Numerical Approximation of a One-Dimensional Space Fractional Advection-Dispersion Equation with Boundary Layer

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Abstract

Finite element computations for singularly perturbed convection-diffusion equations have long been an attractive theme for numerical analysis. In this article, we consider the singularly perturbed fractional advection-dispersion equation (FADE) with boundary layer behavior. We derive a theoretical estimate which shows that the under-resolved case corresponds to \( \epsilon < h^{\alpha-1} \), where \( \alpha \) is the order of the diffusion operator. We also present a numerical method for solving such an FADE in which the boundary layer is incorporated into the finite element basis, and provide numerical experiments which show that knowledge of the boundary layer (either analytically or through a direct numerical simulation) can be used to greatly enhance the efficiency of the finite element method.

Key words: Fractional diffusion, fractional advection-dispersion equation, fractional derivatives, finite element method, boundary layers.

1 Introduction

In this article, we present analysis and computational experiments for the singularly perturbed fractional advection-dispersion equation in one spatial dimension:

\[
-\epsilon D(p_a D_x^{\alpha-2} + q_x D_b^{\alpha-2})Du - u_x = f, \quad \text{in } \Omega,
\]

\[
u = 0, \quad \text{on } \partial \Omega.
\]
where $\Omega$ is the real interval $(a, b)$, $1 < \alpha \leq 2$ is the order of the fractional dispersion operator, with skewness parameters defined by $p, q$ satisfying $p+q = 1$, and $\epsilon \ll 1$.

In short, the FADE results from considering diffusion processes in which the jump phenomena are governed by stochastic processes which converge (in distribution) to stable laws of non-Gaussian type. In this case, the diffusion is said to follow a Lévy process. Lévy processes have two distinct characteristics. First, a Lévy distribution is a “heavy-tailed” distribution, which means it possesses infinite variance. Second, a Lévy distribution may be skewed rather than symmetric. The FADE can be derived using a probabilistic argument or can be said to result from a fractional Fick’s law [1–3].

The main motivating application of the FADE is in porous media flow. Many authors have noted that data collected at the MADE site [4] follow the form of an FADE more closely than alternative, integer-order models [5,6]. Another very notable application of the FADE is in modeling magnetically confined turbulent plasmas [7,8]. Many authors have proposed alternative physical models for many phenomena which involve fractional derivatives instead of their integer-order counterparts. For a review of several of these, see [9].

Numerical solutions of (1) can be found in a variety of ways. Initially, in [5,6,10], the Grünwald formula was applied to obtain finite difference approximations. However, in the works [11–16], a variational approach to the FADE was undertaken in which a mathematical theory has been developed in order to extend variational methods and more specifically Galerkin finite element methods to the FADE.

However, several real-world flow phenomena are convection-dominated. Such flows may involve a number of sharp transition regions for which numerical approximations are not very easily obtained. One example in the literature where a fractional dispersion operator appears in a partial differential equation which models convection-dominated flow is in the study of the fractional version of Burgers equation [17]. There are numerous techniques in which the finite element method may be enhanced in order to better capture the true nature of the flow. Two examples of popular methods used to resolve the transition or boundary layers are the streamline-diffusion finite element method [18] and a posteriori error estimation [19].

Friedrichs [20] was the first to study the mathematical nature of boundary layers, and he showed that the singularly perturbed convection diffusion problem

$$-\epsilon u_{xx} - u_x = 1; \quad u(0) = 0; \quad u(1) = 0,$$

possesses a boundary layer on the order of $e^{-x/\epsilon}$. In this paper, we wish to study the nature of the boundary layers for the FADE (1) and employ a method as
presented in [21–23], in which the usual finite element basis is augmented with the boundary layer function in order to improve the computational results.

The article is outlined as follows. In Section 2, we present notation and preliminaries, in which we review the definitions of the Riemann-Liouville fractional differential and integral operators, the FADE, and the theory behind the variational solution to the FADE. In Section 3, we present the main error estimate, which indicates that the singularly perturbed FADE (1) is under-resolved if \( \epsilon < h^{\alpha-1} \). In Section 4, we outline a computational procedure by which the usual finite element basis is augmented with a boundary-layer basis function, and give examples of particular boundary-layers found in FADEs. In Section 5, we present some computational experiments which show that the method outlined in Section 4 yields approximate solutions which are far superior to those which are found in usual finite element simulations in porous media flows with boundary layer behavior.

2 Preliminaries

In this section, we present preliminary information which is essential to developing the variational solution of the FADE and the piecewise polynomial finite element approximations thereof. For the complete analysis of the variational solution of the FADE in one spatial dimension, see [12].

Fractional derivatives have a mathematical history which is as old and celebrated as their integer-order counterparts. Following are the definitions of the left and right Riemann-Liouville fractional integral and differential operators. Note that by \( D \) we mean the usual differential operator. For a more complete discussion of the mathematical properties of fractional derivatives, see [9,24,25].

Definition 1 [Left Riemann-Liouville Fractional Integral]. Let \( u \) be defined on the interval \((a,b)\) and \( \sigma > 0 \). Then the left Riemann-Liouville fractional integral of order \( \sigma \) is defined to be

\[
aD_x^{-\sigma}(u(x)) := \int_a^x \frac{(x-\xi)^{\sigma-1}}{\Gamma(\sigma)} u(\xi) d\xi.
\]  

(2)

Definition 2 [Left Riemann-Liouville Fractional Derivative]. Let \( u \) be defined on the interval \((a,b)\), \( \mu > 0 \), \( n \) be the smallest integer greater than \( \mu \) \((n - 1 \leq \mu < n)\), and \( \sigma = n - \mu \). Then the left Riemann-Liouville fractional
derivative of order \( \mu \) is defined to be
\[
a D^\alpha_x u(x) := D^n a D_x^{-\sigma} u(x) = \frac{d^n}{dx^n} \left( \int_a^x \frac{(x - \xi)^{\sigma-1}}{\Gamma(\sigma)} u(\xi) d\xi \right).
\] (3)

**Definition 3 [Right Riemann-Liouville Fractional Integral].** Let \( u \) be defined on the interval \((a, b)\) and \( \sigma > 0 \). Then the right Riemann-Liouville fractional integral of order \( \sigma \) is defined to be
\[
x D_b^{-\sigma} (u(x)) := \int_x^b \frac{(\xi - x)^{\sigma-1}}{\Gamma(\sigma)} u(\xi) d\xi.
\] (4)

**Definition 4 [Right Riemann-Liouville Fractional Derivative].** Let \( u \) be defined on the interval \((a, b)\), \( \mu > 0 \), \( n \) be the smallest integer greater than \( \mu \) \((n-1 \leq \mu < n)\), and \( \sigma = n - \mu \). Then the right Riemann-Liouville fractional derivative of order \( \mu \) is defined to be
\[
x D_b^\mu u(x) := (-1)^n \frac{d^n}{dx^n} \left( \int_x^b \frac{(\xi - x)^{\sigma-1}}{\Gamma(\sigma)} u(\xi) d\xi \right).
\] (5)

Central to the analysis of the FADE are the following very useful Fourier transform, semi-group, and adjoint properties of fractional integral operators [25]. For \( 0 < \sigma, \hat{\sigma} < 1 \) and \( u = 0 \) on \( \partial \Omega \), the following hold:

- (Semi-Group Property L) \( a D_x^{-\sigma} a D_x^{-\hat{\sigma}} u = a D_x^{-\sigma - \hat{\sigma}} u \), \( \sigma, \hat{\sigma} > 0 \).
- (Semi-Group Property R) \( x D_b^{-\sigma} x D_b^{-\hat{\sigma}} u = x D_b^{-\sigma - \hat{\sigma}} u \), \( \sigma, \hat{\sigma} > 0 \).
- (Adjoint Property) \( (a D_x^{-\sigma} u, v)_{L^2(a,b)} = (u, x D_b^{-\sigma} v)_{L^2(a,b)} \).
- (Commutative Property L) \( a D_x^\sigma Du = D a D_x^{-\sigma} u \), \( \sigma > 0 \).
- (Commutative Property R) \( x D_b^{-\sigma} Du = D x D_b^{-\sigma} u \), \( \sigma > 0 \).

In addition, we will employ Lemma 8 in [14], which states that if \( u = 0 \) on \( \partial \Omega \), then
\[
(-Du, v)_{L^2(a,b)} = (x D_b^{(1-\mu)} u, a D_x^\mu v)_{L^2(a,b)},
\] (11)
for all \( 0 < \mu < 1 \), where \( u, v \) are in sufficiently regular function spaces such that the integration in the inner product makes sense.

In this article, we present preliminary analysis and computations for (1). We derive the variational form for (1) by multiplying through by an arbitrary test function, integrating over \( \Omega \), and integrating by parts in each term. That is:
\[
a(u, v) := \epsilon p (a D_x^{\alpha-2} u_x, v_x) + \epsilon q (x D_b^{\alpha-2} u_x, v_x) + (u, v_x).
\]
Then, we have immediately by Theorem 3.5 in [12] that there exists a unique $u \in H_{0}^{\alpha/2}(\Omega)$ such that

$$a(u, v) = (f, v), \quad \forall \ v \in H_{0}^{\alpha/2}(\Omega).$$

(12)

Existence and uniqueness of a solution to (12) rely heavily upon the definitions and equivalence of several semi-norms defined with respect to left and right fractional differential operators with the fractional order Sobolev norm of equivalent order. We define the fractional order Sobolev norm in terms of the Fourier transform, as in [26].

In the later analysis, we will utilize the fact that the semi-norms

$$|u|_{H_{0}^{\beta}(\Omega)} := \|\|\omega|^{\beta}F(u)\|_{L^{2}(\Omega)},$$

$$|u|_{J_{L,0}^{\beta}(\Omega)} := \|aD_{x}^{\beta}u\|_{L^{2}(\Omega)},$$

and

$$|u|_{J_{R,0}^{\beta}(\Omega)} := \|x^{\beta}D_{b}^{\beta}u\|_{L^{2}(\Omega)},$$

are equivalent when considering the corresponding spaces as the closure of $C_{0}^{\infty}(\Omega)$ distributions with respect to the norm defined by

$$\|\cdot\|_{*} := \left(\|\cdot\|_{L^{2}(\Omega)}^{2} + |\cdot|_{\ast}^{2}\right)^{1/2},$$

where by “*” we mean that we define the norms in (13)-(15), by squaring, adding the square of the $L^{2}$ norm, and taking the square root.

In addition to the equivalence of the semi-norms (13), (14), (15), coercivity of the variational form (12) requires an equivalence of a symmetric semi-norm:

$$|u|_{J_{S,0}^{\beta}(\Omega)} := \left|\left(aD_{x}^{\beta}u, xD_{b}^{\beta}u\right)\right|^{1/2}$$

(16)

Note that as in [12], the equivalence of (16) to (13)-(15) only holds when $\beta \neq \frac{1}{2} + n, n = 0, 1, 2, \ldots$, i.e.

$$C_{1}|u|_{H_{0}^{\beta}(\Omega)} \leq |u|_{J_{S,0}^{\beta}(\Omega)} \leq C_{2}|u|_{H_{0}^{\beta}(\Omega)}.$$  

(17)

In order to prove the main error estimate, we consider the usual finite element formulation of (12). Let $\{S^{h}\}$ denote a family of partitions of $\Omega$, with grid parameter $h$. Associated with $S^{h}$, define the finite-dimensional subspace $X^{h} \subset H_{0}^{\alpha}(\Omega)$ to be the space of piecewise polynomials of order $m - 1$, where $m \geq 2 \in \mathbb{N}$.
Therefore, the Galerkin finite element approximation \( u^h \) to \( u \) is defined to be the (unique) \( u^h \in X^h \) such that
\[
a(u^h, v^h) = (f, v^h), \quad \forall \ v^h \in X^h.
\] (18)

Denote by \( U \) the piecewise polynomial interpolant of \( u \) in \( S^h \). The finite-dimensional subspace \( X^h \) and the interpolant \( U \) are chosen specifically so that they satisfy the approximation property over fractional Sobolev spaces [27].

**Lemma 1 [Approximation Property].** Let \( u \in H^r(\Omega), 0 < r \leq m, \) and \( 0 \leq s \leq r \). Then there exists a constant \( C_A \) depending only on \( \Omega \) such that
\[
\|u - U\|_{H^s(\Omega)} \leq C_A h^{r-s} \|u\|_{H^r(\Omega)}.
\] (19)

### 3 Error estimate

In this section, we present the analysis which indicates the under-resolved nature of the singularly perturbed versions of both the integer and fractional order advection-dispersion equations. Before proceeding with the analysis of the fractional order problem, let us recall how the result is derived for the integer order case.

Let \( \Omega := (a, b) \) be a finite open interval. The integer order problem with boundary layer is:
\[
-\epsilon u_{xx} - u_x = f, \quad \text{in} \ \Omega, \\
u = 0, \quad \text{on} \ \partial \Omega.
\] (20)

**Lemma 2.** Let \( u \in H^2(\Omega) \) be the exact solution of (20) and \( u^h \) be its corresponding piecewise linear finite element approximation. Then we have the following error estimate, where \( C \) is a constant independent of \( u, h, \) and \( \epsilon \):
\[
|u - u^h|_{H^1(\Omega)} \leq C \left( h + \frac{h^2}{\epsilon} \right) \|u\|_{H^2(\Omega)}.
\] (21)

**Proof.** First, we define the bilinear form corresponding to the integer-order problem as:
\[
a_{\text{int}}(u, v) := \epsilon (u_x, v_x) + (u, v_x).
\]
Then \( u \) satisfying (20) is the (unique) \( u \in H^1_0(\Omega) \) such that
\[
a_{\text{int}}(u, v) = (f, v), \quad \forall \ v \in H^1_0(\Omega),
\]
and \( u^h \) is the (unique) \( u^h \in X^h \) such that
\[
a_{\text{int}}(u^h, v^h) = (f, v^h), \quad \forall \ v^h \in X^h.
\]
Now, setting \( U \) to be the piecewise linear interpolant of \( u \in X^h \), and notice that
\[
a_{\text{int}}(u^h - u, v^h) = 0, \quad \forall \ v^h \in X^h
\]
implies that
\[
a_{\text{int}}(u^h - U, u^h - U) = a_{\text{int}}(u - U, u^h - U).
\]
Now, as \( u^h - U = 0 \) on \( \partial \Omega \), we have that
\[
(u^h - U, u^h_x - U_x) = 0.
\]
Therefore,
\[
\epsilon |u^h - U|_{H^1(\Omega)}^2 \leq \epsilon |u - U|_{H^1(\Omega)} |u^h - U|_{H^1(\Omega)} + |u - U|_{L^2(\Omega)} |u^h - U|_{H^1(\Omega)}.
\]
Dividing through by \( |u^h - U|_{H^1(\Omega)} \), \( \epsilon \), and applying the usual approximation properties for \( U \),
\[
\|u - U\|_{L^2(\Omega)} \leq C_A h^2 \|u\|_{H^2(\Omega)},
\]
\[
|u - U|_{H^1(\Omega)} \leq C_A h \|u\|_{H^2(\Omega)},
\]
we obtain the stated result. \( \square \)

We now proceed in proving the main error result, an estimate which indicates the under-resolved nature of the boundary value problem (1).

**Theorem 3.** Let \( u \in H^2(\Omega) \) be the exact solution of (1) (and therefore (12)) and \( u^h \) be the piecewise linear finite element approximation satisfying (18). Then we have the following error estimate, where \( C \) is a constant independent of \( u, h, \) and \( \epsilon \):
\[
|u - u^h|_{H^{\alpha/2}(\Omega)} \leq C \left( h^{2-\alpha/2} + \frac{h^{1+\alpha/2}}{\epsilon} \right) \|u\|_{H^2(\Omega)}.
\]

**Proof.** Proceeding in the same way as Lemma 2, we immediately have that
\[
a(u^h - u, v^h) = 0, \quad \forall \ v^h \in X^h.
\]
Setting $U$ to be the piecewise linear interpolant of $u$, we have, as before

$$a(u^h - U, u^h - U) = a(u - U, u^h - U).$$

Again noticing that

$$(u^h - U, u^h_x - U_x) = 0,$$

we have

$$\begin{align*}
& \quad p \epsilon \left( a D^{\alpha/2}_x (u^h - U)_x, (u^h - U)_x \right) \\
&+ q \epsilon \left( x D^{\alpha/2}_b (u^h - U)_x, (u^h - U)_x \right) \\
&= p \epsilon \left( a D^{\alpha/2}_x (u - U)_x, (u^h - U)_x \right) \\
&+ q \epsilon \left( x D^{\alpha/2}_b (u - U)_x, (u^h - U)_x \right) \\
&+ (u - U, u^h_x - U_x). \\
\end{align*}$$

We first work with the left-hand side of (22). Using the semi-group and adjoint properties (6), (7), (8), we have

$$\begin{align*}
& \quad p \epsilon \left( a D^{\alpha/2}_x (u^h - U)_x, (u^h - U)_x \right) \\
&+ q \epsilon \left( x D^{\alpha/2}_b (u^h - U)_x, (u^h - U)_x \right) \\
&= p \epsilon \left( a D^{\alpha/2}_x (u^h - U)_x, (u^h - U)_x \right) \\
&+ q \epsilon \left( x D^{\alpha/2}_b (u^h - U)_x, (u^h - U)_x \right) \\
&+ (u - U, u^h_x - U_x) \\
&= \epsilon \left( x D^{\alpha/2}_b (u^h - U)_x, a D^{\alpha/2}_x (u^h - U)_x \right) \\
&+ \epsilon \left( x D^{\alpha/2}_b (u^h - U)_x, b D^{\alpha/2}_x (u^h - U)_x \right) \\
&+ \epsilon \left( x D^{\alpha/2}_b (u^h - U)_x, a D^{\alpha/2}_x (u^h - U)_x \right) \\
&+ \epsilon \left( x D^{\alpha/2}_b (u^h - U)_x, b D^{\alpha/2}_x (u^h - U)_x \right) \\
&= \epsilon \left( x D^{\alpha/2}_b (u^h - U)_x, a D^{\alpha/2}_x (u^h - U)_x \right)
\end{align*}$$

Now, utilizing Lemma 3.1 in [12] and noting that $u = 0$ on $\partial \Omega$, we have

$$\text{LHS of (22)} = -\epsilon \left( a D^{\alpha/2}_x (u^h - U), x D^{\alpha/2}_b (u^h - U) \right)$$

$$= \epsilon |u^h - U|^2_{H^{\alpha/2}_0(\Omega)}$$

$$\geq C_1 \epsilon |u^h - U|^2_{H^{\alpha/2}_0(\Omega)},$$

by the equivalence of (16) and (13), i.e. (17).

Now, we use similar arguments in order to bound the right-hand side of (22). First, using the semi-group and adjoint properties repeatedly, we have
and the norm equivalences of (13), (14), and (15), we have

Using the triangle inequality, the Cauchy-Schwartz inequality in each term,
and the norm equivalences of (13), (14), and (15), we have

Finally, dividing through by \( \| u^h - U \|_{H^{\alpha/2}(\Omega)} \), \( \epsilon \), and using the fractional approximation properties

we obtain the stated result.
Remark. From Theorem 3, we can observe that the under-resolved case for the singularly perturbed FADE corresponds to the case that $\varepsilon < h^{\alpha-1}$.

4 The Boundary Layer Basis Method

In this section, we present the fractional version of the boundary layer basis method, as well as present analytical examples which exhibit the nature of the boundary layers for the FADE as compared to the integer order advection dispersion equation.

In the works [21–23], the FEM basis can be augmented by the explicit, analytic function of the boundary layer. The formula for the boundary layer is defined to be the explicit solution to the ordinary differential equation:

$$-\varepsilon u_{xx} - u_x = 1; \quad u(0) = 0; \quad u(1) = 0,$$

which is given by:

$$u(x) = -x + \frac{1 - e^{-x/\varepsilon}}{1 - e^{-1/\varepsilon}}.$$

In this section, we outline the computational strategy by which the boundary layer basis method (BL) may be implemented computationally. The BL method may be generally outlined as the solution of the abstract Dirichlet variational problem

$$\langle Au, v \rangle = \langle f, v \rangle, \quad \forall v \in X. \quad (27)$$

where $A$ represents some abstract operator (in our case a singularly-perturbed fractional dispersion operator), $X$ represents an appropriate function space and, if necessary, $\langle \cdot, \cdot \rangle$ is interpreted in a duality sense.

Step 1. Perform an analytical calculation or a costly direct numerical simulation (DNS) in order to obtain an exact, or almost nearly exact, solution $\tilde{\phi}$ to the variational problem

$$\langle A\tilde{\phi}, v \rangle = \langle 1, v \rangle, \quad \forall v \in X.$$

Step 2. Choose a finite element basis $V_N := \{\phi_i\}_{i=1}^N$ over which the variational problem (27) is to be solved.

Step 3. Define the new finite element basis as $\tilde{V}_N := \{\tilde{\phi}_i\}_{i=1}^N \cup \tilde{\phi}$. 

**Step 4.** Solve the variational problem (27) over the new finite element basis. Find the unique $u^h \in \tilde{V}_N$ such that

$$(Au^h, v^h) = (f, v^h), \quad \forall v^h \in \tilde{V}_N, \quad u^h = 0, \quad \text{on } \partial \Omega,$$

where $\Pi$ is the projection operator onto $\tilde{V}_N$.

Our computational experiments center around defining an FADE in which the boundary layer can be found explicitly. What is interesting to note is that graphically, the numerical instability caused by the boundary layer follows that of Theorem 3: that smaller values of $\alpha$ lead to more instability.

We begin with the following version of the singularly perturbed FADE, which is

$$-\epsilon D_0D_x^{\alpha-2}D\tilde{\phi} - \tilde{\phi}_x = 1; \quad \tilde{\phi}(0) = 0; \quad \tilde{\phi}(1) = 0, \quad (28)$$

and $\alpha$ is the order of the dispersion operator, $1 < \alpha \leq 2$.

We will obtain a solution to this equation using the Laplace transform.

$$\mathcal{L}\{-\epsilon D_0D_x^{\alpha-2}D\tilde{\phi}\} + \mathcal{L}\{-\tilde{\phi}_x\} = \frac{1}{s}.$$

First, using the Laplace transform property of the left Riemann-Liouville fractional integral

$$\mathcal{L}\{0 \ D^{\alpha-2}_x \ u\} = s^{-\alpha} \mathcal{L}\{u\},$$

and the boundary condition to obtain:

$$-\epsilon s \mathcal{L}\{D\tilde{\phi}\} + \epsilon C_1 - s \mathcal{L}\{\tilde{\phi}\} + u(0) = \frac{1}{s}.$$

Note that the constant $C_1$ is equal to

$$C_1 = \left[0 \ D^{\alpha-2}_x \tilde{\phi}\right]_{x=0},$$

but the constant $C_1$ will later be used in order to ensure that the boundary condition is satisfied at $x = 1$.

Next, we employ the Laplace transform property of the left Riemann-Liouville fractional integral

$$\mathcal{L}\{0 \ D^{-\sigma}_x \ u\} = s^{-\sigma} \mathcal{L}\{u\},$$

and the boundary condition to obtain:

$$-\epsilon s^{\alpha-1} \mathcal{L}\{D\tilde{\phi}\} + \epsilon C_1 - s \mathcal{L}\{\tilde{\phi}\} = \frac{1}{s}.$$
Again, using the derivative property of the Laplace transform, we have

\[-\epsilon s^{\alpha} \mathcal{L}\{\tilde{\phi}\} + \epsilon s^{\alpha-1}\tilde{\phi}(0) + \epsilon C_1 - s\mathcal{L}\{\tilde{\phi}\} = \frac{1}{s}\]

which implies

\[-\epsilon s^{\alpha} \mathcal{L}\{\tilde{\phi}\} - s\mathcal{L}\{\tilde{\phi}\} = \frac{1}{s} - \epsilon C_1.\]

Finally, we are able to divide through and obtain the expression:

\[\mathcal{L}\{\tilde{\phi}\} = \frac{\epsilon C_1}{\epsilon s^{\alpha} + s} - \frac{1}{\epsilon s^{\alpha+1} + s^2}.\]  

(29)

In order to determine the value of the function \(\tilde{\phi}\), we will exploit equation (1.80) in Podlubny’s monograph [9], which reads

\[\int_0^\infty e^{-sx}x^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\pm ax^\alpha)dx = \frac{k!s^{\alpha-\beta}}{(s^\alpha \mp a)^{k+1}},\]  

(30)

where \(E_{\alpha,\beta}^{(k)}(t)\) represents the \(k^{th}\) derivative of the two-parameter Mittag-Leffler function with parameters \(\alpha, \beta\):

\[E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.\]

We rewrite the first term in (29) and notice that it can be cast in terms of (30), that is:

\[\mathcal{L}^{-1}\left\{\frac{C_1 s^{-1}}{s^{\alpha-1} + 1/\epsilon}\right\} = C_1 x^{\alpha-1} E_{\alpha-1,\alpha}(-x^{\alpha-1}/\epsilon).\]

Next, we are able to perform an analogous operation on the second term of (29), which is

\[\mathcal{L}^{-1}\left\{\frac{s^{-2}/\epsilon}{s^{\alpha-1} + 1/\epsilon}\right\} = \frac{x^{\alpha}}{\epsilon} E_{\alpha-1,\alpha+1}(-x^{\alpha-1}/\epsilon).\]

Finally, we are able to arrive at the formula:
Fig. 1. Plots of the boundary layer function $\tilde{\phi}$ in (28) for $\epsilon = 10^{-1}$ and $\alpha = 1.5, 1.75, 2.00$. Notice that smaller values of $\alpha$ imply sharper boundary layers.

$$
\tilde{\phi}(x) = C_1 x^{\alpha-1} E_{\alpha-1,\alpha}( -x^{\alpha-1}/\epsilon ) - \frac{x^\alpha}{\epsilon} E_{\alpha-1,\alpha+1}( -x^{\alpha-1}/\epsilon ),
$$

(31)

where $C_1$ is chosen such that $\tilde{\phi}(1) = 0$, that is

$$
C_1 = \frac{E_{\alpha-1,\alpha+1}( -1/\epsilon )}{\epsilon E_{\alpha-1,\alpha}( -1/\epsilon )}.
$$

In the sequel, we will present numerical simulations which utilize the Mittag-Leffler function programmed in Matlab, MLF.m, [28]. Figure 4 displays boundary layers $\tilde{\phi}$ for (28), and $\epsilon = 10^{-1}$. Notice that the boundary layers are sharper for smaller values of $\alpha$, the order of the diffusion operator.

5 Numerical Experiments

In this section, we present three examples which illustrate the utility of the boundary layer basis method for the FADE in one spatial dimension. The first example is one in which the boundary layer follows the formula (31) and an exact solution to the problem may be derived using Laplace transforms. We
compare the traditional finite element method versus the BL method graphically and numerically. The second example involves the same FADE except that the right hand side is chosen such that the exact solution is unknown. The third and final example illustrates how the (BL) method can be generalized to FADEs whose dispersion terms contain both left and right fractional integral operators.

The computational experiments were carried out using MATLAB and the BL method was implemented using the MATLAB function MLF.m [28]. Note that because of the very sharp nature of the boundary layers, inner products involving the boundary layer basis function were evaluated utilizing MATLAB’s “quad” command, which employs an adaptive Simpson’s rule to accurately evaluate one dimensional integrals over finite intervals to desired precision.

Example 1. For this example, we approximate the solution $u$ to the boundary value problem

$$-\epsilon D_0 D_x^{\alpha-2} Du - u_x = x; \quad u(0) = 0; \quad u(1) = 0,$$

(32)

and apply the (BL) method with $\tilde{\phi}$ defined as in (31). Using the Laplace transform properties, specifically (30), we can determine that the exact solution $u$ to (32) is given by

$$u(x) = C E_1 x^{\alpha-1} E_{\alpha-1,\alpha}(-x^{\alpha-1}/\epsilon) - \frac{x^{\alpha+1}}{\epsilon} E_{\alpha-1,\alpha+2}(-x^{\alpha-1}/\epsilon),$$

(33)

where $C_{E1}$ is given by

$$C_{E1} = \frac{E_{\alpha-1,\alpha+2}(-1/\epsilon)}{\epsilon E_{\alpha-1,\alpha}(-1/\epsilon)}.$$

Figures 2 and 3 illustrate the superiority of the (BL) method to the finite element method for values $\alpha = 1.75$, $n = 8, 16$ and $\epsilon = 10^{-1}$ and $\epsilon = 10^{-3}$ respectively. It is clear from the pictures that knowledge of the boundary layer significantly improves the accuracy of the finite element computations.

Table 1 shows the successive $L^2$ and $H^1$ errors for the traditional finite element method with $\epsilon = 10^{-1}$, whereas Table 2 shows the errors for the BL method. Again it is clear that knowledge of the boundary layer greatly improves the magnitude and convergence rates of the discretization errors in both the $H^1$ and $L^2$ errors.

In [21], for the integer order problem with boundary layer, (20), the authors are able to prove the following estimate:

$$\| u - u^h \|_{L^2(\Omega)} \leq C h \| u \|_{H^2(\Omega)},$$

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where $C$ is independent of $h, \epsilon$. Further detailed analysis is required to determine how this theoretical result extends to the fractional case. It is not clear at this point what the theoretical asymptotic convergence rates (independent of $\epsilon$) will be for the BL method as applied to FADEs.

Further, we present numerical results for both the traditional and the BL methods. Table 1 and Table 2 show the $L^2$ and $H^1$ norms for different stepsizes $h$ and for $\alpha = 1.5$ and $\alpha = 1.75$.

### Table 1
Numerical Results for the traditional method in Example 1 for $\alpha = 1.5, 1.75$, $\epsilon = 10^{-1}$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$|u - u_h|_{L^2}$ ($\alpha = 1.5$)</th>
<th>$|u - u_h|_{H^1}$ ($\alpha = 1.5$)</th>
<th>$|u - u_h|_{L^2}$ ($\alpha = 1.75$)</th>
<th>$|u - u_h|_{H^1}$ ($\alpha = 1.75$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/4$</td>
<td>2.978397 · 10^{-1}</td>
<td>3.665819 · 10^{+0}</td>
<td>8.884897 · 10^{-2}</td>
<td>1.475286 · 10^{0}</td>
</tr>
<tr>
<td>$1/8$</td>
<td>9.635949 · 10^{-2}</td>
<td>4.065295 · 10^{+0}</td>
<td>3.471371 · 10^{-2}</td>
<td>1.743033 · 10^{0}</td>
</tr>
<tr>
<td>$1/16$</td>
<td>4.707397 · 10^{-2}</td>
<td>3.653285 · 10^{+0}</td>
<td>1.431950 · 10^{-2}</td>
<td>1.673007 · 10^{0}</td>
</tr>
<tr>
<td>$1/32$</td>
<td>2.520975 · 10^{-2}</td>
<td>3.911990 · 10^{+0}</td>
<td>5.661807 · 10^{-3}</td>
<td>1.527183 · 10^{0}</td>
</tr>
<tr>
<td>$1/64$</td>
<td>1.322505 · 10^{-2}</td>
<td>4.143803 · 10^{+0}</td>
<td>2.208042 · 10^{-3}</td>
<td>1.388581 · 10^{0}</td>
</tr>
</tbody>
</table>

### Table 2
Numerical Results for the BL method in Example 1 for $\alpha = 1.5, 1.75$, $\epsilon = 10^{-1}$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$|u - u_h|_{L^2}$ ($\alpha = 1.5$)</th>
<th>$|u - u_h|_{H^1}$ ($\alpha = 1.5$)</th>
<th>$|u - u_h|_{L^2}$ ($\alpha = 1.75$)</th>
<th>$|u - u_h|_{H^1}$ ($\alpha = 1.75$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/4$</td>
<td>4.542422 · 10^{-3}</td>
<td>6.252533 · 10^{-2}</td>
<td>7.607834 · 10^{-3}</td>
<td>7.717261 · 10^{-2}</td>
</tr>
<tr>
<td>$1/8$</td>
<td>2.071944 · 10^{-3}</td>
<td>4.649594 · 10^{-2}</td>
<td>2.325904 · 10^{-3}</td>
<td>6.841074 · 10^{-2}</td>
</tr>
<tr>
<td>$1/16$</td>
<td>7.011151 · 10^{-4}</td>
<td>4.239459 · 10^{-2}</td>
<td>6.950787 · 10^{-4}</td>
<td>4.197245 · 10^{-2}</td>
</tr>
<tr>
<td>$1/32$</td>
<td>2.405733 · 10^{-4}</td>
<td>3.092093 · 10^{-2}</td>
<td>1.703308 · 10^{-4}</td>
<td>2.195632 · 10^{-2}</td>
</tr>
<tr>
<td>$1/64$</td>
<td>7.644914 · 10^{-5}</td>
<td>1.995941 · 10^{-2}</td>
<td>4.142305 · 10^{-5}</td>
<td>1.073634 · 10^{-2}</td>
</tr>
</tbody>
</table>

### Example 2.
For this example, we approximate the solution $u$ to the boundary value problem

$$-\epsilon D_0D_x^{\alpha-2}Du - u_x = 1 + \sin(4\pi x); \quad u(0) = 0; \quad u(1) = 0,$$

and apply the (BL) method with $\tilde{\phi}$ defined as in (31). Notice that for this example, we do not find the exact solution, but we can still note the superiority of the BL method.

Figures 4 and 5 illustrate the superiority of the (BL) method to the finite
Fig. 2. A comparison of the traditional and BL methods applied to the FADE in Example 1 with $\epsilon = 10^{-1}$ and $f := x$. Left: FEM approximations by the traditional method with $h = 1/8, 1/16$ versus the exact solution. Right: FEM approximations by the BL method with $h = 1/8, 1/16$ versus the exact solution.

Fig. 3. A comparison of the traditional and BL methods applied to the FADE in Example 1 with $\epsilon = 10^{-3}$ and $f := x$. Left: FEM approximations by the traditional method with $h = 1/8, 1/16$ versus the exact solution. Right: FEM approximations by the BL method with $h = 1/8, 1/16$ versus the exact solution.

element method for values $\alpha = 1.75$, $n = 8, 16, 32$ and $\epsilon = 10^{-1}$ and $\epsilon = 10^{-3}$ respectively.

**Example 3.** In this example, we explore how the (BL) method utilized above generalizes for FADEs which contain right fractional integrals or both left and right fractional integrals. We consider in this example the boundary value problem

$$-\epsilon D_x D_t^{\alpha-2} D u - u_x = f; \quad u(0) = 0; \quad u(1) = 0.$$  \hfill (35)

In Section 4, we saw how to use a Laplace transform argument to find an analytical representation for the boundary layer function when (1) contains only the left sided fractional integral operator. Continuing in a similar fashion,
we define the boundary layer function for the initial boundary value problem (35) as the solution $\tilde{\varphi}_2$ to the boundary value problem

$$-\varepsilon D_{x}D_{1}^{\alpha-2} D\tilde{\varphi}_2 - (\tilde{\varphi}_2)_x = 1; \hspace{0.5cm} \tilde{\varphi}_2(0) = 0; \hspace{0.5cm} \tilde{\varphi}_2(1) = 0. \hspace{0.5cm} (36)$$

In order to obtain an analytical representation for $\tilde{\varphi}_2$, we introduce the auxiliary problem

$$-\varepsilon D_{0}D_{x}^{\alpha-2} D\tilde{\varphi}_3 + (\tilde{\varphi}_3)_x = 1; \hspace{0.5cm} \tilde{\varphi}_3(0) = 0; \hspace{0.5cm} \tilde{\varphi}_3(1) = 0, \hspace{0.5cm} (37)$$

and notice that under a substitution, $\tilde{\varphi}_2(1-x) = \tilde{\varphi}_3(x)$.

Using a Laplace transform argument similar to that in Section 4, we notice
that
\[ \mathcal{L}\{\tilde{\phi}_3\} = \frac{\epsilon C_1}{\epsilon s^\alpha - s} - \frac{1}{\epsilon s^{\alpha+1} - s^2}. \] (38)

And therefore,
\[ \tilde{\phi}_3(x) = C_1 x^{\alpha-1} E_{\alpha-1,\alpha}(x^{\alpha-1}/\epsilon) - \frac{x^\alpha}{\epsilon} E_{\alpha-1,\alpha+1}(x^{\alpha-1}/\epsilon), \] (39)

where
\[ C_1 = \frac{E_{\alpha-1,\alpha+1}(1/\epsilon)}{\epsilon E_{\alpha-1,\alpha}(1/\epsilon)}. \]

which immediately implies that
\[ \tilde{\phi}_2(x) = C_1 (1-x)^{\alpha-1} E_{\alpha-1,\alpha}((1-x)^{\alpha-1}/\epsilon) - \frac{(1-x)^\alpha}{\epsilon} E_{\alpha-1,\alpha+1}((1-x)^{\alpha-1}/\epsilon). \] (40)

Remark. Note that for the generic version of the FADE (1) with both left and right fractional integrals, the boundary layer function is given by \( p \tilde{\phi} + q \tilde{\phi}_2 \).

We repeat the numerical experiments in Examples 1 and 2 for the right-sided version of the FADE. Figures 6 and 7 illustrate the superiority of the (BL) method to the finite element method for values \( \alpha = 1.75, n = 8, 16, 32, \epsilon = 10^{-1}, \) and \( f := x, f := 1 + 4 \sin(\pi x) \), respectively.

We have only included calculations in this example for the relatively large value of \( \epsilon = 10^{-1} \). Although the representation of the boundary layer function \( \tilde{\phi}_2 \) in (40) is mathematically correct, the utility of this representation in our Matlab code is limited as values of the Mittag-Leffler function are very large for positive real values of the argument \( x \). For example,
\[
\begin{align*}
E_{0.75,2.75}(10^1) &\approx 1.406683 \times 10^7, \\
E_{0.75,2.75}(10^2) &\approx 1.096269 \times 10^{197}, \\
E_{0.75,2.75}(10^3) &\approx \text{Inf (floating point overflow)}.\end{align*}
\]

It would be of interest in future work to derive numerical approximation schemes for \( \tilde{\phi}_2 \) which are numerically stable for very small values of \( \epsilon \).

Acknowledgement. The author would like to thank Vincent J. Ervin and an anonymous reviewer for helpful suggestions regarding this paper.
Fig. 6. A comparison of the traditional and BL methods applied to the FADE in Example 3 with $\epsilon = 10^{-1}$ and $f := x$. Left: FEM approximations by the traditional method with $h = 1/8, 1/16, 1/32$. Right: FEM approximations by the BL method with $h = 1/8, 1/16, 1/32$.

Fig. 7. A comparison of the traditional and BL methods applied to the FADE in Example 3 with $\epsilon = 10^{-3}$ and $f := 1 + \sin(4\pi x)$. Left: FEM approximations by the traditional method with $h = 1/8, 1/16, 1/32$. Right: FEM approximations by the BL method with $h = 1/8, 1/16, 1/32$.

References


