of recurrence relations where real sequences of positive real numbers were involved. The convergence of the Newton iteration was then guaranteed from the fact that it is a Cauchy sequence.

Firstly, we take into account the situation in which condition (4) is considered (see [3]). We suppose that \( r_0 = F'(x_0)^{-1} \in \mathcal{L}(Y, X) \) exists for some \( x_0 \in \Omega \), where \( \mathcal{L}(Y, X) \) is the set of bounded linear operators from \( Y \) into \( X \). We also assume the following:

(C1) \( \|r_0\| \leq \beta \),
(C2) \( \|F(x) - F(x_0)\| \leq \eta \),
(C3) \( \|F'(x) - F'(y)\| \leq \omega(\|x - y\|), x, y \in \Omega \), where \( \omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is a continuous and non-decreasing function such that \( \omega(0) = 0 \),
(C4) a continuous and non-decreasing function \( h : [0, 1] \rightarrow \mathbb{R}_+ \) exists, such that \( \omega(tz) \leq h(t)\omega(z) \), with \( t \in [0, 1] \) and \( z \in [0, \infty) \).

Note that condition (C4) does not involve any restriction, since \( h \) always exists, such that \( h(1) = 1 \), as a consequence of \( \omega \) is a non-decreasing function. We can even consider \( h(t) = \sup_{z \geq 0} \omega(tz)/\omega(z) \).

**Theorem 2.1.** Let \( X \) and \( Y \) be two Banach spaces and \( F : \Omega \subseteq X \rightarrow Y \) a once Fréchet differentiable operator in an open convex domain \( \Omega \). We suppose that \( F''(x_0)^{-1} \in \mathcal{L}(Y, X) \) exists for some \( x_0 \in \Omega \) and conditions (C1)–(C4) hold. If \( a = \beta\omega(\eta) < \min(1 - h(b), 1/(1 + H)) \), where \( b = Ha/(1 - a) \) and \( H = \int_0^1 h(t) \, dt \). If \( B(x_0, R) \subseteq \Omega \), where \( R = \eta/(1 - b) \), then the Newton sequence, starting from \( x_0 \), converges to a solution \( x^* \) of the equation \( F(x) = 0 \), the solution \( x^* \) and the iterates \( x_n \) belong to \( B(x_0, R) \). Moreover, if there exists a positive root \( r \) of the equation

\[
2\beta\omega(R + r) \int_{1/2}^1 h(t) \, dt = 1,
\]

the solution \( x^* \) of the equation \( F(x) = 0 \) is then unique in \( \Omega_0 = B(x_0, r) \cap \Omega \).
Secondly, we state a semilocal convergence result for Newton’s process, where condition (5) is now taken into account. We assume (C1), (C2),

(\(\mathcal{C}_3\)) \(\|F''(x)\| \leq \tilde{o}(\|x\|), x, y \in \Omega, \) where \(\tilde{o} : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}\) is a continuous real function such that \(\tilde{o}(0) \geq 0\) and \(\tilde{o}\) is monotonous (i.e. non-decreasing or non-increasing),

(\(\mathcal{C}_4\)) The equation

\[
3\beta \eta \varphi(t) - 2\beta \eta^2 \varphi(t) - 2t + 2\eta = 0
\]

has at least one positive root, where

\[
\varphi(t) = \begin{cases} 
\tilde{o}(\|x_0\| + t) & \text{if } \tilde{o} \text{ is non-decreasing,} \\
\tilde{o}(\|x_0\| - t) & \text{if } \tilde{o} \text{ is non-increasing.}
\end{cases}
\]

We denote the smallest positive root of this equation by \(\tilde{R}\). Note that the \(\tilde{R}\) root must be less than \(\|x_0\|\) if \(\tilde{o}\) is non-increasing.

The semilocal convergence theorem for Newton’s process is now as follows (see [4]).

**Theorem 2.2.** Let \(X\) and \(Y\) be two Banach spaces and \(F : \Omega \subseteq X \to Y\) a twice Fréchet differentiable operator in an open convex domain \(\Omega\). We suppose that \(F'(x_0)^{-1} \in \mathcal{L}(Y, X)\) exists for some \(x_0 \in \Omega\) and conditions (C1), (C2), (\(\mathcal{C}_3\)), (\(\mathcal{C}_4\)) and \(B(x_0, \tilde{R}) \subseteq \Omega\) hold. If \(x = \beta \eta \varphi(\tilde{R}) \in (0, \frac{1}{2})\) and \(B(x_0, \tilde{R}) \subseteq \Omega\), then Newton’s sequence, starting from \(x_0\), converges to a solution \(x^*\) of the equation \(F(x) = 0\). Moreover, the solution \(x^*\) of the equation is unique in \(\Omega_0 = B(x_0, \tilde{R}) \cap \Omega\), where \(\tilde{R}\) is the smallest positive root of

\[
\beta \int_0^1 \int_0^1 \varphi(s(\tilde{R} + t(\tilde{R} - \tilde{R}))) \, ds(\tilde{R} + t(\tilde{R} - \tilde{R})) \, dt = 1.
\]

3. **On the R-order of Newton’s method**

In the following, we obtain a priori error bounds for the Newton iteration when it converges to a solution \(x^*\) of the equation \(F(x) = 0\). We use the technique developed in [3] and [4] to study the semilocal convergence results (namely, Theorems 2.1 and 2.2) for the Newton process, that consists of constructing a system of recurrence relations that must be satisfied for some scalar sequences.

Firstly, we analyze the \(R\)-order of convergence for Newton’s method when conditions (C1)–(C4) are satisfied with \(h(t) = t^p\), \(t \in [0, 1]\) and \(p \in [0, 1]\), so that \(\omega(tz) \leq t^p \omega(z)\), for \(z > 0, t \in [0, 1]\) and \(p \in [0, 1]\). We see that the \(R\)-order of convergence is at least \(1 + p\).

From (C1)–(C4) with \(h(t) = t^p\), \(t \in [0, 1]\) and \(p \in [0, 1]\), we consider \(a_0 = \beta \omega(\eta)\). Observe that if \(x_1 \in \Omega\) and \(a_0 < 1\), we have

\[
\|I - \Gamma_0 F'(x_1)\| \leq \|\Gamma_0\| \|F'(x_0) - F'(x_1)\| \leq \beta \omega(\eta) = a_0 < 1.
\]

By the Banach lemma, \(\Gamma_1 = F'(x_1)^{-1}\) exists and \(\|\Gamma_1\| \leq f(a_0)\|\Gamma_0\|\), where

\[
f(x) = \frac{1}{1 - x}.
\]

(7)

Therefore, \(x_1\) is well defined.

Besides, from Taylor’s formula and the Newton iteration, it follows that

\[
F(x_1) = \int_0^1 [F'(x_0 + t(x_1 - x_0)) - F'(x_0)](x_1 - x_0) \, dt,
\]

\[
\|F(x_1)\| = \left(\int_0^1 \omega(t\eta) \, dt\right) \|x_1 - x_0\| \leq \frac{\omega(\eta)}{1 + p} \|x_1 - x_0\|,
\]

\[
\|F'(x_1)\| = \left(\int_0^1 \omega(t\eta) \, dt\right) \|x_1 - x_0\| \leq \frac{\omega(\eta)}{1 + p} \|x_1 - x_0\|.
\]
so that
\[
\|x_2 - x_1\| \leq \| \Gamma_1 \| \| F(x_1) \| \leq \frac{a_0}{1 + p} f(a_0) \| x_1 - x_0 \|, \\
\|x_2 - x_0\| \leq \| x_2 - x_1 \| + \| x_1 - x_0 \| \leq \left( 1 + \frac{a_0}{1 + p} f(a_0) \right) \| x_1 - x_0 \|
\]
and
\[
\| \Gamma_1 \| \omega(\| x_2 - x_1 \|) \leq \beta f(a_0) \left( \frac{a_0}{1 + p} f(a_0) \right)^p \omega(\| x_1 - x_0 \|) \leq a_0^{1+p} f(a_0)^{1+p} / (1 + p)^p
\]
provided that \( a_0 \leq (1 + p)/(2 + p) \), since \( f \) is increasing in \((0,1)\).

Note that we can do
\[
a_0^{1+p} f(a_0)^{1+p} / (1 + p)^p = a_1
\]
and define then the following real sequence:
\[
a_n = a_{n-1}^{1+p} f(a_{n-1})^{1+p} / (1 + p)^p, \quad n \geq 1,
\]
that satisfies the properties of the following lemma, whose proof is immediate.

**Lemma 3.1.** If \( a_0 = \beta \omega(\eta) \) satisfies
\[
a_0 \leq \frac{1 + p}{2 + p} \quad \text{and} \quad a_0^p < (1 + p)^p (1 - a_0)^{1+p}, \tag{8}
\]
then

(a) the sequence \( \{a_n\} \) is decreasing,

(b) \( a_n \leq \frac{1 + p}{2 + p} \), for all \( n \geq 0 \).

Since our goal is to obtain a priori error bounds for the Newton iteration when it converges to a solution \( x^* \) of the equation \( F(x) = 0 \), we first present in Lemma 3.2 a system of recurrence relations and in Lemma 3.3 some properties of the real sequence \( \{a_n\} \).

**Lemma 3.2.** If the conditions given in (8) are satisfied, the following items are true for all \( n \geq 1 \):

[I] \( \Gamma_n = F'(x_n)^{-1} \) exists and \( \| \Gamma_n \| \leq f(a_{n-1}) \| \Gamma_{n-1} \| \),

[II] \( \| x_{n+1} - x_n \| \leq \frac{a_{n+1}}{1 + p} f(a_{n-1}) \| x_n - x_{n-1} \| \),

[III] \( \| \Gamma_n \| \omega(\| x_{n+1} - x_n \|) \leq a_n \),

[IV] \( \| x_{n+1} - x_n \| \leq \frac{1 - \Delta^{n+1}}{1 - \Delta} \| x_1 - x_0 \| < R \eta \) where \( \Delta = \frac{a_0}{1 + p} f(a_0) \) and \( R = \frac{1}{1 - \Delta} \).

From a similar way to the mentioned above for \( n = 1 \) and using induction the proof of the previous lemma follows.

**Lemma 3.3.** Let \( f \) be the scalar function given by (7) and \( \gamma = a_1 / a_0 \). If (8) is satisfied, then

(a) \( f(\gamma x) < f(x) \), for \( \gamma \in (0, 1) \) and \( x \in (0,1) \),

(b) \( a_n < \gamma (1 + p)^{n-1} a_{n-1} \) and \( a_n < \gamma (1 + p)^{n-1} f(a_0) \), for all \( n \geq 2 \).

**Proof.** Item (a) is obvious. The proof of (b) follows by invoking the induction hypothesis. Since \( a_1 = \gamma a_0 \) and \( \gamma < 1 \),
\[
a_2 = a_1^{1+p} f(a_1)^{1+p} / (1 + p)^p < (\gamma a_0)^{1+p} f(a_0)^{1+p} / (1 + p)^p = \gamma^{1+p} a_1.
\]
Now, we suppose that \( a_{n-1} < \gamma^{(1+p)^{n-2}} a_{n-2} < \gamma^{(1+p)^{n-1} - 1/p} a_0 \), and consequently
\[
\begin{align*}
&\ a_n = a_{n-1}^1 f(a_{n-1})^{1+p} / (1 + p)^p < \gamma^{(1+p)^{n-1}} a_{n-2}^{1+p} f(a_{n-2})^{1+p} / (1 + p)^p = \gamma^{(1+p)^{n-1}} a_{n-1}^{-1} \\
&< \gamma^{(1+p)^{n-1} - 1/p} a_{n-2} < \cdots < \gamma^{(1+p)^{n-1} - 1/p} a_0.
\end{align*}
\]

The lemma is then complete. \( \square \)

After that, we obtain the following a priori error bounds and the R-order of convergence.

**Theorem 3.4.** Let \( X \) and \( Y \) be two Banach spaces and \( F : \Omega \subseteq X \rightarrow Y \) a once Fréchet differentiable operator in an open convex domain \( \Omega \). We suppose that \( F'(x_0)^{-1} \in L^p(Y, X) \) exists for some \( x_0 \in \Omega \) and conditions (C1)-(C4) hold with \( h(t) = t^p \), \( t \in [0, 1] \) and \( p \in [0, 1] \). If \( a_0 = \beta_0(\eta) \) satisfies (8) and \( B(x_0, R_\eta) \subseteq \Omega \), where \( R = 1/(1 - \Delta) \) and \( \Delta = (a_0/(1 + p)) f(a_0) \), we have the following a priori error estimates:
\[
\|x^* - x_n\| \leq \left( \gamma^{(1+p)^{n-1} - 1/p^2} \right) \frac{\Theta n}{1 - \gamma^{(1+p)^{n-1}/p} \Theta \eta}, \quad n \geq 0,
\]
where \( \gamma = a_1 / a_0 \) and \( \Theta = \Delta / \gamma^{1/p} \). Moreover, Newton’s sequence is of R-order of convergence at least \( 1 + p \).

**Proof.** Take into account (8), \( \gamma = a_1 / a_0 \), \( \Delta = ((a_0)/(1 + p)) f(a_0) \) and \( \Theta = \Delta / \gamma^{1/p} \). For \( m \geq 1 \), we have
\[
\|x_{n+m} - x_n\| \leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\|
\]
\[
\leq \|x_1 - x_0\| \sum_{j=n-1}^{n+m-2} \left( \prod_{i=0}^{j-1} \left( \frac{a_i}{1 + p} f(a_i) \right) \right)
\]
as a consequence of recurrence relation [3] (see Lemma 3.2). From
\[
\prod_{i=0}^{j} \frac{a_i}{1 + p} f(a_i) = \Delta \Theta_{\gamma}^{(1+p)/(p^2)((1+p)^j-1)}
\]
it follows:
\[
\|x_{n+m} - x_n\| \leq \|x_1 - x_0\| \sum_{j=n-1}^{n+m-2} \left( \Delta \Theta_{\gamma}^{(1+p)/(p^2)((1+p)^j-1)} \right)
\]
\[
eq \|x_1 - x_0\| \sum_{i=0}^{m-1} \left( \Theta^{(1+p)/(p^2)((1+p)^i-1)} \right)
\]
and
\[
\|x_{n+m} - x_n\| \leq \|x_1 - x_0\| \Theta^{(1+p)/(p^2)-1/p^2} \sum_{i=0}^{m-1} \left( \Theta_{\gamma}^{(1+p)^{n-1}} / p \right) \gamma_{\gamma}^{(1+p)^n - 1/p^2}
\]
since
\[
\gamma_{\gamma}^{(1+p)^{n+1} - 1/p^2} = \gamma_{\gamma}^{(1+p)^{n-1} - 1/p^2} \gamma_{\gamma}^{(1+p)^n/p^2} ((1+p)^i-1) \leq \gamma_{\gamma}^{(1+p)^{n-1} - 1/p^2} \gamma_{\gamma}^{(1+p)^n/p^2} \gamma_{\gamma}^{(1+p)^n/p^2}
\]
Consequently,
\[
\|x_{n+m} - x_n\| < \frac{1 - (\Theta_{\gamma}^{(1+p)^n}/p)^m}{1 - \Theta_{\gamma}^{(1+p)^n}/p} \Theta^{(1+p)^n - 1/p^2}.
\]

By letting \( m \rightarrow \infty \) in (10), we obtain (9).
Now, from (9), it follows that
\[ \| x^* - x_n \| \leq \frac{\eta}{\gamma^{1/p^2}(1 - \Theta)} (\gamma^{1/p^2})^{(1+p)^n}, \quad n \geq 0. \]
and the \( R \)-order of convergence of the Newton sequence is therefore at least \( 1 + p. \)

**Remark.** Notice that if \( F' \) is \((K, p)\)-Hölder continuous in \( \Omega \), then \( \omega(z) = Kz^p, K \geq 0, p \in [0, 1] \) and \( \omega(tz) \leq t^p \omega(z). \)

So, the \( R \)-order of convergence is at least \( 1 + p \), as we already know (see [3]). Secondly, we see the \( R \)-order of convergence is two if conditions \((C_1), (C_2), (C_3)\) and \((C_4)\) hold. Under these conditions, see [4], the scalar sequence is now
\[ x_n = \frac{2}{n} f(x_{n-1})^2 / 2, \quad n \geq 1, \]
where \( x_0 = \beta n \varphi(\tilde{R}) \) and \( f \) is defined in (7). The sequence \( \{x_n\} \) is decreasing if \( x_0 < \frac{1}{2} \), and satisfies, for all \( n \geq 2, \)
\[ x_n < \tilde{\gamma}^{n-1} x_{n-1} \quad \text{and} \quad x_n < \tilde{\gamma}^{n-1} x_0, \]
where \( \tilde{\gamma} = x_1 / x_0. \) In this case, the recurrence relations are:

[i] \( \tilde{\Gamma}_n = F'(x_n)^{-1} \) exists and \( \| \tilde{\Gamma}_n \| \leq f(x_{n-1}) \| \tilde{\Gamma}_{n-1} \|, n \geq 1, \)

[ii] \( \| x_{n+1} - x_n \| < \frac{2n}{x_{n-1}} f(x_{n-1}) \| x_n - x_{n-1} \|, n \geq 1, \)

[iii] \( \varphi(\tilde{R}) \| \tilde{\Gamma}_n \| \| x_{n+1} - x_n \| < x_n, n \geq 1, \)

[iv] \( \| x_{n+1} - x_n \| < (1 + \sum_{j=0}^{n} (\| \tilde{\Gamma}_j \| f(x_j))) \| x_n \| < \tilde{R}, n \geq 1. \)

Consequently, the a priori error bounds and the \( R \)-order of convergence are given in the following theorem.

**Theorem 3.5.** Let \( X \) and \( Y \) be two Banach spaces and \( F : \Omega \subseteq X \rightarrow Y \) a twice Fréchet differentiable operator in an open convex domain \( \Omega. \) We suppose that \( F'(x_0)^{-1} \in \mathcal{L}(Y, X) \) exists for some \( x_0 \in \Omega \) and conditions \((C_1), (C_2), (C_3), (C_4)\) hold. If \( x_0 = \beta n \varphi(\tilde{R}) \in (0, \frac{1}{2}) \) and \( B(x_0, \tilde{R}) \subseteq \Omega, \) the a priori error bounds are:
\[ \| x^* - x_n \| \leq (\tilde{\gamma}^{n-1}) \frac{(\tilde{\Lambda} / \tilde{\gamma})^n}{1 - (\tilde{\Lambda} / \tilde{\gamma})^{2^n} \eta}, \quad n \geq 0, \]
where \( \tilde{\gamma} = x_1 / x_0 \) and \( \tilde{\Lambda} = x_0 f(x_0) / 2. \) The \( R \)-order of convergence of the Newton sequence is then at least 2.

The proof of the last theorem follows analogously to the Proof of Theorem 3.4.

4. Example
We now consider the particular case in which \( F' \) is \((K, p)\)-Hölder continuous, namely \( F' \) verifies (2).

Rokne is one of the authors that analyzed the semilocal convergence of the Newton method for operators \( F \) such that \( F' \) satisfies condition (2) [11]. But he does not obtain the \( R \)-order of convergence. We see that we can improve the a priori error bounds given in [11] by the technique presented here.

We then consider the following particular nonlinear Hammerstein integral equation of the second kind:
\[ x(s) = 1 + \int_0^1 G(s, t)x(t)^{3/2} dt, \]
where \( G(s, t) \) is Green’s function
\[ G(s, t) = \begin{cases} \frac{1}{2} s t, & t \leq s, \\ s(1-t), & s \leq t. \end{cases} \]
Table 1
Error bounds \( \|x^* - x_n\| \) for (12)

<table>
<thead>
<tr>
<th>( n )</th>
<th>Bounds (9)</th>
<th>Rokne’s bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.010403</td>
<td>0.0109739</td>
</tr>
<tr>
<td>2</td>
<td>0.000192358</td>
<td>0.000730658</td>
</tr>
<tr>
<td>3</td>
<td>5.43361 \times 10^{-7}</td>
<td>3.23906 \times 10^{-6}</td>
</tr>
<tr>
<td>4</td>
<td>9.00389 \times 10^{-11}</td>
<td>2.15661 \times 10^{-7}</td>
</tr>
<tr>
<td>5</td>
<td>2.11229 \times 10^{-16}</td>
<td></td>
</tr>
</tbody>
</table>

Table 2
New error bounds for (12)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( |x^* - x_n| )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0103999</td>
</tr>
<tr>
<td>2</td>
<td>0.000179584</td>
</tr>
<tr>
<td>3</td>
<td>4.17452 \times 10^{-7}</td>
</tr>
<tr>
<td>4</td>
<td>4.69819 \times 10^{-11}</td>
</tr>
<tr>
<td>5</td>
<td>5.61115 \times 10^{-17}</td>
</tr>
</tbody>
</table>

Observe that solving Eq. (12) is equivalent to solve \( F(x) = 0 \), where

\[
F : \Omega \subseteq C[0, 1] \to C[0, 1], \quad \Omega = \{ x \in C[0, 1]; \ x(s) > 0, \ s \in [0, 1] \},
\]

\[
[F(x)](s) = x(s) - 1 - \int_0^1 G(s, t)x(t)^{3/2} \, dt,
\]

so that

\[
[F'(x)y](s) = y(s) - \frac{3}{2} \int_0^1 G(s, t)x(t)^{1/2}y(t) \, dt.
\]

If we now choose \( x_0(s) = 1 \), we have

\[
p = \frac{1}{2}, \quad \beta = \frac{16}{15}, \quad \eta = \frac{2}{15}, \quad w(z) = 3z^{1/2}/16 \quad \text{and} \quad h(t) = t^{1/2}.
\]

Then \( a_0 = \beta \omega(\eta) = 0.0905 \ldots \) and satisfies (8). Therefore, according to Theorem 3.4, Newton’s sequence is of \( R \)-order \( \frac{3}{2} \), and the a priori error bounds (9) for Eq. (12) are given in Table 1, which improve the ones obtained by Rokne’s technique (see [11]).

Taking into account that estimates regarding consecutive points are optimal to measure \( \|x^* - x_n\| \), we look for an element \( x_k (k > n) \) of the sequence \( \{x_n\} \) such that \( \|x^* - x_k\| \) is small enough, and \( \|x^* - x_n\| \) is measured from the distance between two consecutive points. So,

\[
\|x^* - x_n\| \leq \|x^* - x_{n+1}\| + \|x_{n+1} - x_{n+1-1}\| + \cdots + \|x_{n+1} - x_n\|, \quad j \geq 1, \ n \geq 1,
\]

and the error given in (9) is then improved. If we then consider

\[
\|x^* - x_n\| \leq \|x^* - x_{n+1}\| + \|x_{n+1} - x_n\|, \quad n \geq 1,
\]

we obtain better bounds than by (9), see Table 2.

References