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Analysis of power-law exponents by maximum-likelihood maps

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Maximum-likelihood exponent maps have been studied as a technique to increase the understanding and improve the fit of power-law exponents to experimental and numerical simulation data, especially when they exhibit both upper and lower cut-offs. This technique is tested for seismological data, acoustic emission data and avalanches in numerical simulations of the 3D-Random Field Ising model. In the different examples we discuss the nature of the deviations observed in the exponent maps and some relevant conclusions are drawn for the physics behind each phenomenon.

Application of The Method

Maximum Likelihood :

Critical systems use to produce **power-law** distributions:

$$\mathbf{g(x)dx} = \frac{\mathbf{x^{-\gamma}}}{\zeta(\gamma)}\mathbf{dx} \qquad \mathbf{x_{min} < x < x_{max}}$$

where $\zeta(\gamma)$ denotes the normalization factor.

These kind of distribution is **scale-free** in all the range from X_{min} to X_{max} and all the meaningful information is stored in the exponent value γ .

The Maximum Likelihood Method is the most efficient way to evaluate this exponent[1]. Being $\{X_i\}$ the set of n measurements, the exponent can be found solving the relation

$$0 = \frac{\partial \ln \mathcal{L}(\hat{\gamma})}{\partial \hat{\gamma}} = -\sum_{\{X_i\}} \ln(X_i) - n\frac{\zeta'(\hat{\gamma})}{\zeta(\hat{\gamma})}$$

Exponent Maps :

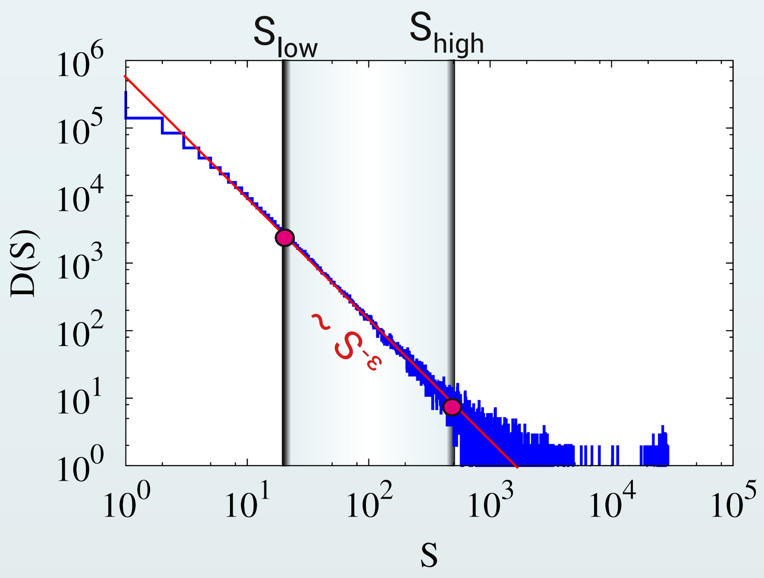
We evaluate the exponent inside the restricted (X_{low} , X_{high}) interval instead of (X_{low} , ∞) as usual. For continuous data, this correspond to the relation:

$$\sum_{\{X_{low}<X_i<X_{high}\}}^{N} \ln(X_i) = \frac{N}{\hat{\gamma}-1} + \frac{N(X_{high}^{1-\hat{\gamma}}\ln X_{high} - X_{low}^{1-\hat{\gamma}}\ln X_{low})}{X_{high}^{1-\hat{\gamma}} - X_{low}^{1-\hat{\gamma}}}$$

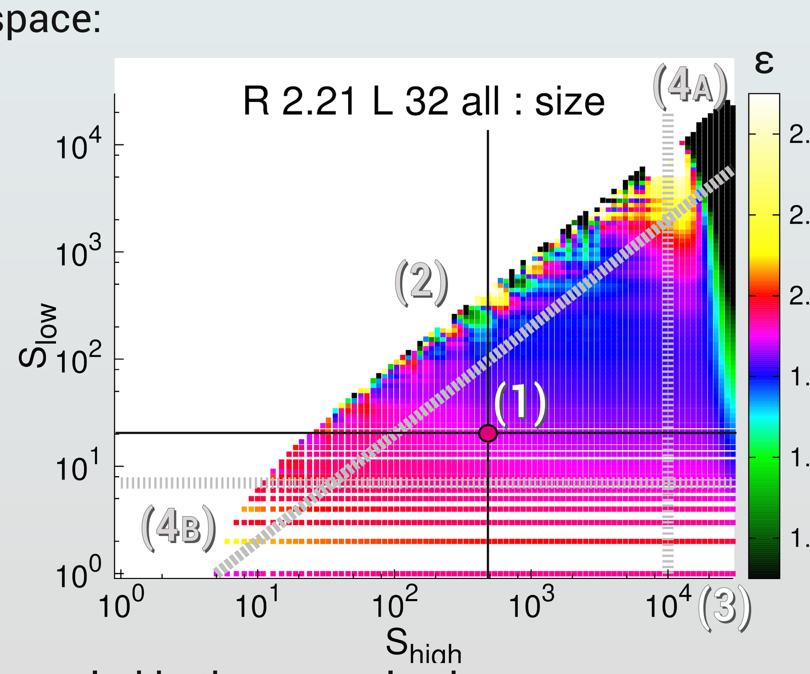
For discrete data, where we have the frequencies f(k) of occurrence of each value k:

$$\sum_{k=K_{low}}^{K_{high}} f(k)\ln(k) = N\frac{\sum_{K_{low}}^{K_{high}} k^{-\hat{\gamma}}\ln(k)}{\sum_{K_{low}}^{K_{high}} k^{-\hat{\gamma}}}$$

In the following example the measurements $\{X_i\}$ correspond to the sizes $\{S_i\}$ of the avalanches in the **3D-GRFIM**:



The obtained values can be represented in a 2D colored map in the S_{high}, S_{low} space:



(1) point sampled in the example above

(2) Noisy region. Low on statistics

(3) Evaluation with all the data

(4a, 4b) Regions affected by finite-size anomalies

References :

[1] M.L. Goldstein, S.A. Morris, G.G.Yen, Eur. Phys. Journ. B **41**, 255-258 (2004)

[2] B. Gutenberg, C. F. Richter, Bull. Seismol. Soc. Am. **34**, 185 (1944)

[3] Ekhard K.H. Salje et al., Phil. Mag. Lett. **91**, 554-560 (2011)

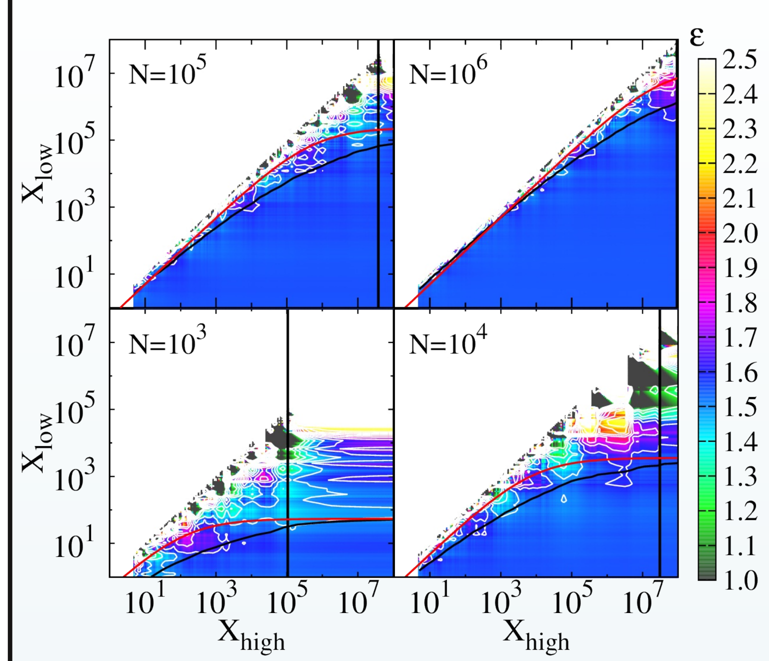
[4] James P. Sethna et al., Phys. Rev. Lett. **70**, 3347 (1993)

[5] F.J. Perez-Reche, E.Vives, Phys. Rev. B **70**, 214422 (2004)

[6] http://arxiv.org/abs/1202.2043

Working Example Maps on Continuum Data

Synthetic Data :



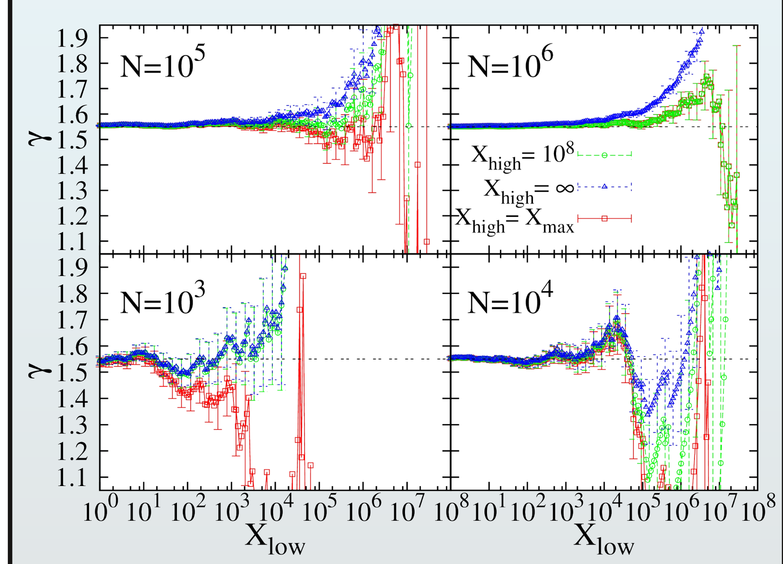
In order to test the performance of the map we generate a set of synthetic data distributed according to a power-law.

White Contour Lines correspond to the discrete values in 0.1 intervals

Black Vertical Line: Is the value of the greatest signal obtained

Red Line mark the theoretical threshold for error greater than 0.5

Black Line show the actual threshold for an error greater than 0.5.



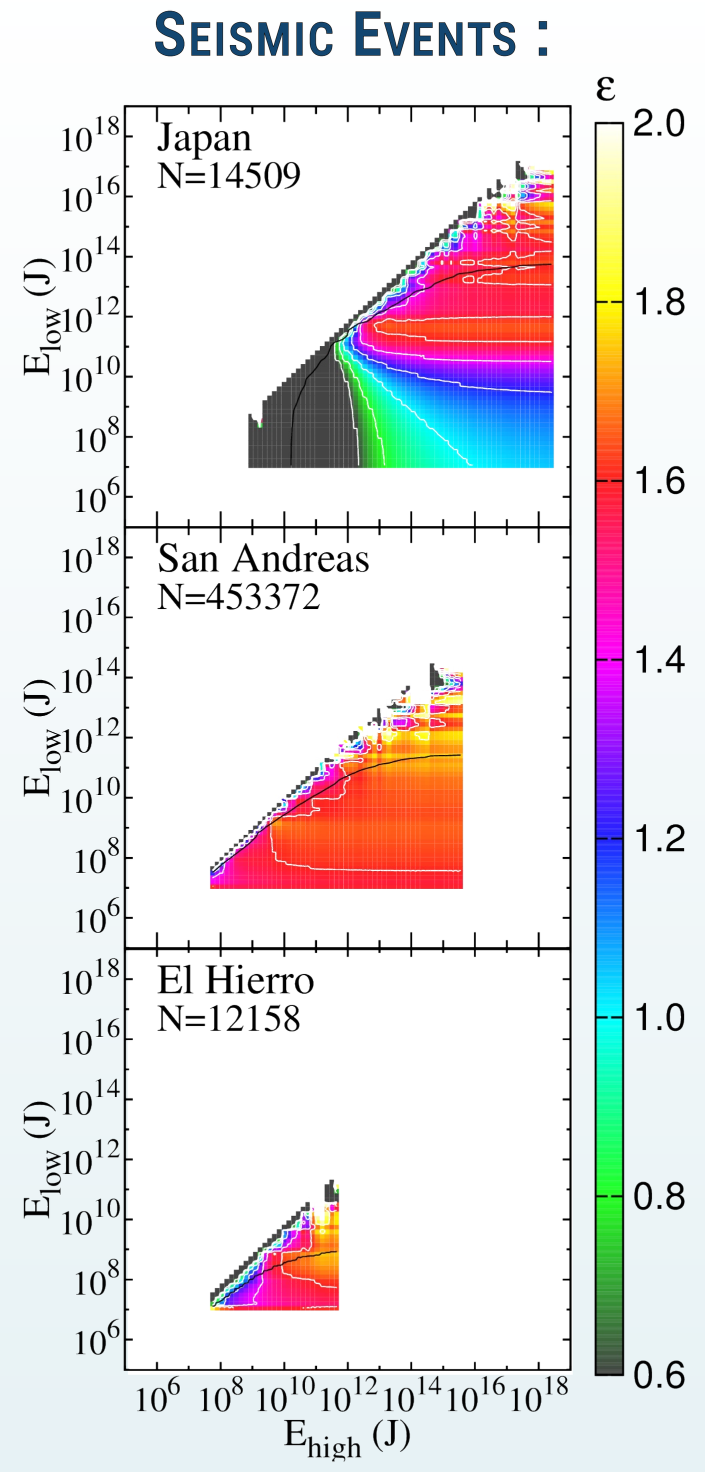
Profiles of X_{low} at a fixed X_{high} at different values:

Green: X_{high} at the known or imposed cutoff of the distribution

Blue: the usual criteria stated in ref. 1, X_{high} -> ∞.

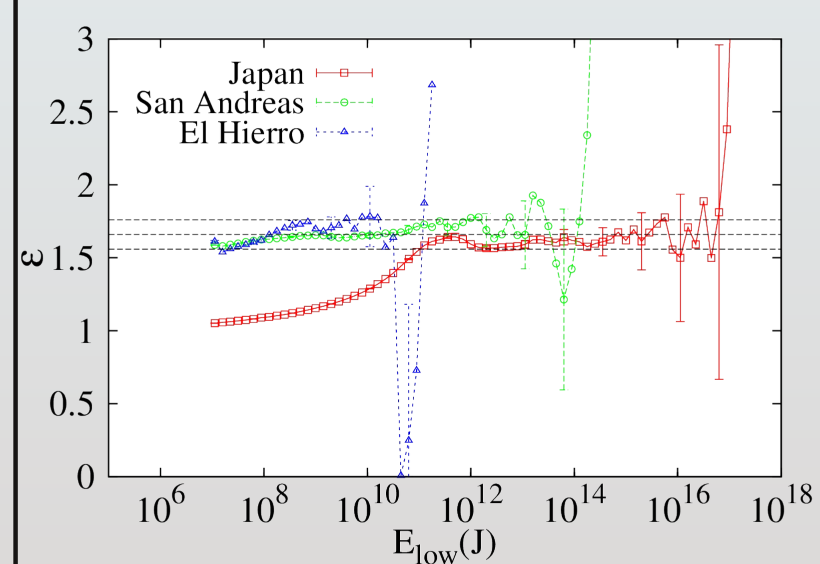
Red: X_{high} at the biggest value obtained in the distribution.

Seismic Events :



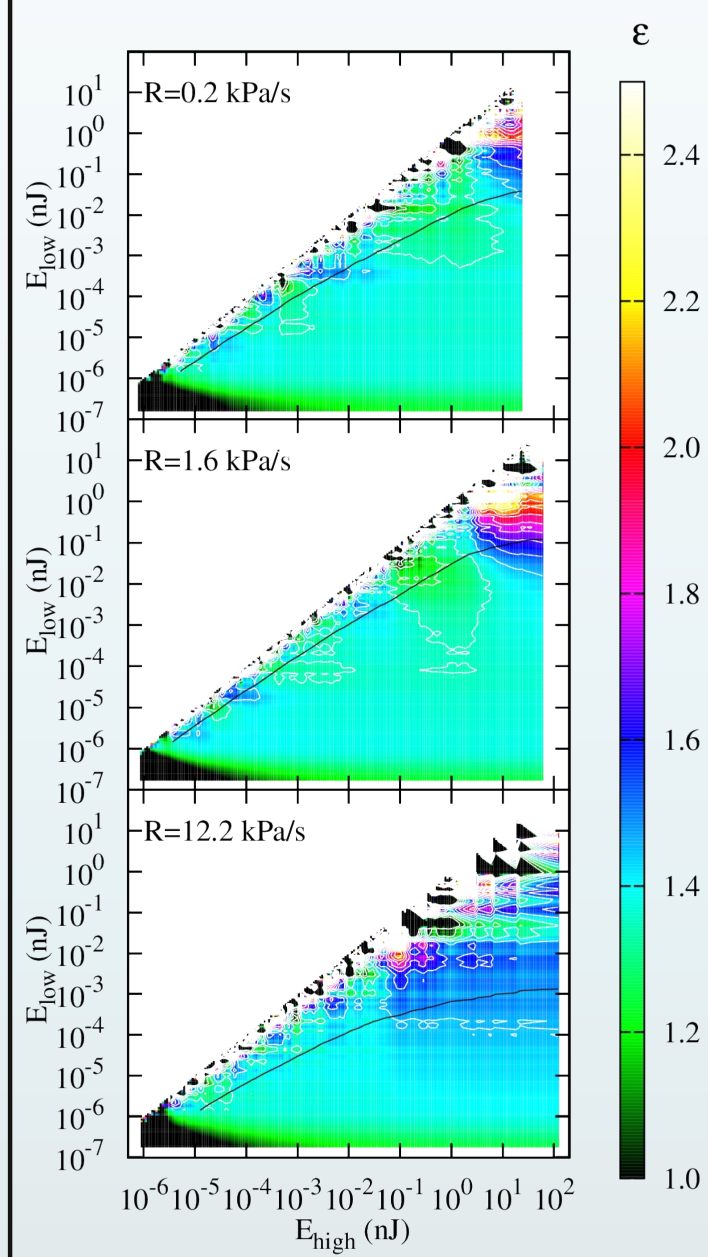
The **Gutenberg–Richter Law**[2] states that the frequency of earthquakes with energy E follow a power-law distribution with exponent ε = 1+b/1.5 ~ 1.66 , where b is known as the Guttenberg-Richter exponent.

The analysis over the data from different areas show the scale-free nature of this law.



Compression of The porous Material Vycor:

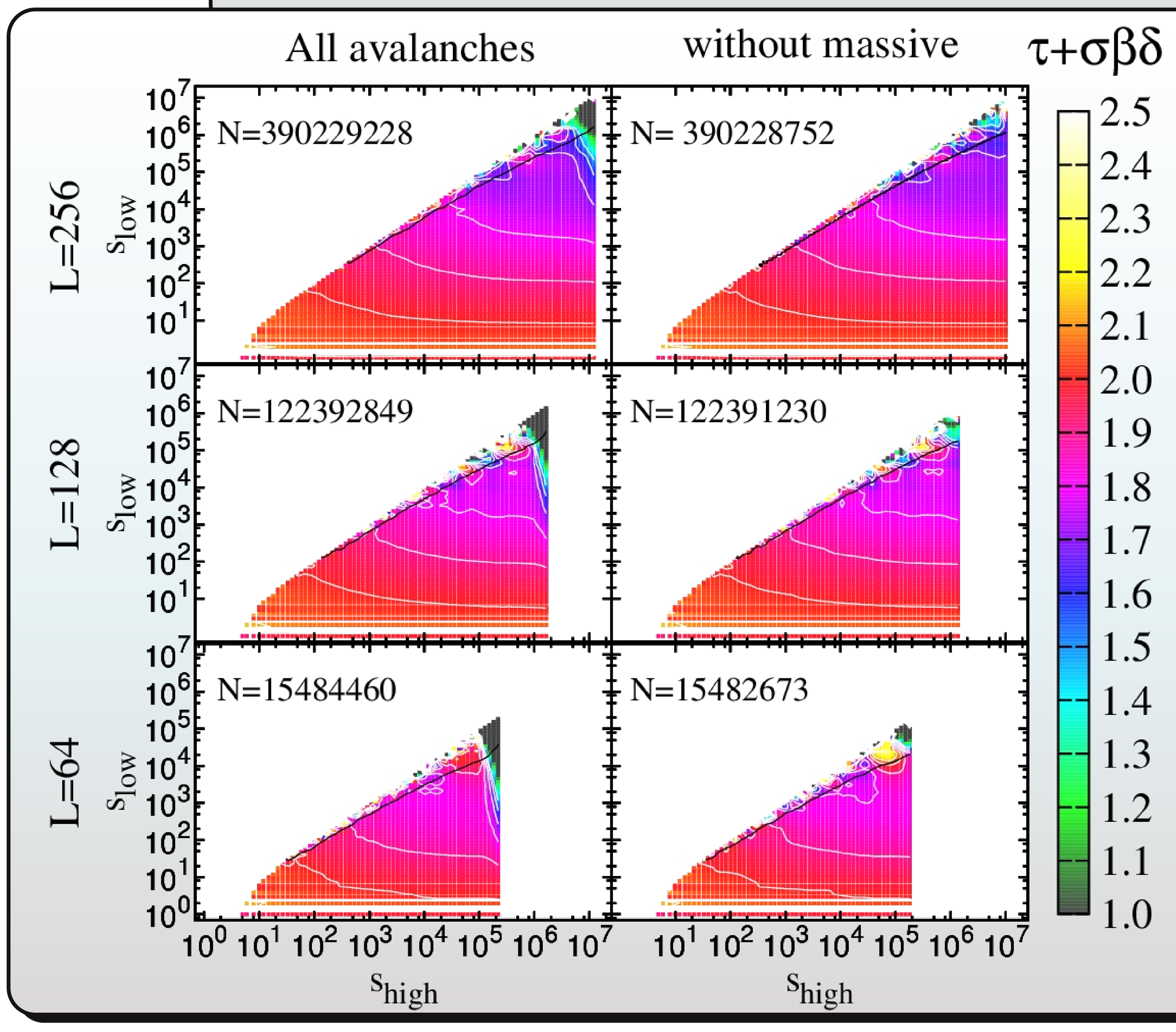
In most experimental data, power-law distributions appear truncated in the large event region with an exponential cutoff. This effect cause a drastic increase in the exponent evaluation in this region.



This map represent the energy of the signals obtained in a **Acoustic Emission** experiment, where a sample of **Vycor** was compressed at a constant rate[3].

The extension of the plateau fit the hypothesis of scale-free behaviour.

Working Example Maps on Discrete Data



Avalanche Size Distribution Function in Numerical Simulation of the 3D-GRFIM :

In numerical simulations the finite size of the lattice may truncate the power law down to a maximum value. This cause the appearance of a sharp distribution in the large-event region that drastically distorts the power-law if the evaluation range is not properly selected.

This example show the size distribution of avalanches obtained in the simulation of the **3D-Gaussian Random Field Ising Model**, a prototype model widely used in the study of avalanche dynamics.[4]

There are methods to exclude the massive signals that cause the distortion.[5] The exponent-maps helped us to check their performance.