ON THE ATTRACTIVITY OF SOLUTIONS FOR A CLASS OF
MULTI-TERM FRACTIONAL FUNCTIONAL DIFFERENTIAL
EQUATIONS

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Abstract. In this paper, we present some alternative results concerning with the ex-
istence and attractivity dependence of solutions for a class of nonlinear fractional func-
tional differential equations. In our consideration, we apply the well-known Schauder
fixed point theorem in conjunction with the technique of measure of noncompactness.
Moreover, we provide examples to illustrate the effectiveness of the obtained results.

1. Introduction

During the last two decades, a great interest has been devoted to the study of fractional
differential equations and such class pulled the interest of so many authors towards itself,
motivated by their extensive use in mathematical modelling. Fractional calculus, which is
as old as classical calculus, has found important applications in the study of problems in
acoustics, thermal systems, rheology or modelling of materials and mechanical systems.
Moreover, in some areas of science like signal processing, identification systems, control
theory or robotics, fractional differential operators seem more suitable to model than the
classical integer order operators. Due to this, fractional differential equations have also
been used in models about biochemistry (modelling of polymers and proteins), electrical
engineering (transmission of ultrasound waves), medicine (modelling of human tissue
under mechanical loads), etc. Thus, differential and integral equations of fractional
order play nowadays a very important role in describing some real world phenomena.

In the recent years, the theory of fractional differential equations has been analytically
investigated by a big number of very interesting and novel papers (see [4, 5, 6, 16, 19,
20, 23, 24, 25]). Moreover, fractional order differential equations, as functional differen-
tial equations [15], are related with causality principles and recently attractivity of
solutions of fractional differential equations and fractional functional equations has been
intensively investigated (see [8, 9, 10, 11, 12, 13, 17] for more details).

The aim of this paper is to study the existence of solutions of a class of multi-term
fractional functional equations in the space of bounded and continuous functions on an
unbounded interval. Moreover, we investigate some important properties of the solutions
related with the concept of attractivity of solutions.
Consider the initial value problem (IVP for short) of the following fractional functional differential equation:

\[
\begin{cases}
  cD^\alpha u(t) = \sum_{i=1}^{m} cD^{\alpha_i} f_i(t, u_t) + f_0(t, u_t), & t > t_0, \\
  u(t) = \varphi(t), & t_0 - \sigma \leq t \leq t_0,
\end{cases}
\]

(1)

where \( cD^\alpha \) denotes Caputo’s fractional derivative of order \( \alpha > 0, \sigma \) is a positive constant, \( \varphi \in C([t_0-\sigma, t_0], \mathbb{R}) \) and for each \( i = 1, 2, \ldots, m \), \( cD^{\alpha_i} \) is the Caputo fractional derivative of order \( 0 < \alpha_i < \alpha \) and \( f_i: I \times C([-\sigma, 0], \mathbb{R}) \rightarrow \mathbb{R} \), such that \( I = [t_0, \infty) \), is a given function. We also consider for any \( t \in I \) the function \( u_t: [-\sigma, 0] \rightarrow \mathbb{R} \) given by \( u_t(s) = u(t + s) \) for each \( s \in [-\sigma, 0] \).

By using the classical Schauder fixed point principle and the concept of measure of noncompactness, we show that Eq. (1) has attractive solutions under rather general and convenient assumptions. We note that employing classical Schauder fixed point principle, we give an alternative result using a control function and new imposed conditions without applying the concept of measure of noncompactness.

The organization of this paper is as follows. In Section 2, we recall some useful preliminaries. In Section 3, we provide some assumptions and a lemma to present the main result of such section for Eq. (1) using the Schauder fixed point principle. In Section 4, we first recall some auxiliary facts about the concept of measure of noncompactness and related notations, then we study the existence of solution for Eq. (1) applying a generalized version of the well-known Darbo type fixed point theorem together with the technique of measure of noncompactness. Finally, in Section 5 some illustrative examples are given to show the practicability of the obtained results.

2. Preliminaries

This section is devoted to recall some essential definitions and auxiliary facts in fractional calculus. We also define some concepts related to attractivity of Eq. (1) together with the Schauder fixed point theorem, which will be needed further on (c.f. [20]).

**Definition 1.** [18, 21] The Riemann-Liouville fractional integral of order \( \gamma > 0 \) with the lower limit \( t_0 \in \mathbb{R} \) for a function \( f \) is defined as:

\[
^{rl} I^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{1-\gamma}} ds, \quad t > t_0,
\]

provided that the right-hand side is point-wise defined on \([t_0, \infty)\), where \( \Gamma(\cdot) \) is the gamma function.

**Definition 2.** [18, 21] The Riemann-Liouville fractional derivative of order \( n - 1 < \gamma < n \) with the lower limit \( t_0 \in \mathbb{R} \) for a function \( f \in C^n([t_0, \infty), \mathbb{R}) \) can be written as:

\[
^{rl} D^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{\gamma+1-n}} ds, \quad t > t_0, \quad n \in \mathbb{N}.
\]

**Definition 3.** [18] Caputo’s fractional derivative of order \( n - 1 < \gamma < n \) for a function \( f \in C^n([t_0, \infty), \mathbb{R}) \) can be written as

\[
cD^{\gamma} f(t) = \begin{cases} 
  ^{rl} D^{\gamma} \left( f(s) - \sum_{k=0}^{n-1} \frac{f^{(k)}(t_0)}{\Gamma(k-\gamma+1)} (s-t_0)^{k-\gamma} \right) (t), & t > t_0, \quad n \in \mathbb{N},
\end{cases}
\]
Definition 4. [13, Definition 2.5] The solution \( u(t) \) of IVP (1) is attractive if there exist a constant \( b_0(t_0) \) such that \( |\varphi(s)| \leq b_0 \) (for all \( s \in [t_0 - \sigma, t_0] \)) implies that \( u(t) \to 0 \) as \( t \to \infty \).

Definition 5. [17, Definition 1] The solution \( u(t) \) of IVP (1) is said to be globally attractive, if there are

\[
\lim_{t \to \infty} (u(t) - v(t)) = 0,
\]

for any solution \( v = v(t) \) of IVP (1).

Remark 1. In Section 4 we consider another definition related to concept of attractivity of solutions (see Definition 8).

Theorem 1 (Schauder Fixed Point Theorem). [22, Theorem 4.1.1] Let \( U \) be a nonempty and convex subset of a normed space \( B \). Let \( T \) be a continuous mapping of \( U \) into a compact set \( K \subset U \). Then \( T \) has a fixed point.

3. Attractivity of Solutions with Schauder Fixed Point Principle

Throughout this section we investigate Eq. (1) using the Schauder fixed point theorem under the following assumptions:

\( (H_0) \) For each \( i = 0, 1, \ldots, m \), the function \( f_i(t, u) \) is Lebesgue measurable with respect to \( t \) on \( [t_0, \infty) \) and \( f_i(t, \psi) \) is continuous with respect to \( \psi \) on \( C([-\sigma, 0], \mathbb{R}) \).

\( (H_1) \) There exists a strictly decreasing function \( \mathcal{H}: \mathbb{R}^+ \to \mathbb{R}^+ \) which vanishes at infinity such that for all \( t \in I = [t_0, \infty) \),

\[
\left| \varphi(t_0) + \sum_{i=0}^{m} \frac{1}{\Gamma(\alpha_i - \alpha)} \int_{t_0}^{t} (t-s)^{\alpha_i-\alpha-1} f_i(s, u_s) ds \right| \leq \mathcal{H}(t-t_0).
\]

\( (H_2) \) There is a constant \( \beta \) such that for each \( i = 1, 2, \ldots, m \), we have that \( f_i \in L^{1/\beta}(I, C([-\sigma, 0], \mathbb{R})) \) with

\[
\beta \in \left( 0, \min_{0 \leq i \leq m} \alpha - \alpha_i \right).
\]

Under the condition \( (H_0) \), the equivalent representation for Eq. (1) is

\[
(2)(t) = \left\{ \begin{array}{ll}
\varphi(t_0) + \sum_{i=0}^{m} \frac{1}{\Gamma(\alpha_i - \alpha)} \int_{t_0}^{t} (t-s)^{\alpha_i-\alpha-1} f_i(s, u_s) ds, & t > t_0, \\
\varphi(t), & t_0 - \sigma \leq t \leq t_0,
\end{array} \right.
\]

where \( \alpha_0 = 0 \) and \( 0 < \alpha_i < \alpha \) for \( i = 1, 2, \ldots, m \). Now we define the operator \( \mathcal{F} \) as

\[
[\mathcal{F}u](t) = \left\{ \begin{array}{ll}
\varphi(t_0) + \sum_{i=0}^{m} \frac{1}{\Gamma(\alpha_i - \alpha)} \int_{t_0}^{t} (t-s)^{\alpha_i-\alpha-1} f_i(s, u_s) ds, & t > t_0, \\
\varphi(t), & t_0 - \sigma \leq t \leq t_0,
\end{array} \right.
\]

for every \( u \in C([t_0 - \sigma, \infty), \mathbb{R}) \).

It is clear that \( u(t) \) is a solution of Eq. (1) if it is a fixed point of the operator \( \mathcal{F} \). Now applying the operator \( \mathcal{F} \) and the imposed conditions as above we have the following lemma.
Lemma 1. Suppose that $f_i(t,u)$ satisfies conditions $(H_0) - (H_2)$. Then Eq. (1) has at least one solution in $C([t_0 - \sigma, \infty), \mathbb{R})$.

Proof. Let us define the set $S \subset C([t_0 - \sigma, \infty), \mathbb{R})$ by

$$S = \{u : u \in C([t_0 - \sigma, \infty), \mathbb{R}) \text{ and } |u(t)| \leq \mathcal{H}(t - t_0) \text{ for all } t \geq t_0\}.$$

Obviously, the set $S$ is a nonempty, closed, bounded and convex subset of $C([t_0 - \sigma, \infty), \mathbb{R})$. To prove that Eq. (1) has a solution it only needs to show that the operator $\mathcal{F}$ has a fixed point in $S$. First, we show that $S$ is $\mathcal{F}$-invariant. This is easily obtained by condition $(H_1)$. Now we should prove that $\mathcal{F}$ is continuous. To do this, let $(u^n)_{n \in \mathbb{N}}$ be a sequence of functions such that $u^n \in S$ for all $n \in \mathbb{N}$ and $u^n \to u$ as $n \to \infty$. Obviously, by the continuity of $f_i(t,u_i)$ we have

$$\lim_{n \to \infty} f_i(t,u^n_i) = f_i(t,u_i), \quad \text{for all } t > t_0 \text{ and } i = 0, 1, \ldots, m.$$

Suppose that $\varepsilon > 0$ is given. Since $\mathcal{H}$ is strictly decreasing, then for some $T > t_0$ we get

$$(3) \quad \mathcal{H}(t - t_0) < \frac{\varepsilon}{2}, \quad \text{for all } t > T.$$

For $t_0 < t \leq T$, we obtain

$$|\mathcal{F} u^n (t) - [\mathcal{F} u] (t)|$$

$$\leq \sum_{i=0}^{m} \frac{1}{\Gamma(\alpha - \alpha_i)} \int_{t_0}^{t} (t - s)^{\alpha - \alpha_i - 1} |f_i(s,u^n_s) - f_i(s,u_s)| \, ds$$

$$\leq \sum_{i=0}^{m} \frac{1}{\Gamma(\alpha - \alpha_i)} \left( \int_{t_0}^{t} (t - s)^{\alpha - \alpha_i - 1} \, ds \right)^{1-\beta} \left( \int_{t_0}^{t} |f_i(s,u^n_s) - f_i(s,u_s)|^{\frac{1}{\beta}} \, ds \right)^{\beta}$$

$$\leq \sum_{i=0}^{m} \frac{(1 - \beta)^{1-\beta}}{\Gamma(\alpha - \alpha_i)(\alpha - \alpha_i - \beta)^{1-\beta}} (T - t_0)^{\alpha - \alpha_i - \beta} \left( \int_{t_0}^{t} |f_i(s,u^n_s) - f_i(s,u_s)|^{\frac{1}{\beta}} \, ds \right)^{\beta}$$

$$\leq \sum_{i=0}^{m} \frac{(1 - \beta)^{1-\beta}}{\Gamma(\alpha - \alpha_i)(\alpha - \alpha_i - \beta)^{1-\beta}} (T - t_0)^{\alpha - \alpha_i} \sup_{t_0 < t \leq T} |f_i(s,u^n_s) - f_i(s,u_s)|,$$

which vanishes as $n \to \infty$. On the other hand, since $S$ is $\mathcal{F}$-invariant, then (3) yields

$$|\mathcal{F} u^n (t) - [\mathcal{F} u] (t)| \leq 2 \mathcal{H}(t - t_0) < \varepsilon, \quad \text{for all } t > T.$$

Consequently, for $t > t_0$ we infer that

$$|\mathcal{F} u^n (t) - [\mathcal{F} u] (t)| \to 0 \quad \text{as } n \to \infty.$$

If $t \in [t_0 - \sigma, t_0]$, we obviously have that $|\mathcal{F} u^n (t) - [\mathcal{F} u] (t)| = 0$. Hence, the continuity of $\mathcal{F}$ has been proved.

Next, we show that $\mathcal{F}(S)$ is equicontinuous. Suppose that $\varepsilon > 0$ is given, $t_1, t_2 > t_0$ and $t_1 < t_2$. Let $t_1, t_2 \in (t_0, T]$ where $T > t_0$ is chosen such that (3) holds. Applying
condition \((H_2)\) we have
\[
|\mathcal{F}u\{t_2\} - [\mathcal{F}u]\{t_1\}| \\
\leq \sum_{i=0}^{m} \frac{1}{\Gamma(\alpha - \alpha_i)} \left| \int_{t_0}^{t_2} (t_2 - s)^{\alpha - \alpha_i - 1} f_i(s, u_s) \, ds - \int_{t_0}^{t_1} (t_1 - s)^{\alpha - \alpha_i - 1} f_i(s, u_s) \, ds \right| \\
\leq \sum_{i=0}^{m} \frac{1}{\Gamma(\alpha - \alpha_i)} \left[ \int_{t_0}^{t_1} |(t_1 - s)^{\alpha - \alpha_i - 1} - (t_2 - s)^{\alpha - \alpha_i - 1}| |f_i(s, u_s)| \, ds \right] \\
+ \int_{t_1}^{t_2} (t_2 - s)^{\alpha - \alpha_i - 1} |f_i(s, u_s)| \, ds \\
\leq \sum_{i=0}^{m} \frac{1}{\Gamma(\alpha - \alpha_i)} \left[ \left( \int_{t_0}^{t_1} |(t_1 - s)^{\alpha - \alpha_i - 1} - (t_2 - s)^{\alpha - \alpha_i - 1}| \frac{1}{1 - \beta} \, ds \right)^{1 - \beta} \left( \int_{t_0}^{t_1} |f_i(s, u_s)| \frac{1}{\beta} \, ds \right)^{\beta} \right] \\
+ \left( \int_{t_1}^{t_2} (t_2 - s)^{\alpha - \alpha_i - 1} \frac{1}{\beta} \, ds \right)^{1 - \beta} \left( \int_{t_1}^{t_2} |f_i(s, u_s)| \frac{1}{\beta} \, ds \right)^{\beta} \\
\leq \sum_{i=0}^{m} \frac{1 - \beta}{\Gamma(\alpha - \alpha_i)(\alpha - \alpha_i - \beta)} \left[ \left| t_2 - t_1 \right|^{\alpha - \alpha_i - \beta} + \left| (t_1 - t_0) \frac{\alpha - \alpha_i - \beta}{1 - \beta} - (t_2 - t_0) \frac{\alpha - \alpha_i - \beta}{1 - \beta} \right|^{1 - \beta} \right] \\
\times \left( \int_{t_0}^{T} |f_i(s, u_s)| \frac{1}{\beta} \, ds \right)^{\beta} + (t_2 - t_1)^{\alpha - \alpha_i - \beta} \left( \int_{t_0}^{T} |f_i(s, u_s)| \frac{1}{\beta} \, ds \right)^{\beta},
\]
which tends to zero when \(t_1 \to t_2\). Now, consider \(t_1, t_2 > T\). Then, taking into account the fact that \(S\) is \(\mathcal{F}\)-invariant and \((3)\), we obtain that
\[
|\mathcal{F}u\{t_2\} - [\mathcal{F}u]\{t_1\}| \\
= \sup_{u \in S} \left\{ |u(t)| : t > T \right\} = 0.
\]
Note that for the case \(t_0 < t_1 < T < t_2\) we have the following implication
\[
(t_1 \to t_2) \implies (t_1 \to T) \land (t_2 \to T).
\]
This, together with the above discussion implies
\[
|\mathcal{F}u\{t_2\} - [\mathcal{F}u]\{t_1\}| \leq \|\mathcal{F}u\{t_2\} - [\mathcal{F}u]\{T\} \| + \|\mathcal{F}u\{T\} - [\mathcal{F}u]\{t_1\}\| \to 0 \quad \text{as} \quad t_1 \to t_2.
\]
Consequently, we conclude that \(\mathcal{F}(S)\) is equicontinuous on each compact interval \([t_0, T]\) for all \(T > 0\). Moreover, since \(\mathcal{F}(S) \subset S\) and from the definition of \(S\) it is clear that
\[
\lim_{T \to \infty} \sup_{u \in \mathcal{F}(S)} \left\{ |u(t)| : t > T \right\} = 0.
\]
Therefore, \(\mathcal{F}(S)\) is a relatively compact set in \(C([t_0 - \sigma, \infty), \mathbb{R})\) and all the conditions of Schauder fixed point theorem are fulfilled.
Hence the operator $\mathcal{F}$, as a self map on $S$, has a fixed point in this set. This fact shows that Eq. (1) has at least one solution in $S$. □

Now we are prepared to formulate our main existence result.

**Theorem 2.** Suppose that conditions $(H_0)$–$(H_1)$ are satisfied, then IVP (1) admits at least one attractive solution in the sense of Definition 4.

**Proof.** According to the previous lemma, there is at least one solution of Eq. (1) belonging to $S$. On the other hand, in order to prove the attractivity, using the property of function $\mathcal{H}$, we infer that all functions in $S$ vanish at infinity and hence the solution of Eq. (1) tends to zero as $t \to \infty$. This makes the proof completed. □

**Remark 2.** Note that conclusion of Theorem 2 does not imply globally attractivity of solutions in the sense of Definition 5.

4. **Uniform Local Attractivity of Solutions with Measure of Noncompactness**

This section is devoted to the study of solutions of Eq. (1) in Banach space $BC([t_0 - \sigma, \infty))$ consisting of all real functions defined, continuous and bounded on the interval $[t_0 - \sigma, \infty)$, via the technique of measure of noncompactness. It is focused on an alternative way to construct some sufficient conditions (quite distinct from the ones in previous section) for solvability of Eq. (1). More precisely, we look for assumptions concerning the functions involved in Eq. (1) which guarantee that this equation has solutions belonging to $BC([t_0 - \sigma, \infty))$ and also being locally attractive on $[t_0 - \sigma, \infty)$. In the following we gather some definitions and auxiliary facts which will be needed further on.

Let $E$ be a Banach space, $\overline{X}$ and Conv $X$ stand for the closure and the convex closure of $X$ as a subset of $E$, respectively. Further, denote by $\mathcal{M}_E$ the family of all nonempty bounded subsets of $E$ and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact sets. Also suppose that $B(x, r)$ is the closed ball centred at $x$ with radius $r$ and the symbol $B_r$ stands for the ball $B(\theta, r)$ such that $\theta$ is the zero element of the Banach space $E$.

In the following definition we recall the notion of measure of noncompactness which has been initially introduced by Banaś and Goebel [8].

**Definition 6.** [8, Definition 3.1.3] A mapping $\mu : \mathcal{M}_E \to \mathbb{R}^+$ is said to be a measure of noncompactness in $E$ if it satisfies the following conditions:

(i) The family $\ker \mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathcal{N}_E$.

(ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

(iii) $\mu(\overline{X}) = \mu(X)$.

(iv) $\mu(\text{Conv } X) = \mu(X)$.

(v) For all $\lambda \in [0, 1]$,

$$\mu(\lambda X + (1 - \lambda) Y) \leq \lambda \mu(X) + (1 - \lambda) \mu(Y).$$

(vi) If $(X_n)_{n \in \mathbb{N}}$ is a sequence of closed sets from $\mathcal{M}_E$ such that $X_{n+1} \subset X_n$ for all $n = 1, 2, \ldots$ and $\lim_{n \to \infty} \mu(X_n) = 0$,

then the intersection set

$$X_\infty = \bigcap_{n=1}^{\infty} X_n$$

is nonempty.
The family ker \( \mu \) described in (i) is said to be the kernel of the measure of noncompactness \( \mu \).

**Definition 7.** Let \( \mu \) be a measure of noncompactness in \( E \). The mapping \( T : C \subseteq E \rightarrow E \) is said to be a \( \mu_E \)-contraction if there exists a constant \( 0 < k < 1 \) such that
\[
\mu(T(W)) \leq k \mu(W),
\]
for any bounded closed subset \( W \subseteq C \).

**Theorem 3** (Darbo-Sadovskii) [8] Let \( C \) be a nonempty, bounded, closed, and convex subset of a Banach space \( E \) and let the continuous mapping \( T : C \rightarrow C \) be a \( \mu_E \)-contraction. Then \( T \) has at least one fixed point in \( C \).

From the point of view of historical remarks, we note that G. Darbo [14] initially introduced condition (4) for any arbitrary measure of noncompactness \( \mu \) and he presented a similar result if the continuous mapping \( T \) is being a \( \mu \)-contraction. Very recently Aghajani, Banaś and Pourhadi (see [1, 2]) have extended Darbo’s fixed point theorem using control functions and presented new results which one of them is applied in this section.

Measure of noncompactness has been applied in some class of fractional differential equations in several papers. For instance, Aghajani, Trujillo and one of the authors have used this tool for Cauchy problem as a classic fractional differential equations in Banach spaces in [3], Banaś and O’Regan in [9] have studied existence of a solutions for a nonlinear quadratic Volterra equation of fractional order and Balachandran, Park and Diana Julie [7] have done something similar for a nonlinear fractional functional integral equation of fractional order with deviating arguments.

We will use the following fixed point theorem.

**Theorem 4.** [1, Theorem 2.2] Let \( C \) be a nonempty, bounded, closed, and convex subset of a Banach space \( E \) and let \( T : C \rightarrow C \) be a continuous function satisfying
\[
\mu(T(W)) \leq \phi(\mu(W))
\]
for each \( W \subseteq C \), where \( \mu \) is an arbitrary measure of noncompactness and \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is a monotone increasing (not necessarily continuous) function with \( \lim_{n \to \infty} \phi^n(t) = 0 \) for all \( t \in \mathbb{R}^+ \). Then \( T \) has at least one fixed point in \( C \).

In what follows, we will work in the Banach space \( BC(\mathbb{R}_{t_0-\sigma}) \), where \( t_0 \) and \( \sigma \) are given in (1). Such functional space is furnished with the standard norm \( \| u \| = \sup \{ |u(t)| : t \geq t_0 - \sigma \} \). For further purposes, we introduce a measure of noncompactness in the space \( BC(\mathbb{R}_{t_0-\sigma}) \), which is constructed similar to the one in the space \( BC(\mathbb{R}^+) \) (for more information see [8, Chapter 9] and references therein).

To do this, let \( B \) be a bounded subset of \( BC(\mathbb{R}_{t_0-\sigma}) \) and \( T > t_0 - \sigma \) given. For \( u \in B \) and \( \varepsilon > 0 \) we denote by \( \omega_{t_0-\sigma}^T(u, \varepsilon) \) the modulus of continuity of the function \( u \) on the interval \( [t_0 - \sigma, T] \), i.e.
\[
\omega_{t_0-\sigma}^T(u, \varepsilon) = \sup \{ |u(t) - u(s)| : t, s \in [t_0 - \sigma, T], |t - s| \leq \varepsilon \}.\]
Now, let us take
\[ \omega^T_{t_0-\sigma}(B, \varepsilon) = \sup \{ \omega^T_{t_0-\sigma}(u, \varepsilon) : u \in B \}, \]
\[ \omega_{t_0-\sigma}(B) = \lim_{\varepsilon \to 0} \omega^T_{t_0-\sigma}(B, \varepsilon), \]
\[ \omega_{t_0-\sigma}(B) = \lim_{T \to \infty} \omega^T_{t_0-\sigma}(B). \]

If \( t \geq t_0 - \sigma \) is a fixed number, let us denote
\[ B(t) = \{ u(t) : u \in B \} \]
and
\[ \text{diam} \ B(t) = \sup \{ |u(t) - v(t)| : u, v \in B \}. \]

Finally, consider the mapping \( \mu \) defined on the family \( \mathcal{M}_{BC(\mathbb{R}_{t_0-\sigma})} \) by the formula
\[ (6) \quad \mu(B) = \omega_{t_0-\sigma}(B) + \limsup_{t \to \infty} \text{diam} \ B(t). \]

Similarly to the measure of noncompactness constructed for \( BC(\mathbb{R}^+) \), one can show that the mapping \( \mu \) is a measure of noncompactness in the space \( BC(\mathbb{R}_{t_0-\sigma}) \) (see also [8]).

Let us point out that, as we will show, information about \( \ker \mu \) is very useful. In this case, the kernel \( \ker \mu \) consists of nonempty and bounded sets \( X \) of functions such that functions belonging to \( X \) are locally equicontinuous on \( \mathbb{R}^+ \) and the thickness of the bundle formed by functions from \( X \) tends to 0 at infinity.

Now, let us assume that \( \Omega \) is a nonempty subset of the space \( BC(\mathbb{R}_{t_0-\sigma}) \) and \( Q \) is an operator defined on \( \Omega \) with values in \( BC(\mathbb{R}^+) \).

Consider the following operator equation:
\[ (7) \quad x(t) = [Qx](t), \quad \text{for all } t \in \mathbb{R}_{t_0-\sigma}. \]

**Definition 8.** [9, Definition 2] We say that solutions of (7) are locally attractive if there exists a closed ball \( B(u_0, r) \) in the space \( BC(\mathbb{R}_{t_0-\sigma}) \) such that for arbitrary solutions \( u = u(t) \) and \( v = v(t) \) of (7) belonging to \( B(u_0, r) \cap \Omega \) we have that
\[ (8) \quad \lim_{t \to \infty} (u(t) - v(t)) = 0. \]

In the case when limit (8) is uniform with respect to set \( B[u_0, r] \cap \Omega \), i.e., when for each \( \varepsilon > 0 \) there exists \( T > 0 \) such that
\[ |u(t) - v(t)| \leq \varepsilon \quad \text{for all } u, v \in B[u_0, r] \cap \Omega \quad \text{and} \quad t \geq T, \]
we will say that solutions of IVP (1) are uniformly locally attractive.

**Remark 3.** As has been remarked in [9], observe that global attractivity of solutions implies local attractivity, but the converse implication is not true.

**Remark 4.** Let us note that the concept of uniform local attractivity of solutions is equivalent to the concept of asymptotic stability introduced in [10, 11] limited to the space \( BC(\mathbb{R}_{t_0-\sigma}) \). So we can use these concepts interchangeably.

Our considerations are based on the following hypotheses:

\( (H_3) \) For each \( i = 0, 1, \ldots, m \), function \( f_i : \mathbb{R}_{t_0-\sigma} \times C([t_0-\sigma, t_0], \mathbb{R}) \to \mathbb{R} \) is continuous and there exists a continuous function \( h_i : \mathbb{R}_{t_0-\sigma} \to \mathbb{R}^+ \) such that
\[ (9) \quad |f_i(t, u) - f_i(t, v)| \leq h_i(t) \mathcal{H}(\|u - v\|), \]
where \( \mathcal{H} : \mathbb{R}^+ \to \mathbb{R}^+ \) is a super-additive function, i.e., \( \mathcal{H}(a) + \mathcal{H}(b) \leq \mathcal{H}(a + b) \) for all \( a, b \geq 0 \).
(H₄) Suppose that for each \( i = 0, 1, \ldots, m \), the following constants exist:
\[
A_i := \sup_{t \in \mathbb{I}} \int_{t_0}^{t}(t-s)^{\alpha_i-1}h_i(s) \, ds < \infty, \quad B_i := \sup_{t \in \mathbb{I}} \int_{t_0}^{t}(t-s)^{\alpha_i-1}|f_i(s,0)| \, ds < \infty;
\]
and, in addition,
\[
\lim_{n \to \infty} \lambda_A^n \mathcal{H}^n(t) = 0 \quad \text{for all} \quad t > 0,
\]
where
\[
\lambda_A = \sum_{i=0}^{m} \frac{A_i}{\Gamma(\alpha - \alpha_i)} < 1.
\]

(H₅) There exists a positive solution \( r_0 \) of the inequality
\[
(10) \quad \sup_{t \in [t_0 - \sigma, t_0]} |\varphi(t)| + \lambda_A \mathcal{H}(r) + \lambda_B \leq r,
\]
where
\[
\lambda_B = \sum_{i=0}^{m} \frac{B_i}{\Gamma(\alpha - \alpha_i)}.
\]

Now we are prepared to formulate our main result as follows.

**Theorem 5.** Under the assumptions (H₃)–(H₅) Eq. (1) has at least one solution in \( BC(\mathbb{R}_{t_0-\sigma}) \). Moreover, solutions of (1) are uniformly locally attractive.

**Proof.** First of all, we consider the operator \( \mathcal{F} \) as defined in the former section by the formula
\[
[\mathcal{F}u](t) = \begin{cases} 
\varphi(t_0) + \sum_{i=0}^{m} \frac{1}{\Gamma(\alpha - \alpha_i)} \int_{t_0}^{t}(t-s)^{\alpha_i-1}f_i(s,u_s) \, ds, & t > t_0, \\
\varphi(t), & t_0 - \sigma \leq t \leq t_0,
\end{cases}
\]
for all \( u \in BC(\mathbb{R}_{t_0-\sigma}) \). Observe that in view of conditions (H₃)–(H₅) the function \( \mathcal{F}u \) is continuous on \( \mathbb{R}_{t_0-\sigma} \). Also, we see that \( BC(\mathbb{R}_{t_0-\sigma}) \) is \( \mathcal{F} \)-invariant. Indeed, for any \( u \in BC(\mathbb{R}_{t_0-\sigma}) \) and \( t > t_0 \) we get
\[
|[\mathcal{F}u](t)| \leq |\varphi(t_0)| + \sum_{i=0}^{m} \frac{1}{\Gamma(\alpha - \alpha_i)} \int_{t_0}^{t}(t-s)^{\alpha_i-1}|f_i(s,u_s)| \, ds
\]
\[
\leq |\varphi(t_0)| + \sum_{i=0}^{m} \frac{1}{\Gamma(\alpha - \alpha_i)} \int_{t_0}^{t}(t-s)^{\alpha_i-1}(|f_i(s,u_s) - f_i(s,0)| + |f_i(s,0)|) \, ds
\]
\[
\leq |\varphi(t_0)| + \sum_{i=0}^{m} \frac{1}{\Gamma(\alpha - \alpha_i)} \int_{t_0}^{t}(t-s)^{\alpha_i-1}h_i(s) \mathcal{H}(|u_s|) \, ds
\]
\[
+ \sum_{i=0}^{m} \frac{1}{\Gamma(\alpha - \alpha_i)} \int_{t_0}^{t}(t-s)^{\alpha_i-1}|f_i(s,0)| \, ds
\]
\[
\leq |\varphi(t_0)| + \lambda_A \mathcal{H}(|u|) + \lambda_B,
\]
which shows that \( \mathcal{F}u \) is bounded on \( [t_0, \infty) \) and linking with the fact that \( \varphi \in C([t_0 - \sigma, t_0], \mathbb{R}) \) we infer that \( \mathcal{F}u \in BC(\mathbb{R}_{t_0-\sigma}) \) and so \( \mathcal{F} \) transforms \( BC(\mathbb{R}_{t_0-\sigma}) \) into itself.

On the other hand, using condition (H₅) there exists a number \( r_0 > 0 \) which enjoys in (10). For such number, the operator \( \mathcal{F} \) transforms the ball \( B_{r_0} \) of \( BC(\mathbb{R}_{t_0-\sigma}) \) into itself.
Let us now take a nonempty subset $X$ of the ball $B_{r_0}$ and fix $x, y \in X$ quite arbitrarily. Then, for fixed $t > t_0$ we obtain

$$
|\{\mathcal{F}u\}(t) - \{\mathcal{F}v\}(t)| \leq \sum_{i=0}^{m} \frac{1}{\Gamma(\alpha - \alpha_i)} \int_{t_0}^{t} (t-s)^{\alpha-\alpha_i-1} |f_i(s, u_s) - f_i(s, v_s)| \, ds
$$

which also implies that

$$
\limsup_{t \to \infty} \text{diam} (\mathcal{F}(X))(t) \leq \lambda_A \mathcal{H} (\limsup_{t \to \infty} \text{diam} X(t)).
$$

Further, let us take $T > t_0$ as fixed and $\varepsilon > 0$. Also, suppose that $u \in X$ is chosen and let $t_1, t_2 \in (t_0, T)$ such that $|t_1 - t_2| \leq \varepsilon$. Without loss of generality we may assume that $t_1 < t_2$. Then, considering our hypothesis, we get

$$
|\{\mathcal{F}u\}(t_2) - \{\mathcal{F}u\}(t_1)| \leq \sum_{i=0}^{m} \frac{1}{\Gamma(\alpha - \alpha_i)} \left( \int_{t_0}^{t_1} (t_1-s)^{\alpha-\alpha_i-1} |f_i(s, u_s)| \, ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-\alpha_i-1} |f_i(s, u_s)| \, ds \right)
$$

for $u \in X \subseteq B_{r_0}$, where the notations in the last term as above are given by

$$
\omega^T_1 (f_i, \alpha_i, \varepsilon) = \sup \left\{ \int_{t_0}^{t_1} |(t_1-s)^{\alpha-\alpha_i-1} - (t_2-s)^{\alpha-\alpha_i-1}| |f_i(s, u_s)| \, ds : t_1, t_2 \in [t_0 - \sigma, T], |t_1 - t_2| \leq \varepsilon, ||u|| \leq r_0 \right\},
$$

$$
\omega^T_2 (f_i, \alpha_i, \varepsilon) = \sup \left\{ \int_{t_1}^{t_2} (t_2-s)^{\alpha-\alpha_i-1} \left( h_i(s) \mathcal{H}(r_0) + |f_i(s, 0)| \right) \, ds : t_1, t_2 \in [t_0 - \sigma, T], |t_1 - t_2| \leq \varepsilon \right\}.
$$

Now, taking into account that the function $f_i(s, u_s)$ is uniformly continuous on the set $[t_0 - \sigma, T] \times B_{r_0}$ for all $i = 0, 1, \ldots, m$, we easily get the following inequality

$$
\omega^T_{t_0-\sigma} (\mathcal{F}(X)) \leq \lambda_A \mathcal{H} (\omega^T_{t_0-\sigma} (X)),
$$
which together with (11) and super-additivity of $\mathcal{H}$ implies that

$$
\mu(\mathcal{F}(X)) = \omega t_0 - \sigma \mathcal{H}(\omega t_0(X)) + \lambda A \mathcal{H}(\limsup_{t \to \infty} \text{diam}(X(t))) \\
\leq \lambda A \mathcal{H}(\mu(X)).
$$

Now, since $\mu$ as given by (6) defines a measure of noncompactness on $BC(\mathbb{R}_t^0 - \sigma)$ then, the recent inequality together with Theorem 4 shows that Eq. (1) has a solution in Banach space $BC(\mathbb{R}_t^0 - \sigma)$.

To prove that all solutions of Eq. (1) are uniformly locally attractive in the sense of Definition 8 let us put $B^1_{r_0} = \text{Conv} \mathcal{F}(B_{r_0})$, $B^2_{r_0} = \text{Conv} \mathcal{F}(B^1_{r_0})$, and so on, where $B_{r_0}$ is the ball with radius $r_0$ and center zero in the space $BC(\mathbb{R}_t^0 - \sigma)$. We simply observe that $B^1_{r_0} \subset B_{r_0}$ and $B^n_{r_0} \subset B^n_{r_0}$ for $n = 1, 2, \ldots$ and also the sets of this sequence are closed, convex and nonempty. Besides, in view of the recent inequality we obtain that

$$
\mu(B^n_{r_0}) \leq \lambda A \mathcal{H}(\mu(B_{r_0})) \quad \text{for any} \quad n = 1, 2, \ldots
$$

Combining the fact that $\mu(B_{r_0}) \geq 0$ and condition $(H_4)$ with the above inequality we get

$$
\lim_{n \to \infty} \mu(B^n_{r_0}) = 0.
$$

Therefore, using the definition of measure of noncompactness we infer that the set $B = \bigcap_{n=1}^{\infty} B^n_{r_0}$ is nonempty, bounded, closed and convex. The set $B$ is $\mathcal{F}$-invariant and the operator $\mathcal{F}$ is continuous on such set. Moreover, keeping in mind the fact that $B \in \ker \mu$ and the characterization of sets belonging to $\ker \mu$ we conclude that all solutions of Eq. (1) are uniformly locally attractive in the sense of Definition 8. This completes the proof.

\begin{remark}
It is clear that this also proves that Eq. (1.1) has at least one attractive solution in the sense of Definition 4.
\end{remark}

5. Concrete Examples

In this section, we present some examples to illustrate our main results obtained in the former sections.

\begin{example}
Consider the fractional functional differential equation

$$
\begin{align*}
\frac{c}{D} u(t) &= \frac{c}{D} \left((s + 1)^{-\frac{1}{2}} e^{-\sin u(s-1)}\right) (t) + \sin \frac{u(t-1)}{3} \left(|u(t-1)| + t + 1\right)^{-\frac{3}{2}}, \\
\frac{u(t)}{t} &= te^{-t}, \quad t \in [-1, 0].
\end{align*}
$$
\end{example}
Obviously, one can show that condition \((H_0)\) holds. To prove that condition \((H_1)\) is satisfied, since \(u(0) = 0\) we have the following relation for all \(t > 0\):

\[
\left| \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t (t - s)^{-\frac{1}{2}} \sin \frac{u(s - 1)}{3} \left(|u(s - 1)| + s + 1\right)^{-\frac{5}{2}} ds \right|
\]

\[
+ \frac{1}{\Gamma\left(\frac{1}{6}\right)} \int_0^t (t - s)^{-\frac{5}{6}} (s + 1)^{-\frac{5}{6}} e^{-\sin u(s - 1)} ds \right| (13)
\]

\[
\leq \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t (t - s)^{-\frac{1}{2}} s^{-\frac{5}{2}} ds + \frac{1}{\Gamma\left(\frac{1}{6}\right)} \int_0^t (t - s)^{-\frac{5}{6}} s^{-\frac{5}{2}} ds
\]

\[
= \frac{t^{-\frac{3}{10}}}{\Gamma\left(\frac{1}{2}\right)} \int_0^1 s^{-\frac{5}{2}} (1 - s)^{-\frac{1}{2}} ds + \frac{t^{-\frac{1}{6}}}{\Gamma\left(\frac{1}{6}\right)} \int_0^1 s^{-\frac{5}{6}} (1 - s)^{-\frac{5}{6}} ds.
\]

Bring to mind that for any \(\alpha, \beta \in \mathbb{R}^+\) we have the following identity

\[
\int_0^1 s^{\alpha-1} (1 - s)^{\beta-1} ds = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)},
\]

together with (13) implies that

\[
\frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t (t - s)^{-\frac{1}{2}} \sin \frac{u(s - 1)}{3} \left(|u(s - 1)| + s + 1\right)^{-\frac{5}{2}} ds
\]

\[
+ \frac{1}{\Gamma\left(\frac{1}{6}\right)} \int_0^t (t - s)^{-\frac{5}{6}} (s + 1)^{-\frac{5}{6}} e^{-\sin u(s - 1)} ds \leq \mathcal{H}(t),
\]

where

\[
\mathcal{H}(t) := \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{10}\right)} t^{-\frac{3}{10}} + \frac{\sqrt{\pi}}{\Gamma\left(\frac{5}{6}\right)} t^{-\frac{3}{2}}, \quad t > 0,
\]

which is clearly a strictly decreasing function on \(\mathbb{R}^+\) and shows that condition \((H_1)\) holds. Finally, it remains to prove that condition \((H_2)\) holds too. To do this, let \(\beta = \frac{1}{10} \in (0, \min\{\frac{1}{2}, \frac{1}{6}\}\)) then we obtain

\[
\int_0^\infty \left[ \sin \frac{u(s - 1)}{3} \left(|u(s - 1)| + s + 1\right)^{-\frac{5}{2}} \right]^{\frac{1}{3}} ds \leq \int_0^\infty (s + 1)^{-8} ds = \frac{1}{5}.
\]

which shows that \(f_0(s, u) = \sin \frac{u(s - 1)}{3} \left(|u(s - 1)| + s + 1\right)^{-\frac{4}{5}} \in L^\frac{1}{3}(I, C([-1, 0], \mathbb{R}))\).

Similarly, for \(f_1(s, u) = (s + 1)^{-\frac{1}{2}} e^{-\sin u(s - 1)}\) we infer that

\[
\int_0^\infty \left[ (s + 1)^{-\frac{1}{2}} e^{-\sin u(s - 1)} \right]^{\frac{1}{2}} ds \leq \int_0^\infty (s + 1)^{-5} ds = \frac{1}{4}.
\]

This prove that \(f_1(s, u) \in L^{\frac{1}{2}}(I, C([-1, 0], \mathbb{R}))\). Therefore, all conditions of Theorem 2 are satisfied and so the solution of Eq. (12) is existent and also attractive.

In the following example, we present another class of fractional differential equations which shows the practicability of Theorem 5.

**Example 2.** Consider the fractional functional differential equation of the form

\[
\begin{cases}
  cD^\alpha u(t) = cD^\frac{1}{2} \left( \sin \frac{u(s - \frac{\pi}{2})}{2\pi(s + 3)^\frac{3}{2}} \right)(t) + cD^\frac{1}{2} \left( \cos \frac{u(s - \frac{\pi}{2})}{2\pi(s + 2)^\frac{3}{2}} \right)(t), & t > 0, \\
u(t) = \frac{t}{(t + 3)^2}, & t \in [-\frac{\pi}{2}, 0].
\end{cases}
\]
It is clear that condition \((H_3)\) holds. It only needs to take the following replacement:

\[
h_1(t) = \frac{1}{2\pi} t^{-\frac{1}{2}}, \quad h_2(t) = \frac{1}{2\pi} t^{-\frac{1}{2}}, \quad \mathcal{H}(t) = t.
\]

To justify condition \((H_4)\) we have

\[
A_1 = \sup_{t \in I} \frac{1}{2\pi} \int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{3}{2}} ds = \frac{1}{2\pi} \Gamma \left( \frac{1}{3} \right) \Gamma \left( \frac{2}{3} \right) = \frac{1}{2\pi} \frac{2\pi}{\sqrt{3}} = \frac{1}{\sqrt{3}} < \infty,
\]

\[
A_2 = \sup_{t \in I} \frac{1}{2\pi} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds = \frac{1}{2\pi} \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right) = \frac{1}{2} < \infty.
\]

Hence, we have that

\[
\lim_{n \to \infty} \lambda_A^n \mathcal{H}^n(t) = \lim_{n \to \infty} \lambda_A^n t = 0 \quad \text{for all} \quad t > 0,
\]

where

\[
\lambda_A = \frac{1}{\sqrt{3} \Gamma \left( \frac{2}{3} \right)} + \frac{1}{2\sqrt{\pi}} \approx 0.7084 < 1.
\]

Finally, we proceed to show the existence of solution. For \((10)\) in condition \((H_5)\) since \(B_1 = 0\) and

\[
B_2 := \sup_{t \in I} \int_0^t (t-s)^{-\frac{1}{2}} \frac{1}{2\pi(s+2)^{\frac{3}{2}}} ds \leq \frac{1}{2\pi} \sup_{t \in I} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds = \frac{1}{2} \implies \lambda_B \leq \frac{1}{2},
\]

we obtain a positive solution \(r_0\) for

\[
(15) \quad \sup_{t \in [-1,0]} \left| \frac{t}{(t+3)^2} \right| + \lambda_A r + \lambda_B \leq 0.25 + 0.8r + 0.5 \leq r
\]

by considering \(r_0 \geq 3.75\). Now the solution of Eq. \((14)\) is existent and uniformly locally attractive since all conditions of Theorem 5 are satisfied.

Remark 6. We note that verification of the existence of solution for Eq. \((14)\) using Theorem 2 seems very unlikely since finding a vanishing and strictly decreasing function \(\mathcal{H}\) in \((H_1)\) is difficult for such equation.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.
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