ARRANGEMENTS OF PLANES IN SPACE

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This paper presents a variety of formulas for the number of cells, faces, and edges, bounded and unbounded, that are formed by an arbitrary set of planes in $E^3$. Using an elegant geometric method described in 1966 by Brousseau, we first prove a version in $E^3$ of the general partition formulas established algebraically by Zaslavsky in 1975. From these formulas we deduce two families of inclusion-exclusion formulas for the counters, the first similar to formulas outlined by Roberts in 1889, the second related to formulas given by Steiner in 1826. We conclude with some non-trivial new bounds for the counters of an arbitrary arrangement in $E^3$ and two specific examples.

1. Introduction

The study of division-of-space problems has a long history to which many people have contributed. We begin by mentioning some of those contributions that bear directly on our results.

In 1826, Steiner [15] showed by recursion that if planes in Euclidean three-space fall into $s$ parallel families having $n_1, n_2, \ldots, n_s$ planes respectively, if there are no multiple points and no multiple lines, and if the parallel families are in general position (that is, a selection of one plane from each parallel family produces an arrangement of $s$ planes in general position), then there are

$$1 + \sigma_1 + \sigma_2 + \sigma_3 \text{ cells,}$$
$$-1 + \sigma_1 - \sigma_2 + \sigma_3 \text{ bounded cells,}$$
$$\sigma_1 + 2\sigma_2 + 3\sigma_3 \text{ faces,}$$
$$\sigma_1 - 2\sigma_2 + 3\sigma_3 \text{ bounded faces,}$$
$$\sigma_2 + 3\sigma_3 \text{ edges,}$$
$$-\sigma_2 + 3\sigma_3 \text{ bounded edges, and}$$
$$\sigma_3 \text{ vertices,}$$

where $\sigma_1, \sigma_2,$ and $\sigma_3$ are the first three elementary symmetric functions on the $s$
variables $n_1, n_2, \ldots, n_k$. (Steiner actually states just the first two of these formulas.) In particular, if the planes are in general position, then $s = n$, $n_i = 1$ for each $i$, and $\sigma_1 = n$, $\sigma_2 = \binom{n}{2}$, and $\sigma_3 = \binom{3}{3}$.

Sixty years later, Roberts [14] gave more or less explicit formulas for the number of cells, faces, and edges, bounded and unbounded, that are formed by arbitrary sets of planes in space. In the style of the day, Roberts never states completely general formulas, but relies instead on instructions that describe what is to be done in the various circumstances. He begins with Steiner's formulas for plane: in general position and subtracts from them the parts lost because of the various kinds of "degeneracies" in the arrangement. In the process, however, some parts are taken away more than once and must be replaced, so his formulas have an inclusion-exclusion character. Roberts' arguments, which are essentially recursive, are a blend of insightful intuitive geometry, unsupported contentions, and unclear language, and the entire development is a marvel of geometric intuition.

In important recent work, Zaslavsky [16] has employed the algebraic machinery of lattice theory and Möbius inversion to obtain general partition formulas for arbitrary topological dissections. The special case of arrangements of hyperplanes in Euclidean (and projective) spaces of arbitrary dimension is studied at length in Zaslavsky [17], where the counters are expressed first in terms of certain characteristic polynomials associated with the intersection structure of the set of flats, and later, more explicitly in terms of the number of incidence chains among the flats of various dimensions.

Another, more geometric method of attacking these problems in $\mathbb{E}^3$ was described in 1966 by Brousseau [4] (and rediscovered in 1976 by Freeman [7]). This elegant method, which employs an auxiliary "sweeping" plane, is closely related to an argument used in 1955 by Hadwiger [10] in studying Euler's relation. (See also the lucid proof of Euler's formula for polytopes in $\mathbb{E}^3$ given by Grünbaum [9, especially p. 135]). The sweep argument leads directly to perspicuous "additive" formulas for the counters of an arbitrary arrangement in terms of such natural structural data as the number of planes through each point and each line and information about what lines of intersection pass through which points of intersection.

It turns out that by a change of notation these additive formulas can be transformed directly into Zaslavsky's incidence-chain formulas, so the sweep method gives a direct, geometric proof of Zaslavsky's formulas in $\mathbb{E}^3$ that is independent of lattice theory. It seems likely that Zaslavsky's formulas for arrangements in $\mathbb{E}^d$ can be established inductively by an analogous sweep-hyperplane argument.\footnote{This is indeed the case. In a recently published summary article (Arrangements of hyperplanes; matroids and graphs, in Proc. Tenth Southeastern Conf. on Combinatorics, Graph Theory, and Computing (Boca Raton, 1979), Vol. II, pp. 895–911, Utilitas Math. Publ., Winnipeg, Man., 1979), Zaslavsky attributes to Curtis Greene the observation that the sweep argument in $\mathbb{E}^d$ is justified by Corollary 2.2 of Zaslavsky [17].}
In this paper we use the sweep method to establish these additive formulas, and from them we deduce a variety of other formulas and some non-trivial new bounds for the counters.

After two short sections to recall some familiar preliminary results in the plane and to set our notation, we present the additive formulas in Section 4. In Section 5 we state and prove inclusion-exclusion formulas for the counters of an arbitrary arrangement that are similar to those Roberts apparently had in mind, and, in the case in which Roberts is most explicit, we show that our formula for the bounded cells is equivalent to his. We obtain these formulas by transforming the additive formulas into inclusion-exclusion form with the help of four basic combinatorial identities for the incidence structure of the complex. The resulting formulas seem to be simpler and to have clearer structure than those implied by Roberts.

We deduce a set of related inclusion-exclusion formulas that handle parallelism in a different way in Section 6. These convenient formulas, which we call "symmetric" because they involve the elementary symmetric functions, are deduced from the earlier inclusion-exclusion formulas through an application of the general addition formula for the binomial coefficients. They are related to Steiner's formulas (1) in much the same way that the inclusion-exclusion formulas are related to the formulas for arrangements in general position.

In Section 7 we deduce new upper and lower bounds for the counters of an arbitrary arrangement, and in Section 8 we conclude with two simple examples based on the cube.

Throughout we follow an ad hoc notational style that is appropriate for three-dimensional space rather than specialize a more general notation. There clearly are analogous inclusion-exclusion and symmetric formulas for arrangements in higher-dimensional Euclidean spaces, but they will not be easy to write down.

2. Arrangements of lines in the plane

This section is devoted to the results in the plane that are needed in what follows. A more extensive study of the two-dimensional case in a somewhat more general setting appears in Alexanderson and Wetzel [1].

Suppose an arrangement (i.e., a finite set) \( \mathfrak{A} \) of \(|\mathfrak{A}| = n \) lines is given in the (Euclidean) plane. To avoid trivial special cases we assume that \( n \geq 2 \), although some of the formulas hold for \( n = 1 \) as well. For each point \( P \), let \( \lambda(P) \) be the number of lines of \( \mathfrak{A} \) that pass through \( P \). If \( \lambda(P) \geq 2 \) we say that \( P \) is \( \text{determined} \) by \( \mathfrak{A} \) and has \( \text{multiplicity} \ \lambda(P) \), and we call \( P \) a \( \text{vertex} \) of \( \mathfrak{A} \). Let \( \mathcal{V} \) be the set of vertices of \( \mathfrak{A} \), and write \( V = \#\mathcal{V} \).

The \( \text{regions} \) of \( \mathfrak{A} \) are the open, connected components of the complement of \( \bigcup(\mathfrak{A}) \). Each is an open, convex polygon. We denote by \( R' \) the number of bounded regions, by \( R'' \) the number of unbounded regions, and by \( R = R' + R'' \) the total number of regions.
The lines of $\mathcal{A}$ are cut by the other lines into pieces that we call segments, the components of $\bigcup (\mathcal{A}) - \mathcal{P}$. We denote by $S'$ the number of bounded segments, by $S''$ the number of unbounded segments (i.e., rays), and by $S = S' + S''$ the total number of segments.

An arrangement $\mathcal{A}$ is linear if its lines are all parallel (i.e., if the normal vectors to the lines of $\mathcal{A}$ do not span $\mathbb{E}^2$). For such arrangements $R' = S' = 0$, and there are as many unbounded regions and segments as there are segments and points formed on a normal cross-section line, viz., $n + 1$ and $n$ respectively. An arrangement is linear if and only if $\mathcal{P} = \emptyset$.

The counters of an arrangement are related by Euler's formulas $R - S + V = 1$ and $R' - S' + V = 1$. The first of the following formulas is a rearrangement of Euler's formula due to Brousseau [4]. The formulas all appear, in a different notation, in Zaslavsky [17, p. 61].

**Theorem 1.** For any arrangement of $n \geq 2$ lines in the Euclidean plane $\mathbb{E}^2$,

$$
R = 1 + n + \sum_{\mathcal{P}} (\lambda(P) - 1), \tag{2}
$$

$$
S = n + \sum_{\mathcal{P}} \lambda(P), \tag{3}
$$

and if the arrangement is not linear,

$$
R' = 1 - n + \sum_{\mathcal{P}} (\lambda(P) - 1), \tag{4}
$$

$$
S' = -n + \sum_{\mathcal{P}} \lambda(P), \tag{5}
$$

$$
R'' = S'' = 2n.
$$

**Proof.** Move the arrangement to the sphere by stereographic projection. Counting the point-edge incidences of the resulting map on the sphere in two ways gives (3) immediately. The map on the sphere has $v = p + 1$ vertices, $e = S$ edges, and $f = R$ faces; and Euler's formula $f - 2 - v + e$ together with (3) gives (2). A non-linear arrangement in the plane evidently has $S'' = 2n$ rays and $R'' = 2n$ unbounded regions, so (4) and (5) follow from (2) and (3).

Writing $\mathcal{P}^* = \{P \in \mathcal{P} : \lambda(P) \geq 3\}$, we can recast these formulas in the following useful forms, which we call the *simplicity* formulas.

**Corollary 2.** For any arrangement of $n \geq 2$ lines in $\mathbb{E}^2$,

$$
R = 1 + n + p + \sum_{\mathcal{P}^*} (\lambda(P) - 2),
$$

$$
S = n + 2p + \sum_{\mathcal{P}^*} (\lambda(P) - 2);
$$
and if the arrangement is not linear,

\[ R' = 1 - n + p + \sum_{P} (\lambda(P) - 2), \]
\[ S' = -n + 2p + \sum_{P} (\lambda(P) - 2). \]

The inequalities

\[ R \geq 1 + n + p, \quad R' \geq 1 - n + p, \]
\[ S \geq n + 2p, \quad S' \geq -n + 2p, \]

follow at once.

Other useful forms of these formulas appear in Alexanderson and Wetzei [1].

3. Arrangements of planes in three-space

Let \( \mathcal{A} \) be an arrangement (i.e., a finite set) of \( |\mathcal{A}| = n \) planes in Euclidean three-space \( \mathbb{E}^3 \). To avoid trivial special cases we suppose throughout that \( n \geq 3 \), although many of the formulas hold for \( n = 2 \) and \( n = 1 \). For each point \( P \) let \( \lambda(P) \) be the number of planes of \( \mathcal{A} \) that pass through \( P \). If among the \( \lambda(P) \) planes through \( P \) there are three whose intersection is exactly \( P \), we say that \( P \) is determined by \( \mathcal{A} \) and has multiplicity \( \lambda(P) \), and we call \( P \) a vertex of \( \mathcal{A} \). A vertex \( P \) is simple if \( \lambda(P) = 3 \) and multiple if \( \lambda(P) \geq 4 \). Let \( \mathcal{V} \) be the set of points determined by \( \mathcal{A} \), and let \( V = |\mathcal{V}| \). Finally, let \( \mathcal{V}^* = \{ P \in \mathcal{V}: \lambda(P) \geq 4 \} \) be the set of multiple vertices of \( \mathcal{A} \).

For each line \( L \), let \( \lambda(L) \) be the number of planes of \( \mathcal{A} \) that pass through \( L \). If \( \lambda(L) \geq 2 \) we say that \( L \) is determined by \( \mathcal{A} \) and has multiplicity \( \lambda(L) \), and we call \( L \) simple if \( \lambda(L) = 2 \) and multiple if \( \lambda(L) \geq 3 \). Let \( \mathcal{L} \) be the set of lines determined by \( \mathcal{A} \), and let \( l = |\mathcal{L}| \) be the number of such lines. Let \( \mathcal{L}^* = \{ L \in \mathcal{L}: \lambda(L) \geq 3 \} \) be the set of multiple lines of \( \mathcal{A} \). For each point \( P \) in \( \mathcal{V} \), let \( \mathcal{L}(P) = \{ L \in \mathcal{L}: P \in L \} \), and let \( I(P) = |\mathcal{L}(P)| \). Finally, let \( \mathcal{L}^*(P) = \{ L \in \mathcal{L}(P): \lambda(L) \geq 3 \} \) be the set of multiple lines that pass through \( P \).

The cells of \( \mathcal{A} \) are the open, connected components of the complement of \( \bigcup (\mathcal{A}) \). Each is an open, convex polyhedron. We denote by \( C' \) the number of bounded cells, by \( C'' \) the number of unbounded cells, and by \( C = C' + C'' \) the total number of cells. The faces of these polyhedra, which are open, convex polygons, are the faces of \( \mathcal{A} \). We denote by \( F' \) the number of bounded faces, by \( F'' \) the number of unbounded faces, and by \( F = F' + F'' \) the total number of faces. Similarly, the edges of the faces, which are open line segments and rays, are called the edges of \( \mathcal{A} \). We denote by \( E' \) the number of bounded edges, by \( E'' \) the number of unbounded edges, and by \( E = E' + E'' \) the total number of edges. To complete the notational pattern, we write \( V \) for the number of vertices of the arrangement, viz., \( p \). Note that we consistently use * to mean bounded, " to mean unbounded, and * to mean multiple.
An arrangement $\mathcal{A}$ in $\mathbb{E}^3$ is planar if there is a line to which all the planes of $\mathcal{A}$ are parallel (i.e., if the normal vectors of all the planes of $\mathcal{A}$ do not span $\mathbb{E}^3$). It is clear that such an arrangement forms as many cells, faces, and edges as there are regions, segments, and points, respectively, formed by the arrangement of lines induced on a cross-section plane; and it forms no bounded cells, faces, or edges at all. One can easily see that an arrangement is planar if and only if $\mathcal{P} = \emptyset$.

4. Additive formulas in three-space

The counters $C, C', C'', F, F', F'', E, E', E''$, and $V$ are determined by the functions $\lambda$ and the incidence structure as reflected in the sets $\mathcal{L}(P)$ and are given by the following fundamental formulas. The first is mentioned without detailed proof in Kerr and Wetzel [11], and the formulas for arrangements having no multiple lines are deduced using the sweep method in Kerr and Wetzel [12]. All nine formulas are given, in a different notation, in Zaslavsky [17, p. 61]. We agree that sums over empty sets have the value zero.

**Theorem 3.** For any arrangement of $n \geq 3$ planes in $\mathbb{E}^3$,

$$C = 1 + n + \sum_{\mathcal{L}} (1 + \lambda(L)) + \sum_{\mathcal{P}} \left[ 1 - \lambda(P) + \sum_{\mathcal{L}(P)} (1 + \lambda(L)) \right].$$  \hspace{1cm} (6)

$$F = n + \sum_{\mathcal{L}} \lambda(L) + \sum_{\mathcal{P}} \left[ -\lambda(P) + \sum_{\mathcal{L}(P)} \lambda(L) \right],$$  \hspace{1cm} (7)

$$E = \sum_{\mathcal{L}} 1 + \sum_{\mathcal{P}} \sum_{\mathcal{L}(P)} 1 = l + \sum_{\mathcal{P}} l(P);$$  \hspace{1cm} (8)

and if the arrangement is not planar,

$$C' = -1 + n - \sum_{\mathcal{L}} (-1 + \lambda(L)) + \sum_{\mathcal{P}} \left[ 1 - \lambda(P) + \sum_{\mathcal{L}(P)} (-1 + \lambda(L)) \right],$$  \hspace{1cm} (9)

$$F' = n - \sum_{\mathcal{L}} \lambda(L) + \sum_{\mathcal{P}} \left[ -\lambda(P) + \sum_{\mathcal{L}(P)} \lambda(L) \right],$$  \hspace{1cm} (10)

$$E' = \sum_{\mathcal{L}} 1 + \sum_{\mathcal{P}} \sum_{\mathcal{L}(P)} 1 = l - \sum_{\mathcal{P}} l(P);$$  \hspace{1cm} (11)

$$C'' = 2 + 2 \sum_{\mathcal{L}} (-1 + \lambda(L)),$$  \hspace{1cm} (12)

$$F'' = 2 \sum_{\mathcal{L}} \lambda(L),$$  \hspace{1cm} (13)

$$F''' = 2 \sum_{\mathcal{L}} 1 = 2!.$$  \hspace{1cm} (14)

**Proof.** Induction arguments can easily be given for the first three formulas, and other arguments are possible. We use the sweep argument because it is both short and pretty.
Let $\pi$ be a plane that is not parallel to any of the planes of $\mathcal{A}$ nor to any of the lines in which those planes meet, and suppose $\pi$ is so far away that all the points, bounded cells, bounded faces, and bounded edges formed by $\mathcal{A}$ lie on the same side. In this initial position, $\pi$ meets the planes of $\mathcal{A}$ in an arrangement of $n$ lines, and there are as many lines through a point $P$ determined by that arrangement of lines as there are planes through the line determined by $\mathcal{A}$ that meets $\pi$ in $P$. According to formulas (2) and (3), this arrangement of lines on $\pi$ has

$$1 + n + \sum_{x} (-1 + \lambda(L))$$

regions,

$$n + \sum_{x} \lambda(L)$$

segments, and

$$\sum_{x} 1 = l$$

points.

Each region on $\pi$ corresponds to (i.e., lies in) a well-defined unbounded cell of $\mathcal{A}$, each segment on $\pi$ corresponds to a well-defined unbounded face of $\mathcal{A}$, and each point on $\pi$ corresponds to a well-defined unbounded edge of $\mathcal{A}$. Consequently, the sweep-plane $\pi$ initially identifies

$$1 + n + \sum_{x} (-1 + \lambda(L))$$
cells,

$$n + \sum_{x} \lambda(L)$$
faces, and

$$\sum_{x} 1 = l$$
edges,

all of which are unbounded.

Now sweep the plane $\pi$ across the arrangement, keeping it always parallel to its initial position. New cells, faces, and edges are encountered precisely at the points determined by $\mathcal{A}$, and a moment's contemplation shows that exactly as many new cells, faces, and edges are added at $P$ as there are bounded regions, segments, and points, respectively, in the arrangement of lines formed on $\pi$ by the $\lambda(P)$ planes through $P$, viz.,

$$1 - \lambda(P) + \sum_{x \in \mathcal{A}(P)} (-1 + \lambda(L))$$
new cells,

$$-\lambda(P) + \sum_{x \in \mathcal{A}(P)} \lambda(L)$$
new faces, and

$$\sum_{x \in \mathcal{A}(P)} 1 = I(P)$$
new edges,

according to formulas (4) and (5). Summing over $\mathcal{A}$ proves formulas (6), (7), and (8).

The cells, faces, and edges initially identified by $\pi$ are all unbounded. If the arrangement $\mathcal{A}$ is not planar, then after $\pi$ has moved through all the points determined by $\mathcal{A}$ there are as many new unbounded cells, faces, and edges as
there are bounded regions, segments, and points on \( \pi \), viz.,

\[
1 - n + \sum_{x} (-1 + \lambda(L)) \ \text{new cells}
\]

\[
-n + \sum_{x} \lambda(L) \ \text{new faces, and}
\]

\[
\sum_{x} 1 = l \ \text{new edges}.
\]

Formulas (12), (13), and (14) follow at once, and (9), (10), and (11) result from subtracting off the unbounded parts.

The Euler formulas for arrangements are immediate consequences of these formulas.

**Corollary 4.** For any arrangement in \( \mathbb{E}^3 \), \( C - F + E - V = 1 \); and for any non-planar arrangement, \( C' - F' + E' - V = -1 \) and \( C'' - F'' + E'' = 2 \).

By changing the notation, we can transform these additive formulas into Zaslavsky's incidence-chain formulas. Let \( P, L, \pi \) range over \( \mathcal{P}, \mathcal{L}, \) and \( \mathcal{A} \), respectively, and write

\[
z_{01} = \left| \{(P, L) : P \subseteq L \} \right| = \sum_{\mathcal{P}} \sum_{x \in \mathcal{P}(P)} 1,
\]

\[
z_{02} = \left| \{(P, \pi) : P \in \pi \} \right| = \sum_{\mathcal{P}} \lambda(P),
\]

\[
z_{12} = \left| \{(L, \pi) : L \subseteq \pi \} \right| = \sum_{\mathcal{L}} \lambda(L),
\]

\[
z_{012} = \left| \{(P, L, \pi) : P \in L \subseteq \pi \} \right| = \sum_{\mathcal{P}} \sum_{x \in \mathcal{L}(P)} \lambda(L).
\]

Then (6) can be written

\[
C = 1 + n - l + p - z_{01} - z_{02} - z_{12} + z_{012},
\]

which is one form in which Zaslavsky presents the incidence-chain formula. The remaining eight formulas can be expressed similarly in these terms.

The formulas of Theorem 3 can be rearranged in the following useful forms, which because of their close connection with the analogous formulas for simple arrangements (see below) we call the simplicity formulas.

**Corollary 5.** For any arrangement of \( n \geq 3 \) planes in \( \mathbb{E}^3 \),

\[
C = 1 + n + l + p + \sum_{x} (-2 + \lambda(L)) + \sum_{x \in \mathcal{P}(P)} \left[ -\lambda(P) + l(P) + \sum_{x \in \mathcal{L}(P)} (-2 + \lambda(L)) \right],
\]

\[
F = n + 2l + 3p + \sum_{x} (-2 + \lambda(L))
\]

\[
+ \sum_{x \in \mathcal{P}(P)} \left[ -\lambda(P) + 2l(P) - 3 + \sum_{x \in \mathcal{L}(P)} (-2 + \lambda(L)) \right],
\]

\[
E = l + 3p + \sum_{x} [-3 + l(P)];
\]
and if the arrangement is not planar,

\[ C' = -1 + n - l + p - \sum_{x \in \mathcal{E}} (-2 + \lambda(L)) + \sum_{x \in \mathcal{E}} \left[ -\lambda(P) + l(P) + \sum_{x \in \mathcal{E}(P)} (-2 + \lambda(L)) \right], \]

\[ F' = n - 2l + 3p - \sum_{x \in \mathcal{E}} (-2 + \lambda(L)) \]

\[ + \sum_{x \in \mathcal{E}} \left[ -\lambda(P) + 2l(P) - 3 + \sum_{x \in \mathcal{E}(P)} (-2 + \lambda(L)) \right], \]

\[ E' = -l + 3p + \sum_{x \in \mathcal{E}} [-3 + l(P)]; \]

\[ C'' = 2 + 2l + 2 \sum_{x \in \mathcal{E}} (-2 + \lambda(L)), \]

\[ F'' = 4l + 2 \sum_{x \in \mathcal{E}} (-2 + \lambda(L)), \]

\[ E'' = 2l. \]

Furthermore, all the summands in these formulas are non-negative.

**Proof.** The formulas themselves are immediate, and the line summands are plainly non-negative. In the sweep proof of the additive formulas, at least one new cell, three new faces, and three new edges are acquired at each point of \( \Phi \). It follows that the point summands are also non-negative.

We call an arrangement in \( \mathbb{E}^3 \) simple if it has no multiple points and no multiple lines. Such an arrangement need not be in general position—there can still be parallel planes and parallel lines of intersection. For simple arrangements the formulas are elegant (cf. Alexanderson and Wetzel [3]).

**Corollary 6.** For a simple arrangement in \( \mathbb{E}^3 \),

\[ C = 1 + n + l + p, \]

\[ F = n + 2l + 3p, \]

\[ E = l + 3p; \]

and if the arrangement is not planar,

\[ C' = -1 + n - l + p, \quad C'' = 2 + 2l, \]

\[ F' = n - 2l + 3p, \quad F'' = 4l, \]

\[ E' = -l + 3p, \quad E'' = 2l. \]

In particular, for \( n \) planes in general position, \( l = \binom{3}{2} \) and \( p = \binom{3}{3} \).

Finally, suppose that the \( n \) planes of a simple arrangement fall into \( s \) parallel families with \( \geq 1 \) planes in the \( i \)th family (we call the \( s \)-tuple \( \langle n_1, n_2, \ldots, n_s \rangle \) the
Steiner data of the arrangement), and suppose the $s$ families are in general position, i.e., selecting one plane from each family gives an arrangement of $s$ planes that are in general position. Then evidently $n = \sigma_1$, $l = \sigma_2$, and $p = \sigma_3$, where $\sigma_1$, $\sigma_2$, $\sigma_3$ are the first three elementary symmetric functions on the variables $n_1, n_2, \ldots, n_s$; and we have Steiner's formulas (1).

5. Inclusion-exclusion formulas in three-space

We have found formulas for the counters of an arrangement of planes in general position. An arrangement can fail to be in general position in four ways: there may be multiple points; there may be multiple lines; there may be parallel planes; and there may be parallel lines of intersection. In this section we deduce inclusion-exclusion formulas that give the counters in terms of the loss from general position caused by each kind of degeneracy.

First we need to introduce notation to describe the degeneracies that involve parallels. For each direction $d$ in $\mathbb{E}^3$ let $\Pi(d)$ be the family of planes having normals in the direction $d$, and write $\lambda(d) = |\Pi(d)|$. If $\lambda(d) \geq 2$ we call $d$ a multiple direction of multiplicity $\lambda(d)$. Let $\mathcal{D}^* = \{d: \lambda(d) \geq 2\}$ be the set of multiple directions.

It is a little more complicated to describe parallel lines of intersection. A set $K = K(d)$ of all the planes in $\mathcal{A}$ whose normals are perpendicular to a given direction $d$ is called a column in the direction $d$ provided it contains at least three planes that form (at least) two distinct parallel lines of intersection. A family of parallel planes and the family of all planes through a multiple line are not columns: a column has to contain three planes two of which intersect in a line that does not lie in the third. (If $\mathbb{E}^3$ is extended by adding an ideal plane at infinity, a column is precisely the set of all planes of $\mathcal{A}$ that are concurrent in an ideal point that is determined by $\mathcal{A}$, but we prefer not to employ this language.) A column is formed whenever a family of parallels is cut by any other plane, but a column need not contain any parallel planes. One can think of a column as an essentially planar subset of the arrangement.

Let $\lambda(K) = |K| \geq 3$ be the number of planes in the column $K$. Let $\mathcal{L}^*(K)$ be the set of all multiple lines formed by the planes of the column $K$, and let $\mathcal{D}^*(K)$ be the set of all multiple directions that are perpendicular to the direction of the column $K$. Thus if $d$ is a direction in $\mathcal{D}^*(K)$ there is a family of $\lambda(d) \geq 2$ parallel planes that lie in $K$. Finally, let $\mathcal{H}$ be the set of all the columns formed by the given arrangement $\mathcal{A}$.

The transition from the additive formulas of Section 4 to the inclusion-exclusion formulas depends on four combinatorial identities, which we collect together in the following lemma. Recall the usual combinatorial convention that $(\uparrow) = 0$ when $\lambda < j$. 
Lemma 7. For any arrangement of planes in $\mathbb{E}^3$,

(a) \[ \binom{n}{2} = \sum_{\langle \mathcal{S} \rangle} \binom{\lambda(\mathcal{L})}{2} + \sum_{\langle \mathcal{S}^* \rangle} \binom{\lambda(\mathcal{D})}{2}, \]

(b) \[ l = \binom{n}{2} - \sum_{\langle \mathcal{S}^* \rangle} \left[ \binom{\lambda(\mathcal{L})}{2} - 1 \right] - \sum_{\langle \mathcal{S}^* \rangle} \binom{\lambda(\mathcal{D})}{2}, \]

(c) \[ l(P) = \binom{\lambda(P)}{2} - \sum_{\langle \mathcal{S}^*(P) \rangle} \left[ \binom{\lambda(\mathcal{L})}{2} - 1 \right], \]

(d) \[ \binom{n}{3} = \sum_{\langle \mathcal{S} \rangle} \left[ \binom{\lambda(P)}{3} - \sum_{\langle \mathcal{S}^*(P) \rangle} \binom{\lambda(\mathcal{L})}{3} \right] + \sum_{\langle \mathcal{S}^* \rangle} \binom{\lambda(\mathcal{L})}{3} \]
\[ + \sum_{\langle \mathcal{S}^* \rangle} \left[ (n - \lambda(d)) \binom{\lambda(d)}{2} + \binom{\lambda(d)}{3} \right] + \sum_{\mathcal{K}} \left( \binom{\lambda(K)}{3} - \sum_{\langle \mathcal{S}^*(\mathcal{K}) \rangle} \binom{\lambda(\mathcal{L})}{3} \right) \]
\[ - \sum_{\mathcal{S}^*(\mathcal{K})} \left[ \lambda(K) - \lambda(d) \right] \binom{\lambda(d)}{2} + \binom{\lambda(d)}{3}. \]

Proof. Two planes are either parallel or intersect in a line, and formula (a) counts the pairs of planes of $\mathfrak{A}$ accordingly. Formula (b) follows at once from (a). Formula (c) is an immediate consequence of the obvious identity

\[ \binom{\lambda(P)}{2} = \sum_{\langle \mathcal{S}^*(P) \rangle} \binom{\lambda(\mathcal{L})}{2}. \]

To prove formula (d), we show that each triple of planes of $\mathfrak{A}$ is counted exactly once by the expression on the right side of (d). There are precisely five possibilities for three different planes:

(i) they intersect in a point,
(ii) they intersect in a line,
(iii) they are parallel,
(iv) exactly two of them are parallel, and
(v) they meet by pairs in three parallel lines.

To facilitate the analysis, we rewrite the right side of (d) in the form

\[ s_1 + s_2 + s_3 + s_4 + s_5 - s_6 - s_7 - s_8 - s_9, \]

where

\[ s_1 = \sum_{\langle \mathcal{S} \rangle} \binom{\lambda(P)}{3}, \quad s_2 = \sum_{\langle \mathcal{S}^* \rangle} \binom{\lambda(\mathcal{L})}{3}, \]

\[ s_3 = \sum_{\mathcal{S}^*} \binom{\lambda(d)}{3}, \quad s_4 = \sum_{\mathcal{S}^*} (n - \lambda(d)) \binom{\lambda(d)}{2}, \]

\[ s_5 = \sum_{\mathcal{K}} \binom{\lambda(K)}{3}, \quad s_6 = \sum_{\langle \mathcal{S}^*(P) \rangle} \binom{\lambda(\mathcal{L})}{3}, \]

\[ s_7 = \sum_{\mathcal{K}} \sum_{\langle \mathcal{S}^*(\mathcal{K}) \rangle} \binom{\lambda(L)}{3}, \quad s_8 = \sum_{\mathcal{K}} \sum_{\langle \mathcal{S}^*(\mathcal{K}) \rangle} \binom{\lambda(d)}{3}, \]

\[ s_9 = \sum_{\mathcal{K}} \sum_{\langle \mathcal{S}^*(\mathcal{K}) \rangle} (\lambda(K) - \lambda(d)) \binom{\lambda(d)}{2}. \]
Now, $s_1 - s_6$ counts the triples of planes of type (i), and $s_5 - s_7 - s_8 - s_9$ counts the triples of type (v). The sums $s_1$, $s_3$, and $s_7$ count the triples of types (ii), (iii), and (iv), respectively. Each triple of planes being counted exactly once by the right side, the formula follows.

Note that the summand of $s_1 - s_6$ at each point $P$ of $\mathcal{P}$ is positive, because there is at least one triple of type (i) at $P$. Similarly the summand at each column $K$ of $s_5 - s_7 - s_8 - s_9$ is non-negative, because it counts the triples of type (v) that lie in $K$.

Inclusion-exclusion formulas now follow in a completely straightforward manner.

**Theorem 8.** For any arrangement of $n \geq 3$ planes in $\mathbb{E}^3$,

\[
C = 1 + n + \binom{n}{2} + \binom{n}{3} - \sum_{\mathcal{P}} \left[ \binom{\lambda(L)}{2} - \sum_{\mathcal{P}_i} \binom{\lambda(L)}{3} \right] \\
- \sum_{\mathcal{P}} \left[ \binom{\lambda(L)}{2} + \binom{\lambda(L)}{3} \right] - \sum_{\mathcal{P}} \left[ (n + 1 - \lambda(d)) \binom{\lambda(d)}{2} + \binom{\lambda(d)}{3} \right] \\
- \sum_{\mathcal{P}} \binom{\lambda(L)}{2} - \sum_{\mathcal{P}_K} \binom{\lambda(L)}{3} - \sum_{\mathcal{P}_P} \left[ (\lambda(K) - \lambda(d)) \binom{\lambda(d)}{2} + \binom{\lambda(d)}{3} \right] \\
F = n + 2 \binom{n}{2} + 3 \binom{n}{3} \\
- \sum_{\mathcal{P}} \left[ -\binom{\lambda(P)}{2} + 3 \binom{\lambda(P)}{3} \right] - \sum_{\mathcal{P}_i} \left[ -\binom{\lambda(L)}{2} + 2 \binom{\lambda(L)}{3} \right] \\
- \sum_{\mathcal{P}} \left[ (3n + 2 - 3\lambda(d)) \binom{\lambda(d)}{2} + 3 \binom{\lambda(d)}{3} \right] - \sum_{\mathcal{P}} \left[ \binom{\lambda(K)}{3} \right] \\
- \sum_{\mathcal{P}_K} \binom{\lambda(L)}{3} - \sum_{\mathcal{P}_P} \left[ (\lambda(K) - \lambda(d)) \binom{\lambda(d)}{2} + \binom{\lambda(d)}{3} \right] \\
E = \binom{n}{2} + 3 \binom{n}{3} \\
- \sum_{\mathcal{P}} \left[ -\binom{\lambda(P)}{2} + 3 \binom{\lambda(P)}{3} \right] - \sum_{\mathcal{P}_i} \left[ 1 - \binom{\lambda(L)}{2} + 3 \binom{\lambda(L)}{3} \right] \\
- \sum_{\mathcal{P}} \left[ -1 + \binom{\lambda(L)}{2} + 3 \binom{\lambda(L)}{3} \right] \\
- \sum_{\mathcal{P}} \left[ (3n + 1 - 3\lambda(d)) \binom{\lambda(d)}{2} + 3 \binom{\lambda(d)}{3} \right] - \sum_{\mathcal{P}} \left[ \binom{\lambda(K)}{3} \right] \\
- \sum_{\mathcal{P}_K} \binom{\lambda(L)}{3} - \sum_{\mathcal{P}_P} \left[ (\lambda(K) - \lambda(d)) \binom{\lambda(d)}{2} + \binom{\lambda(d)}{3} \right] \\
\]
\[ \begin{align*}
V &= \binom{n}{3} - \sum_{\Xi} \left\{ -1 + \binom{\lambda(P)}{3} - \sum_{\Xi^*(P)} \binom{\lambda(L)}{3} \right\} - \sum_{\Xi} \binom{\lambda(L)}{3} \\
&\quad - \sum_{\Xi} \left[ (n-\lambda(d)) \binom{\lambda(d)}{2} + \binom{\lambda(d)}{3} \right] \\
&\quad - \sum_{\Xi} \left\{ \binom{\lambda(K)}{3} - \sum_{\Xi^*(K)} \binom{\lambda(L)}{3} - \sum_{\Xi^*(K)} \left[ (\lambda(K) - \lambda(d)) \binom{\lambda(d)}{2} + \binom{\lambda(d)}{3} \right] \right\}; \\
\end{align*} \]

and if the arrangement is not planar,

\[ \begin{align*}
C' &= -1 + n - \binom{n}{2} + \binom{n}{3} - \sum_{\Xi} \left\{ \binom{\lambda(P)-1}{3} - \sum_{\Xi^*(P)} \binom{\lambda(L)-1}{3} \right\} \\
&\quad - \sum_{\Xi} \left[ \binom{\lambda(L)-1}{3} - \sum_{\Xi^*(L)} \binom{\lambda(d)}{3} \right] \\
&\quad - \sum_{\Xi} \left\{ \binom{\lambda(K)}{3} - \sum_{\Xi^*(K)} \binom{\lambda(L)}{3} - \sum_{\Xi^*(K)} \left[ (\lambda(K) - \lambda(d)) \binom{\lambda(d)}{2} + \binom{\lambda(d)}{3} \right] \right\}.
\end{align*} \]

\[ \begin{align*}
F' &= n - 2 \binom{n}{2} + 3 \binom{n}{3} - \sum_{\Xi} \left\{ \lambda(P) - 2 \binom{\lambda(P)}{2} + 3 \binom{\lambda(P)}{3} \right\} \\
&\quad - \sum_{\Xi^*(P)} \left[ \lambda(L) - 2 \binom{\lambda(L)}{2} + 3 \binom{\lambda(L)}{3} \right] \\
&\quad - \sum_{\Xi^*(L)} \left[ (3n - 2 - 3\lambda(d)) \binom{\lambda(d)}{2} + 3 \binom{\lambda(d)}{3} \right] - \sum_{\Xi} 3 \left\{ \binom{\lambda(K)}{3} \right\} \\
&\quad - \sum_{\Xi^*(K)} \left[ \binom{\lambda(L)}{3} - \sum_{\Xi^*(K)} \left[ (\lambda(K) - \lambda(d)) \binom{\lambda(d)}{2} + \binom{\lambda(d)}{3} \right] \right\};
\end{align*} \]

\[ \begin{align*}
E' &= - \binom{n}{2} + 3 \binom{n}{3} - \sum_{\Xi} \left\{ - \binom{\lambda(P)}{2} + 3 \binom{\lambda(P)}{3} \right\} \\
&\quad - \sum_{\Xi^*(P)} \left[ 1 - \binom{\lambda(L)}{2} + 3 \binom{\lambda(L)}{3} \right] - \sum_{\Xi} \left\{ 1 - \binom{\lambda(L)}{2} + 3 \binom{\lambda(L)}{3} \right\} \\
&\quad - \sum_{\Xi} \left[ (3n - 1 - 3\lambda(d)) \binom{\lambda(d)}{2} + 3 \binom{\lambda(d)}{3} \right] - \sum_{\Xi} 3 \left\{ \binom{\lambda(K)}{3} \right\} \\
&\quad - \sum_{\Xi^*(K)} \left[ \binom{\lambda(L)}{3} - \sum_{\Xi^*(K)} \left[ (\lambda(K) - \lambda(d)) \binom{\lambda(d)}{2} + \binom{\lambda(d)}{3} \right] \right\};
\end{align*} \]

\[ \begin{align*}
C'' &= 2 + 2 \binom{n}{2} - 2 \sum_{\Xi^*} \binom{\lambda(L)-1}{2} - 2 \sum_{\Xi^*} \binom{\lambda(d)}{2}, \\
F'' &= 4 \binom{n}{2} - 2 \sum_{\Xi^*} \left[ -\lambda(L) + 2 \binom{\lambda(L)}{2} \right] - \sum_{\Xi^*} \binom{\lambda(d)}{2}, \\
E'' &= 2 \binom{n}{2} - 2 \sum_{\Xi^*} \left[ -1 + \binom{\lambda(L)}{2} - 2 \sum_{\Xi^*} \binom{\lambda(d)}{2} \right].
\end{align*} \]
Proof. The formula for \( V \) follows at once from identity (d) of the lemma. The formula for \( C \) results from substituting identities (b) and (c) of the lemma and the formula for \( V \) into the simplicity formula for \( C \) and using the identities
\[
\binom{\lambda}{2} - \lambda + 1 = \binom{\lambda - 1}{2},
\]
\[
\binom{\lambda}{3} - \binom{\lambda}{2} + \lambda - 1 = \binom{\lambda - 1}{3}.
\]
(16)
The remaining formulas follow similarly from the simplicity formulas.

It is an instructive exercise to use the sweep argument to derive these formulas directly. Once again, the identities of the lemma play a key role.

Roberts \cite[pp. 418, 419, 421]{Roberts} gives the number of bounded cells formed by an arbitrary arrangement of \( n \geq 3 \) planes by rules that in our notation can be expressed as follows:

\[
C_R = \binom{n-1}{3} - \sum_{x^*} \left\{ \binom{\lambda(P)}{3} - \sum_{x^* \neq P} \left[ (\lambda(P) - 1) \binom{\lambda(L)-1}{2} - 2 \binom{\lambda(L)}{3} \right] \right\}
\]
\[
- \sum_{x^*} \left[ (n-1) \binom{\lambda(L)-1}{2} - 2 \binom{\lambda(L)}{3} \right]
\]
\[
- \sum_{x^*} \left[ (\lambda(d) - 1) \binom{\lambda(d)}{3} - 2 \binom{\lambda(d)+1}{3} \right]
\]
\[
- \sum_{x^*} \left[ \binom{\lambda(K)}{3} - \sum_{x^* \neq K} \left[ \lambda(K) \binom{\lambda(L)-1}{2} - 2 \binom{\lambda(L)}{3} \right] \right]
\]
\[
- \sum_{x^* \neq K} \left[ \lambda(K) \binom{\lambda(d)}{3} - 2 \binom{\lambda(d)+1}{3} \right].
\]
(17)

With the help of (16) and the identity
\[
x\binom{y}{2} - \binom{y+1}{3} = (x-y)\binom{y}{2} + \binom{y}{3},
\]
the difference \( C' - C_R \) between formula (15) and Roberts' formula can be brought to the form

\[
C' - C_R = \sum_{x^*} \sum_{x^* \neq P} (\lambda(L) - \lambda(P)) \binom{\lambda(L)-1}{2} + \sum_{x^*} (n - \lambda(L)) \binom{\lambda(L)-1}{2}
\]
\[
- \sum_{x^*} \sum_{x^* \neq K} (\lambda(K) - \lambda(L)) \binom{\lambda(L)-1}{2}.
\]

A multiple line \( l \) lies in at most one column \( K_L \); and if there is no such column, write \( \lambda(K_L) = \lambda(L) \). Then, reversing the order of summation in the first and last
s. we find
\[ C' \cap C_R = \sum_{2} \left( \frac{\lambda(L)-1}{2} \right) \left[ n - \lambda(K_L) - \sum_{p \in L} (\lambda(P) - \lambda(L)) \right]. \]

The expression in brackets vanishes because both \( n - \lambda(K_L) \) and \( \sum_{p \in L} (\lambda(P) - \lambda(L)) \) count the planes of the arrangement that cross \( L \).

This argument gives a tight proof of Roberts' formula (17) for the bounded cells.

6. Symmetric formulas in three-space

In this section we use an elementary combinatorial identity to rewrite the inclusion-exclusion formulas of Section 5 in a simpler and more convenient form.

Given \( t \geq 1 \) variables \( x_1, x_2, \ldots, x_t \), let \( \sigma_i \) be the \( j \)th elementary symmetric function on the \( x_i \)'s, the sum of the \( \binom{j}{t} \) products of the \( x_i \)'s taken \( j \) at a time. We follow the usual combinatorial conventions that \( \sigma_j = \binom{j}{t} = 0 \) when \( j > t \). Recall the general addition formula for the binomial coefficients:

\[
\binom{\sigma_1}{d} = \sum \binom{x_1}{z_1} \binom{x_2}{z_2} \cdots \binom{x_t}{z_t},
\]

where the sum is over all ordered \( t \)-tuples \( (z_1, z_2, \ldots, z_t) \) of non-negative integers whose sum is \( d \). This formula is immediate on combinatorial grounds: the \( d \)-element subsets of a \( \sigma_1 \)-set partitioned into \( t \) subsets having \( x_1, x_2, \ldots, x_t \) elements respectively are obtained by selecting \( z_1 \) of the \( x_1 \), \( z_2 \) of the \( x_2 \), etc.

Since the sum over those ordered \( t \)-tuples \( (z_1, z_2, \ldots, z_t) \) whose entries are only zeros and ones is precisely \( \sigma_d \), the addition formula can be rewritten in the following form:

\[
\binom{\sigma_1}{d} = \sigma_d + \sum \binom{x_1}{z_1} \binom{x_2}{z_2} \cdots \binom{x_t}{z_t},
\]

where the sum is over all ordered \( t \)-tuples \( (z_1, z_2, \ldots, z_t) \) of non-negative integers whose sum is \( d \) except those whose entries are all zeros and ones. In particular, for \( d = 2 \) and \( d = 3 \) we find

\[
\binom{\sigma_1}{2} = \sigma_2 + \sum_{i=1}^{t} \binom{x_i}{2},
\]

\[
\binom{\sigma_1}{3} = \sigma_3 + \sum_{i=1}^{t} \left[ (\sigma_1 - x_i) \binom{x_i}{2} + \binom{x_i}{3} \right].
\]

It follows that for any coefficients \( \theta_0, \theta_1, \theta_2, \theta_3 \),

\[
\theta_0 + \theta_1 \sigma_1 + \theta_2 \binom{\sigma_1}{2} + \theta_3 \binom{\sigma_1}{3}
\]  

\[
= \theta_0 + \theta_1 \sigma_1 + \theta_2 \sigma_2 + \theta_3 \sigma_3 + \sum_{i=1}^{t} \left[ (\theta_3 \sigma_1 + \theta_2 - \theta_3 x_i) \binom{x_i}{2} + \theta_0 \binom{x_i}{3} \right].
\]

We shall need this identity shortly.
Suppose the $n$ planes in an arrangement $\mathcal{A}$ fall into $s$ parallel families $\Pi_1, \Pi_2, \ldots, \Pi_s$ having $n_1, n_2, \ldots, n_s$ planes respectively, where each $n_i \geq 1$. We call the $s$-tuple $\langle n_1, n_2, \ldots, n_s \rangle$ the Steiner data of the arrangement. For each column $K$ let $I(K)$ be the set of indices $i$ so that the parallel family $\Pi_i$ lies in $K$, and let $s(K) = |I(K)|$. It is convenient to call the $s(K)$-tuple $\langle n_i : i \in I(K) \rangle$ the Steiner data of the column $K$. Let $\sigma_i(K)$ be the $i$th elementary symmetric function on the $s(K)$ integers $n_i$ for $i$ in $I(K)$. In this notation, the column summand in the inclusion-exclusion formulas becomes

\[
\left(\frac{\sigma_1(K)}{3}\right)^3 - \sum_{\mathcal{L} \in \mathcal{I}(K)} \left(\frac{\lambda(\mathcal{L})}{3}\right)^3 - \sum_{i \in I(K)} \left(\sigma_1(K) - n_i\right) \left(\frac{n_i}{2}\right) + \left(\frac{n_i}{3}\right) = \sigma_3(K) - \sum_{\mathcal{L} \in \mathcal{I}(K)} \left(\frac{\lambda(\mathcal{L})}{3}\right)^3,
\]

according to identity (18). Consequently, we have the following useful formulas, which we call the symmetric formulas.

**Theorem 9.** For any arrangement of $n \geq 3$ planes in $\mathbb{E}^3$ having Steiner data $\langle n_1, n_2, \ldots, n_s \rangle$,

\[
C = 1 + \sigma_1 + \sigma_2 + \sigma_3 - \sum_{\mathcal{L} \in \mathcal{I}(K)} \left[\frac{(\lambda(\mathcal{P}) - 1)}{3} - \sum_{\mathcal{L} \in \mathcal{I}(K)} \left(\frac{\lambda(\mathcal{L}) - 1}{3}\right)\right]
\]

\[
- \sum_{\mathcal{L} \in \mathcal{I}(K)} \left[\frac{(\lambda(\mathcal{L}) - 1)}{3} + \left(\frac{\lambda(\mathcal{L})}{3}\right)^3\right] - \sum_{\mathcal{L} \in \mathcal{I}(K)} \left[\sigma_3(K) - \sum_{\mathcal{L} \in \mathcal{I}(K)} \left(\frac{\lambda(\mathcal{L})}{3}\right)^3\right],
\]

\[
F = \sigma_1 + 2\sigma_2 + 3\sigma_3
\]

\[
- \sum_{\mathcal{L} \in \mathcal{I}(K)} \left[\lambda(\mathcal{P}) - 2\left(\frac{\lambda(\mathcal{P})}{2}\right) + 3\left(\frac{\lambda(\mathcal{P})}{3}\right) - \sum_{\mathcal{L} \in \mathcal{I}(K)} \left(\frac{\lambda(\mathcal{L}) - 1}{2}\right) + 3\left(\frac{\lambda(\mathcal{L})}{3}\right)\right]
\]

\[
- \sum_{\mathcal{L} \in \mathcal{I}(K)} \left[-\lambda(\mathcal{L}) + 2\left(\frac{\lambda(\mathcal{L})}{2}\right) + 3\left(\frac{\lambda(\mathcal{L})}{3}\right)\right] - \sum_{\mathcal{L} \in \mathcal{I}(K)} \left[\sigma_3(K) - \sum_{\mathcal{L} \in \mathcal{I}(K)} \left(\frac{\lambda(\mathcal{L})}{3}\right)^3\right],
\]

\[
E = \sigma_1 + 2\sigma_2 + 3\sigma_3 - \sum_{\mathcal{L} \in \mathcal{I}(K)} \left[\lambda(\mathcal{P}) - 2\left(\frac{\lambda(\mathcal{P})}{2}\right) + 3\left(\frac{\lambda(\mathcal{P})}{3}\right) - \sum_{\mathcal{L} \in \mathcal{I}(K)} \left(\frac{\lambda(\mathcal{L}) - 1}{2}\right) + 3\left(\frac{\lambda(\mathcal{L})}{3}\right)\right]
\]

\[
- \sum_{\mathcal{L} \in \mathcal{I}(K)} \left[-1 + \left(\frac{\lambda(\mathcal{L})}{2}\right) + 3\left(\frac{\lambda(\mathcal{L})}{3}\right)\right] - \sum_{\mathcal{L} \in \mathcal{I}(K)} \left[\sigma_3(K) - \sum_{\mathcal{L} \in \mathcal{I}(K)} \left(\frac{\lambda(\mathcal{L})}{3}\right)^3\right],
\]

\[
V = \sigma_1 - \sum_{\mathcal{L} \in \mathcal{I}(K)} \left[-1 + \left(\frac{\lambda(\mathcal{P})}{3}\right) - \sum_{\mathcal{L} \in \mathcal{I}(K)} \left(\frac{\lambda(\mathcal{L})}{3}\right)\right]
\]

\[
- \sum_{\mathcal{L} \in \mathcal{I}(K)} \left(\frac{\lambda(\mathcal{L})}{3}\right) - \sum_{\mathcal{L} \in \mathcal{I}(K)} \left[\sigma_3(K) - \sum_{\mathcal{L} \in \mathcal{I}(K)} \left(\frac{\lambda(\mathcal{L})}{3}\right)^3\right].
\]
Proof. In each case these follow from the corresponding inclusion–exclusion formula via the appropriate instance of identity (19).

The symmetric formulas are related to Steiner's formulas (1) for families of parallels in general position in the same way that the inclusion–exclusion formulas are related to the formulas for planes in general position: decrements caused by the various degeneracies are subtracted off. We shall see in the next section that a13 of these decrements are non-negative.

It is worth noting that the summands at each (finite) multiple point P can be expressed in terms of symmetric functions of the \( l(P) \) variables \( \lambda(L) \) for \( L \) in \( \mathcal{F}(P) \) in precisely the same manner as was the column summand. The resulting formulas seem more complicated and less useful than the symmetric formulas given above, and we omit them.

7. Bounds

It is well known that for given \( n \), the counters \( C, C', F, F', E, E', \) and \( V \) are maximized by arrangements of \( n \) planes in general position. In this section we
deduce lower bounds and better upper bounds from the simplicity and symmetric formulas. The authors would like to acknowledge helpful conversations with George Purdy on the material of this section.

**Theorem 10.** For any arrangement \( \mathfrak{A} \) in \( \mathbb{E}^3 \),

\[
1 + n + l + p \leq C \leq 1 + \sigma_1 + \sigma_2 + \sigma_3,
\]

\[
n + 2l + 3p \leq F \leq \sigma_1 + 2\sigma_2 + 3\sigma_3,
\]

\[
l + 3p \leq E \leq \sigma_2 + 3\sigma_3,
\]

\[
p = V \leq \sigma_3;
\]

and if the arrangement is not planar,

\[
-1 + n - l + p \leq C' \leq -1 + \sigma_1 - \sigma_2 + \sigma_3,
\]

\[
n - 2l + 3p \leq F' \leq \sigma_1 - 2\sigma_2 + 3\sigma_3,
\]

\[
-l + 3p \leq E' \leq -\sigma_1 + 3\sigma_3;
\]

\[
2 + 2l \leq C'' \leq 2 + 2\sigma_2,
\]

\[
4l \leq F'' \leq 4\sigma_2,
\]

\[
2l - E'' \leq 2\sigma_2.
\]

**Proof.** The lower bounds for all the counters except \( C' \) and \( F' \) follow trivially from the simplicity formulas of Corollary 5, since the multiple point and multiple line summands are non-negative. To establish the lower bound for \( C' \), we reverse the double sum in the simplicity formula for \( C' \) and write

\[
C' = -1 + n - l + p + \sum_{P \neq L} (l(P) - \lambda(P)) + \sum_{P \neq L} (V_L - 1)(\lambda(L) - 2),
\]

(20)

where \( V_L = |\mathfrak{P} \cap L| \) is the number of points determined by \( \mathfrak{A} \) that lie on \( L \). Each line \( L \) is crossed by at least one plane since \( \mathfrak{A} \) is not planar, and so the second sum is non-negative. The set of all planes through \( P \) forms a projective plane \( \mathbb{P}^2 \), and the \( \lambda(P) \) lines in that projective plane formed by the planes of \( \mathfrak{A} \) meet to determine \( l(P) \) points. It is well known that the lines of a non-trivial projective arrangement intersect in at least as many points as there are lines (see Grünbaum [8, pp. 11, 12]). Consequently \( l(P) \geq \lambda(P) \), and the first sum in (20) is also non-negative.

A similar argument establishes the lower bound for \( F' \), because \(-\lambda(P) + 2l(P) - 3 = (l(P) - \lambda(P)) + (l(P) - 3) \geq 0.\)

The upper bounds follow from the symmetric formulas once the non-negativity of all the subtractive terms is established. The line summand in each formula is obviously non-negative. The non-negativity of the column summand was noted earlier, immediately following the proof of Lemma 7, and at the same time the non-negativity of the point summand in the formula for \( V \) was observed.
The non-negativity of the remaining point summands follows from an unpublished combinatorial lemma due to Purdy, which we state and prove in a form suitable for our purpose. (Some related inequalities can be found in Erdös and Purdy [6] and in Purdy [13].) Temporarily we use \( t(Q) \) to denote the number of lines of an arrangement in \( \mathbb{P}^2 \) that pass through a vertex \( Q \).

**Lemma 11.** For any non-trivial arrangement of \( m \) lines in the projective plane \( \mathbb{P}^2 \),

\[
\binom{m-1}{3} \geq \sum_Q \binom{t(Q)}{3}.
\]

**Proof.** According to Sylvester’s problem (see Grünbaum [8, p. 16]), there is a simple point in the arrangement. Suppose that such a point is the intersection of lines \( L_1 \) and \( L_2 \). Then there are \( m - 2 \) triples of non-concurrent lines that involve both \( L_1 \) and \( L_2 \), and at least \( \binom{m-2}{2} \) such triples that involve \( L_1 \) or \( L_2 \) (but not both). Since there are

\[
\binom{m}{3} - \sum_Q \binom{t(Q)}{3}
\]

non-concurrent triples in all, it must be so that

\[
\binom{m}{3} - \sum_Q \binom{t(Q)}{3} \geq (m-2) + \binom{m-2}{2} = \binom{m-1}{2},
\]

and the lemma follows.

We return to the proof of the upper bounds. Applying the lemma to the arrangement of \( \lambda(P) \) lines formed in \( \mathbb{P}^2 \) by the planes of \( \mathbb{P} \) through \( P \), we see that

\[
\binom{\lambda(P) - 1}{3} - \sum_{\mathcal{L}(P)} \binom{\lambda(L) - 1}{3} \geq \binom{\lambda(P) - 1}{3} - \sum_{\mathcal{L}(P)} \binom{\lambda(L)}{3} \geq 0;
\]

so the point summand in the symmetric formulas for \( \lambda(P) \) and \( \lambda(P)' \) is non-negative.

The point summand in the symmetric formulas for \( F \) and \( F' \) can be rewritten in the form

\[
\lambda(P) - 2\binom{\lambda(P)}{2} + 3\binom{\lambda(P) - 1}{2} + 3\left[ \left( \binom{\lambda(P)}{3} \right) - \sum_{\mathcal{L}(P)} \binom{\lambda(L)}{3} \right] + \sum_{\mathcal{L}(P)} \left[ 2\binom{\lambda(L)}{2} - \lambda(L) \right],
\]

and by the lemma this is at least

\[
\lambda(P) - 2\binom{\lambda(P)}{2} + 3\binom{\lambda(P) - 1}{2} \geq 0.
\]

Consequently the point summand in the symmetric formulas for \( F \) and \( F' \) is non-negative.
Similarly, the point summand in the symmetric formulas for \( E \) and \( E' \) can be rewritten in the form

\[
- \binom{\lambda(P)}{2} + 3 \binom{\lambda(P)-1}{2} + 3 \left[ \binom{\lambda(P)-1}{3} - \sum_{\mathcal{P}(P)} \binom{\lambda(L)}{3} \right]
\]

\[
+ \sum_{\mathcal{P}(P)} \left[ \binom{\lambda(L)}{2} - 1 \right],
\]

and by the lemma this is at least

\[
- \binom{\lambda(P)}{2} + 3 \binom{\lambda(P)-1}{2} = (\lambda(P) - 1)(\lambda(P) - 3) \geq 0.
\]

Consequently the point summand in the symmetric formulas for \( E \) and \( E' \) is also non-negative.

This completes the proof.

A straightforward examination of the formulas shows that the equalities hold as follows. For \( C, C', F, F', E, E', \) and \( V \) the upper equalities hold precisely for those simple arrangements whose parallel families are in general position.

The upper equality for \( E'' \) and both the upper and lower equalities for \( C'' \) and \( F'' \) hold precisely for non-planar arrangements having no multiple lines.

The lower equalities for \( C \) and \( F \) hold precisely for simple arrangements, and the lower equality for \( E \) holds for simple arrangements and for a single multiple line of multiplicity \( n \). The lower equalities for \( F' \) and \( E' \) hold precisely for non-planar simple arrangements, and the lower equality for \( C' \) holds for non-planar simple arrangements and also for arrangements formed by crossing a planar arrangement of \( n-1 \) planes by a single plane.

It is interesting to note that arrangements exist for which the lower bounds for \( C', F', \) and \( E' \) are negative. Consider, for example, the arrangement formed when a multiple line of multiplicity \( m \geq 3 \) is crossed by \( k \geq 2 \) parallel planes. This arrangement has \( n = m + k, l = mk + 1, \) and \( p = k \); and

\[
-1 + n - l + p = (m-2)(1-k) < 0 = C',
\]

\[
n - 2l + 3p = (m-2)(1-2k) < 0 = F'.
\]

\[
-l + 3p = k(3-m) - 1 < 0 < k - 1 = E'.
\]

Finally, we note that since the multiple point and multiple line decrements in the inclusion–exclusion formulas are the same as those in the symmetric formulas and hence are non-negative, and since the parallel family and column decrements are clearly non-negative, we have an algebraic proof that \( C, C', C'', F, F', F'', E, E', E'', \) and \( V \) are maximized for given \( n \) by arrangements of \( n \) planes in general position.
In this concluding section we briefly investigate two arrangements based on the cube. First we examine the cuboctahedral arrangement, which is obtained by extending the 14 face planes of a cuboctahedron, the polyhedron formed by truncating the vertices of a cube by planes through the midpoints of its edges. Then as a somewhat more complicated example, we study the complete cubical arrangement, comprised of the 20 planes determined by the vertices of a cube. We content ourselves with a brief description of the relevant structure of each arrangement.

The 14 planes of the cuboctahedral arrangement fall into seven parallel pairs, so the arrangement has Steiner data $\langle 2, 2, 2, 2, 2, 2, 2 \rangle$, and $\sigma_2 = 84$ and $\sigma_3 = 280$. By convexity, there are no multiple lines. The planes form three columns with Steiner data $\langle 2, 2 \rangle$ and six columns with Steiner data $\langle 2, 2, 2 \rangle$. There are 42 multiple points, all of multiplicity four: twelve are the vertices of the cuboctahedron, six are the vertices of the octahedron formed by extending the triangular faces, and two are formed on each extended edge of the cube by two opposite extended triangular faces. Using the symmetric formulas, we find the following values:

$$
\begin{align*}
n &= 14, & C &= 289, & C' &= 119, & C'' &= 170, \\
l &= 84, & F &= 710, & F' &= 374, & F'' &= 336, \\
V &= 106, & E &= 528, & E' &= 360, & E'' &= 168.
\end{align*}
$$

The 20 planes of the complete cubical arrangement fall into seven parallel pairs and six singletons, so the arrangement has Steiner data $\langle 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1 \rangle$, and $\sigma_2 = 183$ and $\sigma_3 = 1014$. There are 28 multiple lines: the 12 edges and four space diagonals of the cube are multiple lines of multiplicity three, and the 12 diagonals of the faces of the cube are multiple lines of multiplicity four. The planes form three columns with Steiner data $\langle 2, 2, 1, 1 \rangle$, each of which contains four multiple lines of multiplicity three. Each vertex of the cube is a multiple point of multiplicity nine, the centroid of the cube is a multiple point of multiplicity six, and the centroid of each face is a multiple point of multiplicity seven. There are two further multiple points of multiplicity four on each extended edge of the cube, formed by the slanting planes through the opposite vertices. Using Lemma 7(b) and the symmetric formulas, we find the following values:

$$
\begin{align*}
n &= 20, & C &= 512, & C' &= 248, & C'' &= 264, \\
l &= 91, & F &= 1100, & F' &= 656, & F'' &= 444, \\
V &= 171, & E &= 760, & E' &= 578, & E'' &= 182.
\end{align*}
$$

Some examples using the additive formulas appear in Kerr and Wetzel [12], and an example using Roberts' formula is given in Alexanderson and Wetzel [2].
References