COMPOSITION OPERATORS BETWEEN BLOCH-TYPE SPACES AND MÖBIUS INVARIANT $Q_K$ SPACES

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Abstract. A characterization of the boundeness and compactness of a composition operator $C_\phi f = f \circ \phi$ acting from the Bloch type spaces $B^\alpha$ to the Möbius invariant spaces $Q_K$ is given. In particular, estimates for the essential norm of such an operator are obtained.

1. Introduction and main results

Let $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic map of the unit disc $\mathbb{D} = \{ z : |z| < 1 \}$ into itself. The map $\phi$ induces a linear composition operator $C_\phi f = f \circ \phi$ on the space $H(\mathbb{D})$ of all analytic functions on the unit disc. A fundamental problem in the study of composition operators is to characterize in terms of the function theoretic properties of $\phi$, the boundeness and compactness of the restrictions of $C_\phi$ to various Banach spaces of analytic functions.

Recall that a bounded linear map $T$ from the Banach space $X$ into a Banach space $Y$ is called compact (weakly compact), if it maps the closed unit ball of $X$ onto a relatively compact (a relatively weakly compact) set in $Y$. The essential norm of $T$ is defined to be the distance to the compact operators, that is,

$$\|T\|_e = \inf\{\|T - S\| : S \text{ is compact}\}.$$

Since $\|T\|_e = 0$ if and only if $T$ is compact, estimates for $\|T\|_e$ give conditions for $T$ to be compact.

For $s > -1$, consider the weighted Dirichlet space $D_s$ of all analytic functions of the unit disc $\mathbb{D}$ for which

$$\|f\|_{D_s}^2 = \int_\mathbb{D} |f'(z)|^2 (1 - |z|^2)^s \, dA(z) < \infty.$$

For $s \geq 0$, let $Q_s$ the space of all analytic functions on the unit disc with

$$\|f\|_{Q_s}^2 = \sup_{w \in \mathbb{D}} \int_\mathbb{D} |f'(z)|^2 (1 - |\varphi_w(z)|^2) \, dA(z) < \infty,$$

where $\varphi_w(z) = \frac{w - z}{1 - \overline{w}z}$ is a Möbius map. We note that $Q_s$ is the Möbius invariant space generated by $D_s$, that is

$$\|f\|_{Q_s}^2 = \sup_{w \in \mathbb{D}} \|f \circ \varphi_w\|_{D_s}^2.$$
Note that $Q_0 = \mathcal{D}$ the classical Dirichlet space, $Q_1 = BMOA$, the space of all analytic functions of bounded mean oscillation, and for $s > 1$ the space $Q_s$ coincides with the Bloch space $\mathcal{B}$ of all analytic functions on $\mathbb{D}$ with
\[ \|f\|_\mathcal{B} = \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty. \]

Let $\alpha > 0$. The Bloch-type space $\mathcal{B}^\alpha$ consists of all analytic functions on $\mathbb{D}$ such that
\[ \|f\|_{\mathcal{B}^\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty. \]

We also consider the little Bloch type space $\mathcal{B}_0^\alpha$ of those functions in $\mathcal{B}^\alpha$ with
\[ \lim_{|z| \to 1^{-}} (1 - |z|^2)^\alpha |f'(z)| = 0. \]

For composition operators on weighted Dirichlet spaces $D_s$ with $-1 < s < 1$, a characterization of the boundedness and compactness can be found in [12]. Compactness of $C_\phi$ in $BMOA$ is studied in [1] and [7]. For composition operators between Bloch-type spaces we refer to [3], [4] and [8].

For $1 < p < \infty$ and a nondecreasing function $K : (0, +\infty) \to (0, +\infty)$, consider the Besov type space $B_K^p$ of all analytic functions $f$ on the unit disc for which
\[ \|f\|_{B_K^p} = \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} K(1 - |z|^2) dA(z) < \infty. \]

Let $Q_K(p)$ be the bounded Möbius invariant space generated by $B_K^p$, that is, an analytic function $f$ is in $Q_K(p)$ if
\[ \|f\|_{Q_K(p)} := \sup_{w \in \mathbb{D}} \|f \circ \varphi_w\|_{B_K^p} < \infty. \]

Note that for $p = 2$ and $K(t) = t^s$ with $s \geq 0$ we obtain the $Q_s$ spaces defined before. We assume throughout the paper that
\begin{equation}
(1) \quad \int_{0}^{1} (1 - r^2)^{p-2} K(\log \frac{1}{r}) r \, dr < \infty.
\end{equation}

Otherwise, the space $Q_K(p)$ only contains constant functions (see [9]). Also, from [9], we have that $Q_K(p) \subset \mathcal{B}$.

In this note we study composition operators from $\mathcal{B}^\alpha$ into $Q_K(p)$, and a description of the boundedness and compactness of $C_\phi$ is given in terms of the function $\phi$. We also prove that the essential norm of $C_\phi : \mathcal{B}^\alpha \to Q_K(p)$ is equivalent to the quantity
\[ \lim_{r \to 1^{-}} \sup_{w \in \mathbb{D}} \int_{|\phi(z)| > r} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} (1 - |z|^2)^{p-2} K(1 - |\phi_w(z)|^2) dA(z), \]
generalizing the result given in [2], where the result is obtained for the case $\alpha = 1$, $p = 2$ and $K(t) = t^s$, that is, for composition operators from $\mathcal{B}$ to $Q_s$.

We use the notation $a \lesssim b$ to indicate that there is a constant $C > 0$ such that $a \leq C b$, and the notation $a \approx b$ ($a$ is comparable with $b$) means that $a \lesssim b \lesssim a$.

The paper is organized as follows. In Section 2 we study composition operators from $\mathcal{B}^\alpha$ to $Q_K(p)$ and prove the main results of the paper, and Section 3 is devoted to the study of composition operators from $Q_K(p)$ to $\mathcal{B}^\alpha$. 
2. Composition operators from $\mathcal{B}^\alpha$ to $\mathcal{Q}_K(p)$

We begin this section with a description of when a composition operator from $\mathcal{B}^\alpha$ to $\mathcal{Q}_K(p)$ exists as a bounded operator.

Theorem 1. Let $\alpha \in (0, +\infty)$, $1 < p < \infty$, $\phi : \mathbb{D} \to \mathbb{D}$ analytic, and $K : (0, +\infty) \to (0, +\infty)$ be a nondecreasing function. A composition operator $C_\phi : \mathcal{B}^\alpha \to \mathcal{Q}_K(p)$ is bounded if and only if

$$\sup_{w \in \mathbb{D}} \int_{\mathbb{D}} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p \alpha}} (1 - |z|^2)^{p - 2} K(1 - |\varphi_w(z)|^2) \, dA(z) < \infty.$$

Proof. Let

$$M = \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p \alpha}} (1 - |z|^2)^{p - 2} K(1 - |\varphi_w(z)|^2) \, dA(z).$$

Suppose first that $M < \infty$. Then, given $f \in \mathcal{B}^\alpha$ we have

$$\|C_\phi f\|_{K,p}^p \leq M \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} |f'(|\phi(z)|)|^p |\phi'(z)|^p (1 - |z|^2)^{p - 2} K(1 - |\varphi_w(z)|^2) \, dA(z) \leq M \|f\|_{\mathcal{B}^\alpha}^p,$$

that is, $C_\phi : \mathcal{B}^\alpha \to \mathcal{Q}_K(p)$ is bounded.

Suppose now that $\|C_\phi f\|_{K,p}^p \leq C\|f\|_{\mathcal{B}^\alpha}$ whenever $f$ is in $\mathcal{B}^\alpha$. From Theorem 2.1.1 of [11] there exist two functions $f_1$ and $f_2$ in $\mathcal{B}^\alpha$ such that

$$(|f_1(z)| + |f_2(z)|) \geq (1 - |z|^2)^{-\alpha}, \quad z \in \mathbb{D}.$$

Then

$$M \leq \|C_\phi f_1\|_{K,p}^p + \|C_\phi f_2\|_{K,p}^p \leq C(\|f_1\|_{\mathcal{B}^\alpha}^p + \|f_2\|_{\mathcal{B}^\alpha}^p) < \infty,$$

and the proof is complete. \qed

It is a well known result that, under the usual integral pairing, the dual of the little Bloch space $\mathcal{B}_0$ is isomorphic to the Bergman space $A^1$ of all analytic functions on the unit disc with

$$\int_{\mathbb{D}} |f(z)| \, dA(z) < \infty.$$

We will need a similar result for $\mathcal{B}_0^\alpha$ with another natural pairing.

Lemma 2. The map $h \mapsto \langle \cdot, h \rangle_{\mathcal{B}^\alpha}$ defines an isomorphism from $A^1 \oplus \mathbb{C}$ onto the dual of $\mathcal{B}_0^\alpha$. Here

$$\langle f, g \rangle_{\mathcal{B}^\alpha} = \int_{\mathbb{D}} f'(z)g(z)(1 - |z|^2)^\alpha \, dA(z) + cf(0)$$

for $f \in \mathcal{B}^\alpha$ and $h = (g, c) \in A^1 \oplus \mathbb{C}$.

Proof. Consider the space $A^{-\alpha}$ of all analytic functions on $\mathbb{D}$ with

$$\|f\|_{A^{-\alpha}} := \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f(z)| < \infty,$$

and its closed subspace $A_0^{-\alpha}$ consisting of all functions $f \in A^{-\alpha}$ with

$$\lim_{|z| \to 1} (1 - |z|^2)^\alpha |f(z)| = 0.$$

These spaces are denoted by $A_\infty(\varphi)$ and $A_0(\varphi)$ in [6] with $\varphi(z) = (1 - |z|^2)^\alpha$. Choosing $\psi(z) := 1$, the pair $\{\varphi, \psi\}$ is a normal pair of weight functions in the
Let

We adapt the argument given in [2] to our case. We first show the lower (b) that is,

\[ an \text{ isomorphism from } A_0^{\alpha}, \]

Therefore the duality

\[ (A_0^{\alpha} \oplus \mathbb{C})^* = (A_0^{\alpha})^* \oplus \mathbb{C}^* = A_1 \oplus \mathbb{C} \]
holds with the pairing \((y, h) = \langle f, g \rangle_{A_0} + bc, \) where \(y = (f, b) \in A_0^{\alpha} \oplus \mathbb{C} \) and \(h = (g, c) \in A_1 \oplus \mathbb{C}.\)

We also note that the map \(I : f \to (f', f(0))\) is a linear isometric bijection from \(B_0^\alpha\) to \(A_0^{\alpha} \oplus \mathbb{C},\) with the direct sums endowed with the sum-norm. Therefore the result follows from the above remarks and the fact that

\[ \langle f, h \rangle_{B_0^\alpha} = \langle If, h \rangle \]
holds for \(f \in B_0^\alpha\) and \(h \in A_1 \oplus \mathbb{C}.\)

We also need the following result due to Yamashita (see [10] or [11] p. 13).

Lemma 3. Let \(0 < \alpha < \infty\) and let \(f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}\) be a Hadamard gap series, that is, \(n_{k+1} \geq Cn_k\) for some constant \(C > 1\) and all \(k \in \mathbb{N}.\) Then

(a) \(f \in B_0^\alpha\) if and only if \(\sup_{k} |a_k| n_k^{1-\alpha} < \infty.\)

(b) \(f \in B_0^\alpha\) if and only if \(\lim_{k} |a_k| n_k^{1-\alpha} = 0.\)

Now we are going to study the compactness of a composition operator from \(B_0^\alpha\) to \(Q_K(p)\) when \(K\) is a nondecreasing weight. With the same proof given in [2] (where the case \(\alpha = 1, p = 2, K(t) = t^*\) is proved) we have that a composition operator \(C_\phi : B_0^\alpha \to Q_K(p)\) is compact if and only if it is weakly compact. The following result, where the essential norm of a composition operator from \(B_0^\alpha\) to \(Q_K(p)\) is estimated, is the main result of the paper.

Theorem 4. Let \(\alpha \in (0, +\infty), 1 < p < \infty, \phi : \mathbb{D} \to \mathbb{D}\) analytic, and \(K : (0, +\infty) \to (0, +\infty)\) be a nondecreasing function. Let \(C_\phi\) be a bounded operator from \(B_0^\alpha\) into \(Q_K(p)\). Then

\[ \|C_\phi\|_p^p \approx \lim_{r \to 1^-} \sup_{w \in \mathbb{D}} \int_{|\phi(z)| > r} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^p} (1 - |z|^2)^{p-2} K(1 - |\varphi_w(z)|^2) dA(z). \]

Proof. We adapt the argument given in [2] to our case. We first show the lower estimate of the essential norm. Choose a sequence \(\{\lambda_n\}\) in \(\mathbb{D}\) converging to 1 as \(n \to \infty,\) and let

\[ f_{n, m, \theta}(z) = \sum_{k=1}^{\infty} \frac{2^k}{2^k + 2^m} 2^{-k(1-\alpha)} (\lambda_n e^{i\theta})^{2^k} z^{2^k + 2^m}, \quad 0 \leq \theta \leq 2\pi, \quad n, m \in \mathbb{N} \]

and

\[ f(z) = \sum_{k=1}^{\infty} 2^{-k(1-\alpha)} z^{2^k}. \]

We note that

\[ f_{n, m, \theta}'(z) = 2^m \lambda_n e^{i\theta} f'(\lambda_n e^{i\theta} z) = z^{2^m - 1} \sum_{k=1}^{\infty} 2^{nk} (\lambda_n e^{i\theta})^{2^k}. \]
By Lemma 3 we have that \( f \in B^\alpha \) and \( f_{n,m,\theta} \in B^\alpha_0 \), and normalizing we can assume that \( \|f\|_{B^\alpha} \leq 1 \) and \( \|f_{n,m,\theta}\|_{B^\alpha} \leq 1 \). Given \( F \in (B^\alpha_0)^* \), let \( h = (g,c) \in A^1 \oplus \mathbb{C} \) be such that \( F(f_{n,m,\theta}) = \langle f_{n,m,\theta}, h \rangle_{B^\alpha} \). Then

\[
\sup_{n,\theta} |F(f_{n,m,\theta})| \leq \sup_{n,\theta} \int_D |f'_{n,m,\theta}(z)||g(z)|(1 - |z|^2)^\alpha dA(z)
\]

\[
\leq \int_D \frac{1}{z^{m-1}}|g(z)|dA(z),
\]

which converges to 0 as \( m \to \infty \) by Lebesgue dominated convergence theorem, since \( g \in A^1 \). Therefore

\[
\lim_{m \to \infty} \sup_{n,\theta} |F(f_{n,m,\theta})| = 0.
\]

Now we claim that for any compact operator \( T : B^\alpha \to Q_K(p) \) we have that

\[
\lim_{m \to \infty} \sup_{n,\theta} \|Tf_{n,m,\theta}\|_{K,p} = 0.
\]

Indeed, if not then there is a subsequence \( (m_k)_{k=1}^\infty \) such that for each \( k \) we can find \( n_k \) and \( \theta_k \) such that for all \( k \) we have

\[
\|Tf_{n_k,m_k,\theta_k}\|_{K,p} \geq c > 0,
\]

for some constant \( c \). By (2) we have that \( f_{n_k,m_k,\theta_k} \to 0 \) weakly in \( B^\alpha_0 \) when \( k \to \infty \).

But, since \( T \) is compact, this is a contradiction with (3).

Therefore, if \( T \) is an arbitrary compact operator, we have

\[
\|C_\phi - T\| \geq \lim_{m \to \infty} \sup_{n,\theta} \|\|C_\phi - T\|f_{n,m,\theta}\|_{K,p}
\]

\[
\geq \lim_{m \to \infty} \sup_{n,\theta} (\|\|C_\phi f_{n,m,\theta}\|_{K,p} - \|Tf_{n,m,\theta}\|_{K,p})
\]

\[
= \lim_{m \to \infty} \sup_{n,\theta} \|C_\phi f_{n,m,\theta}\|_{K,p}.
\]

Hence we obtain

\[
\|C_\phi\|^p \geq \lim_{m \to \infty} \sup_{n,\theta}
\left( \sup_{w \in \mathbb{D}} \int_\mathbb{D} |f'_{n,m,\theta}(\phi(z))|^p |\phi'(z)|^p dA_{K,w}(z) \right),
\]

where \( dA_{K,w}(z) = (1 - |z|^2)^{p-2} K(1 - |\varphi_w(z)|^2) dA(z) \).

Now, given \( \varepsilon > 0 \), choose \( m_0 \in \mathbb{N} \) such that for \( m \geq m_0 \) we have

\[
\|C_\phi\|^p + \varepsilon \geq \sup_{w \in \mathbb{D}} \int_\mathbb{D} |\phi(z)|^{2m+1} \left( \sum_{k=0}^{\infty} 2^{nk}(\lambda_n \phi(z))^{2k-1}(e^{i\theta})^{2k} \right) |\phi'(z)|^p dA_{K,w}(z)
\]

for all \( \theta \) and all \( n \). Let \( w \in \mathbb{D} \) be fixed. Integrating with respect to \( \theta \) and using Fubini’s theorem we get

\[
\|C_\phi\|^p \geq \varepsilon
\]

\[
\int_\mathbb{D} |\phi(z)|^{2m+1} \left( \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=0}^{\infty} 2^{nk}(\lambda_n \phi(z))^{2k-1}(e^{i\theta})^{2k} \right) |\phi'(z)|^p dA_{K,w}(z)
\]

By [13], Theorem 8.20, if \( g \) is a function given by a Hadamard gap series, we have

\[
\left( \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p d\theta \right)^{1/p} \geq \left( \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^2 d\theta \right)^{1/2}.
\]
Therefore, by (4) and Parseval identity, we get
\[
\|C_\phi\|^p_k + \varepsilon \\
\geq \int_{D} |\phi(z)|^{2k+1} \left( \frac{1}{2\pi} \int_{0}^{2\pi} \left| \sum_{k=0}^{\infty} 2^{2nk}(\lambda_n \phi(z))^{2k-1}(e^{i\theta})^{2k} \right|^2 \, d\theta \right)^{p/2} |\phi'(z)|^p \, dA_{K,w}(z) \\
= \int_{D} |\phi(z)|^{2k+1} \left( \sum_{k=0}^{\infty} 2^{2nk}|\lambda_n \phi(z)|^{2(2k-1)} \right)^{p/2} |\phi'(z)|^p \, dA_{K,w}(z) \\
\geq \int_{\{\phi > 1 - 2^{-(m+1)}\}} \left( \sum_{k=0}^{\infty} 2^{2nk}|\lambda_n \phi(z)|^{2(2k-1)} \right)^{p/2} |\phi'(z)|^p \, dA_{K,w}(z). \\
\]
for some positive constant \(C'_\alpha\) depending only on \(\alpha\). Indeed, it is straightforward to see that
\[
r^{2k+1-2} \geq \exp\{-2k+2(1-r)^2\}, \quad 1/2 \leq r < 1.
\]
Now, for \(3/4 \leq r < 1\), there is an integer \(k\) such that \(\frac{\alpha}{2} \leq 2^k (1-r) < \frac{\alpha+1}{2}\).
Therefore
\[
2^{2nk} \exp\{-2k+2(1-r)^2\} \geq \left(\frac{\alpha+1}{2}\right)^{2\alpha} e^{-2(\alpha+1)(1-r)^{-2\alpha}}, \\
\]
since for \(\alpha > 0\), the function \(t^{2\alpha} e^{-4t}\) is decreasing in \([\alpha/2, (\alpha + 1)/2]\). Now, the estimate (5) follows from (6) and (7).

Hence, by (5) we have
\[
\sum_{k=0}^{\infty} 2^{2nk}|\lambda_n \phi(z)|^{2(2k-1)} \geq C'_\alpha (1 - |\lambda_n \phi(z)|^2)^{-2\alpha} \\
\]
for all \(z \in D\) with \(|\phi(z)| > 1 - 2^{-(m+1)}\) if \(n\) and \(m\) are big enough. Therefore, by Fatou’s lemma we have
\[
\|C_\phi\|^p_k + \varepsilon \geq \liminf_{n \to \infty} \int_{\{\phi(z) > 1 - 2^{-(m+1)}\}} \frac{|\phi'(z)|^p}{(1 - |\lambda_n \phi(z)|^2)^{\alpha}} \, dA_{K,w}(z) \\
\geq \int_{\{\phi(z) > 1 - 2^{-(m+1)}\}} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{\alpha}} \, dA_{K,w}(z). \\
\]
Thus, since \(w \in D\) is arbitrary, we obtain
\[
\|C_\phi\|^p_k + \varepsilon \geq \limsup_{r \to 1^-} \int_{\{|\phi(z)| > r\}} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{\alpha}} \, dA_{K,w}(z), \\
\]
that proves the lower estimate.

Now we are going to compute the upper estimate. For each \(k \in \mathbb{N}\) define a sequence of compact linear operators \(C_k : B^\alpha \to B^\alpha\) by
\[
C_k f(z) = f\left(\frac{k}{k+1} z\right), \quad z \in D.
\]
Let \( \psi_k(z) = \frac{kz}{k+z} \) so that \( C_k f = f \circ \psi_k \). Then we have
\[
\|C_\phi\|_p^p \leq \|C_\phi - C_\phi C_k\|_p^p = \|C_\phi (I_d - C_k)\|_p^p
\]
which is less than
\[
\sup_{\|f\|_{B^\alpha} \leq 1} \sup_{w \in \mathbb{D}} \int_{\{1/|\phi| > r\}} |(f - f \circ \psi_k)'(\phi(z))|^p |\phi'(z)|^p \, dA_{K,w}(z)
\]
\[
+ \sup_{\|f\|_{B^\alpha} \leq 1} \sup_{w \in \mathbb{D}} \int_{\{|\phi| \leq r\}} |(f - f \circ \psi_k)'(\phi(z))|^p |\phi'(z)|^p \, dA_{K,w}(z)
\]
\[
:= I_k + J_k,
\]
where \( 0 < r < 1 \) is fixed. To estimate the first term \( I_k \) note that for \( \|f\|_{B^\alpha} \leq 1 \) and \( z \in \mathbb{D} \) we have
\[
|f'(z)| \leq (1 - |z|^2)^{-\alpha}.
\]
Since \( f \circ \psi_k \|_{B^\alpha} \leq \|f\|_{B^\alpha} \) we obtain
\[
I_k \leq 2^p \sup_{w \in \mathbb{D}} \int_{\{1/|\phi| > r\}} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} \, dA_{K,w}(z).
\]
Therefore it is enough to show that \( \lim_{k \to \infty} J_k = 0 \). Since \( C_\phi z = \phi \in Q_K(p) \) we get
\[
M := \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} |\phi'(z)|^p (1 - |z|^2)^{p-2} K(1 - |\phi_w(z)|^2) \, dA(z) < \infty.
\]
Therefore
\[
J_k \leq M \sup_{\|f\|_{B^\alpha} \leq 1} \sup_{\{|\phi| \leq r\}} \|g_k'\|_0^p,
\]
where \( g_k = f - f \circ \psi_k \). Since \( g_k \) converges to 0 uniformly on compact subsets of \( \mathbb{D} \), then \( g_k' \) also converges to 0 uniformly on compact subsets of \( \mathbb{D} \). Hence we obtain that
\[
\lim_{k \to \infty} \sup_{\|f\|_{B^\alpha} \leq 1} \sup_{\{|\phi| \leq r\}} |g_k'(\phi(z))|^p = 0,
\]
and the proof is complete. \( \square \)

As an immediate consequence of Theorem 4 we get the following characterization of compact composition operators from \( B^\alpha \) to \( Q_K(p) \).

**Corollary 5.** Let \( 0 < \alpha < \infty \) and \( 1 < p < \infty \). A composition operator \( C_\phi : B^\alpha \to Q_K(p) \) is compact if and only if \( \phi \in Q_K(p) \) and
\[
\lim_{r \to 1^-} \sup_{u \in \overline{D}} \int_{|\phi(z)| > r} \frac{|\phi'(z)|^p}{(1 - |\phi(z)|^2)^{p\alpha}} (1 - |z|^2)^{p-2} K(1 - |\phi_w(z)|^2) \, dA(z) = 0.
\]

If \( \phi : \mathbb{D} \to \mathbb{D} \) is univalent, we can provide some geometric characterizations of the boundedness and compactness of \( C_\phi \). This requires some background on the hyperbolic metric. Recall that the hyperbolic distance \( \beta_D(z, w) \) between two points \( z, w \in \mathbb{D} \) is defined by \( \beta_D(z, w) = \log 2 \frac{1 + \rho(z,w)}{1 - \rho(z,w)} \), where \( \rho(z,w) = |\frac{z - w}{1 - \overline{w}z}| \). We note that
\[
(1 - |z|)^{-1} \leq 2^{\beta_0(0,z)} \leq 2(1 - |z|)^{-1}.
\]
This distance is invariant under Möbius transformations, and therefore transfers to a conformally invariant metric on any simply connected proper subset \( \Omega \) of \( \mathbb{C} \).
If $f : \mathbb{D} \to \Omega$ is any conformal map, the hyperbolic distance on $\Omega$ is given by $\beta_\Omega(w_1, w_2) = \beta_\Omega(z_1, z_2)$ where $w_j = f(z_j)$ for $j = 1, 2$. We denote by $d(z, \partial \Omega)$ the euclidian distance from $z$ to the boundary of $\Omega$.

**Theorem 6.** Let $0 < \alpha < \infty$, $1 < p < \infty$ and let $K$ be a nondecreasing function. Let $f : \mathbb{D} \to \Omega$ be univalent and let $\Omega = \phi(\mathbb{D})$. Then

(i) $C_\phi : \mathcal{B}^\alpha \to \mathcal{Q}_K(p)$ exists as a bounded operator if and only if

$$
\sup_{w \in \Omega} \int_\Omega K(2^{-\beta_\Omega(w, z)}) (1 - |z|^2)^{p\alpha} \, d(z, \partial \Omega)^{p-2} \, dA(z) < \infty.
$$

(ii) $C_\phi : \mathcal{B}^\alpha \to \mathcal{Q}_K(p)$ exists as a compact operator if and only if $\phi \in \mathcal{Q}_K(p)$ and

$$
\lim_{r \to 1^-} \sup_{w \in \Omega} \int_{\Omega \cap \{|z| > r\}} K(2^{-\beta_\Omega(w, z)}) (1 - |z|^2)^{p\alpha} \, d(z, \partial \Omega)^{p-2} \, dA(z) = 0.
$$

**Proof.** Let $\Omega = \phi(\mathbb{D})$. Since $\phi : \mathbb{D} \to \Omega$ is conformal, the fact that $a \in \mathbb{D}$ is equivalent to the fact that $\phi(a) \in \Omega$. If $\psi$ denotes the inverse map of $\phi$, then by (8) we have

$$
1 - |\phi_a(\psi(z))|^2 = 2^{-\beta_\Omega(0, \phi_a(\psi(z)))} = 2^{-\beta_\Omega(\phi(a), z)}.
$$

Also, by Koebe’s distortion theorem we have $(1 - |z|^2)|\phi'(z)| \approx d(\phi(z), \partial \Omega)$. Hence

$$
\int_{\mathbb{D}} \frac{|\phi'(w)|^p K(1 - |\phi_a(w)|^2)}{(1 - |\phi(w)|^2)^{p\alpha}} \, dA(w)
\approx \int_{\mathbb{D}} \frac{|\phi'(w)|^2 K(1 - |\phi_a(w)|^2)}{(1 - |\phi(w)|^2)^{p\alpha}} \, d(\phi(w), \partial \Omega)^{p-2} \, dA(w)
\approx \int_{\Omega} K(1 - |\phi_a(\psi(z))|^2) (1 - |z|^2)^{p\alpha} \, d(z, \partial \Omega)^{p-2} \, dA(z)
\approx \int_{\Omega} K(2^{-\beta_\Omega(w, z)}) (1 - |z|^2)^{p\alpha} \, d(z, \partial \Omega)^{p-2} \, dA(z).
$$

This, together with Theorem 1 and Corollary 5 leads to (i) and (ii). \hfill \Box

3. **Composition operators from $\mathcal{Q}_K(p)$ to $\mathcal{B}^\alpha$**

We begin this section with two lemmas. The first one is standard and can be found for example in [5].

**Lemma 7.** Let $\sigma > -1$ and $a, b > 0$ such that $a + b - \sigma > 2$ and $a - \sigma, b - \sigma < 2$. Then

$$
\int_{\mathbb{D}} \frac{(1 - |\zeta|^2)^\sigma}{|1 - \zeta z|^a |1 - \zeta w|^b} \, dA(\zeta) \lesssim |1 - \bar{w}z|^{2+\sigma-a-b}.
$$

**Lemma 8.** Let $1 < p < \infty$, and let $K$ be a nondecreasing function such that for some $s > 0$ we have that $t^{-s}K(t)$ is increasing for $0 < t \leq 1$. Then for each $w \in \mathbb{D}$ the function

$$
f_w(z) = -\log(1 - \bar{w}z)
$$

belongs to $\mathcal{Q}_K(p)$ with $\|f_w\|_{K,p} \leq C$, where $C$ is a constant independent of $w$. 
Proof. Let \( w \in \mathbb{D} \). By assumption, there is \( s > 0 \) such that \( t^{-s}K(t) \) is increasing for \( 0 < t \leq 1 \). Therefore

\[
\|f_w\|_{K,p} = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f_w'(z)|^p (1 - |z|^2)^{p-2}K(1 - |\varphi_a(z)|^2) \, dA(z)
\]

\[
\leq K(1) \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f_w'(z)|^p (1 - |z|^2)^{p-2} (1 - |\varphi_a(z)|^{2s}) \, dA(z)
\]

\[
\leq K(1) \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} (1 - |z|^{2s+p-2}(1 - |a|^2)^s \, dA(z)
\]

which, by Lemma 7, is bounded by

\[
C \sup_{a \in \mathbb{D}} \frac{(1 - |a|^2)^s}{|1 - aw|^{2s+p-2p+2}} \leq C.
\]

\( \square \)

We note that for \( s > 0 \), the function \( K(t) = t^s \) satisfies the condition for \( K \) given in Lemma 8. Also this condition implies that \( \int_0^1 K(t)/t \, dt < \infty \).

For the case of composition operators from \( Q_K(p) \) to \( B^\alpha \) we have the following description of boundedness and compactness.

Theorem 9. Let \( \alpha \in (0, +\infty) \), \( 1 < p < \infty \) and \( \phi : \mathbb{D} \rightarrow \mathbb{D} \) analytic. Let \( K : (0, +\infty) \rightarrow (0, +\infty) \) be a nondecreasing function such that for some \( s > 0 \) the function \( t^{-s}K(t) \) is increasing for \( 0 < t \leq 1 \). Then

(i) \( C_\phi : Q_K(p) \rightarrow B^\alpha \) is bounded if and only if

\[
\sup_{z \in \mathbb{D}} \frac{|\phi'(z)|}{1 - |\phi(z)|^2} (1 - |z|^2)^\alpha < \infty.
\]

(ii) \( C_\phi : Q_K(p) \rightarrow B^\alpha \) is compact if and only if \( \phi \in B^\alpha \) and

\[
\lim_{r \to 0^+} \sup_{z : |\phi(z)| > r} \frac{|\phi'(z)|}{1 - |\phi(z)|^2} (1 - |z|^2)^\alpha = 0.
\]

Proof. By Lemma 8, if \( w \in \mathbb{D} \) then the function

\[
f_w(z) = -\log(1 - wz)
\]

belongs to \( Q_K(p) \). Then one can repeat the proof for the \( Q_s \) case given in [11]. We omit the details.

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References


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