Numerical comparison of nonlinear subgridscale models via adaptive mesh refinement

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Abstract

In this paper, we provide a method of evaluating the efficacy of nonlinear subgridscale models for use in the large eddy simulation of incompressible viscous flow problems. We compare subgridscale “artificial” viscosity models using a posteriori error estimation and adaptive mesh refinement. Specifically, we compare \( \alpha \)-Laplacian based subgridscale models and discuss the benefits and limitations of different values of \( \alpha \) for some standard benchmark problems for the Navier–Stokes equations.

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1. Introduction

In this paper, we introduce a method of validation for subgridscale “artificial” viscosity terms used in the large eddy simulation (LES) of incompressible flow problems. The method we introduce is based on a posteriori error estimation and adaptive mesh refinement. The a posteriori error estimation technique used is the one found in [1]. The local element error estimator is defined as the residual a posteriori error estimator associated with the incompressible Navier–Stokes equations (NSE) coupled with a “stabilization” term which depends upon the subgridscale viscosity.

It is known that a subgridscale model which introduces artificial viscosity at a rate of \( \mu_0 h_T^{\alpha-1}\|\nabla u\|^{\alpha-2} \), (where \( h_T \) denotes the local mesh parameter, \( u \) is the velocity of the fluid, and \( \mu_0 \) is an appropriately chosen constant), effectively stabilizes the computational simulation in the sense that a Newton iteration is convergent for high Reynolds number flows. We conclude that as \( \alpha \) increases the computational simulation produces flows which are closer to the actual Navier–Stokes flow; however, the average number of Newton iterations required for each overall iteration increases, and it may be more desirable to take values of \( \alpha \) higher than 3, which corresponds to Smagorinsky’s model [2].

The large eddy simulation of viscous incompressible flow problems has gained significant attention, due to its ability to provide realistic approximations for flows which are otherwise unresolvable by direct numerical simulation for computationally feasible grids [3]. We are primarily interested in LES models which are implemented by the introduction of a subgridscale “artificial” viscosity term into the incompressible NSE which effectively adds viscosity
at appropriately chosen locations in the computational domain. All of these models are direct descendents of the model of Smagorinsky [2], which introduces subgridscale viscosity proportional to $|\nabla u|$, the magnitude of the fluid velocity. Other models of interest include: one which relies on $|\nabla u|^{q-2}$, “an $\alpha$-Laplacian model” [4]; one which relies on a nonlinear smoothly increasing function of the magnitude of velocity, $a(|\nabla u|)$, “a bounded nonlinear model” [5]; and those which vary from the aforementioned in that they rely on the magnitude of the rate of deformation tensor $\frac{1}{2}(\nabla u + \nabla u^T)$ or the vorticity tensor $\frac{1}{2}(\nabla u - \nabla u^T)$ [6,3]. Other LES models primarily rely on the approximation of the convolution kernel used in the chosen space filter [7].

Although LES models provide a computational procedure for which numerical simulations such as the Newton iteration are convergent, little is known concerning the qualitative differences in the LES predicted flow and the actual flow. A priori error estimates regarding this difference have been proven only for the case of linear equations with subgridscale terms, such as convection-dominated, convection–diffusion equations or the Oseen problem [5,8,4]. However, the nonlinearity in the incompressible NSE provides for vast differences in qualitative flow phenomena for relatively small changes in the viscosity term(s). Models which provide for quick computational simulation often result in approximate solutions which in no way resemble the actual flow, and models for which the approximated flow might closely resemble the actual flow may not be computationally feasible.

The most effective and attractive method of approximating the error between the approximated and actual flow is that of a posteriori error estimation. A posteriori error estimators provide reasonable approximations to the actual error in cases for which the exact solution is unknown, and provide estimates of the local approximation error, by which adaptive mesh refinement may be performed. In [9], Verfurth provides an elegant and explicit way by which
residual-based a posterior error estimates can be derived for nonlinear equations. The method has been effectively applied to numerous nonlinear models, e.g. incompressible NSE [9], $\alpha$-Laplacian problems [10,9], and viscoelastic fluid flow [11].

Unfortunately the primary deficiency of a posteriori error estimation when applied to the incompressible NSE is that the coarsest mesh must be fine enough in order for the flow to be computationally resolvent. Therefore, adaptive mesh refinement techniques alone are not feasible for high Reynolds number flows. However, combining the methods of subgridscale viscosity modeling and adaptive mesh refinement is attractive, as the computational routine is guaranteed to work for any given mesh, and as the mesh is refined, the introduced subgridscale viscosity decreases, which ultimately leads to a solution which is closer to the actual flow than either method alone could produce. We measure a posteriori error of an approximate solution with subgridscale viscosity using the ordinary a posteriori error estimator for the incompressible NSE (without artificial viscosity). However, in order to show an upper bound for the predicted error, we must also include a stabilization term which relies on the amount of added viscosity [1]. We say that a subgridscale model is good if it both: (i) provides a computationally feasible solution for a reasonable amount of computational resources (storage, time), and (ii) gives relatively small a posteriori error estimate.

In this paper, we follow the a posteriori error estimation technique as in [1] and use the adaptive mesh refinement routine established in [12]. In [1], the authors used a uniform grid to determine which value for $\alpha$ in the $\alpha$-Laplacian model was most appropriate. Instead, for our method we compare values of $\alpha$ using the full adaptive computational routine. Also, we illustrate the conclusions in [1] by showing computational results for some standard benchmark problems, and see that the fully adaptive, large eddy simulation of incompressible viscous flow problems is preferential to other methods in that we generate a flow that is qualitatively and quantitatively closer to the actual physical flow than other methods.

The paper is outlined as follows. In Section 2, we introduce the theory of subgridscale models for viscous incompressible flows. In Section 3, we review a posteriori error estimation procedures for nonlinear equations—specifically, NSE with a subgridscale term. In Section 4, we outline the details of our computational implementation. Finally, in Section 5, we provide numerical experiments which compare various $\alpha$-Laplacian based subgridscale models.

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$, $d = 2$ or 3 with Lipschitz boundary $\partial \Omega$. We denote by $W^{m,p}(\Omega)$ the usual Sobolev spaces [13], and $H^m(\Omega) := W^{m,2}(\Omega)$ with norms $\| \cdot \|$ and semi-norms $| \cdot |$. We also let $C^\infty_0(\Omega)$ denote the set of all distributions $u \in C^\infty(\Omega)$ that vanish outside a compact subset $K$ of $\Omega$, and set $W^{m,p}_0(\Omega)$ (resp. $H^m_0(\Omega)$) to be the closure of $C^\infty_0(\Omega)$ under their respective norms.
2. Subgridscale models for the steady incompressible Navier–Stokes equations

We wish to approximate the solution to the steady-state incompressible Navier–Stokes equations

\[
\begin{align*}
-\text{Re}^{-1} \Delta u + u \cdot \nabla u + \nabla p &= f \quad \text{in } \Omega \\
\nabla \cdot u &= 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega \\
\int_{\Omega} p \, dx &= 0
\end{align*}
\]

for “high” Reynolds numbers. For convenience, we rewrite the NSE in conservative form:

\[
\begin{align*}
-\text{Re}^{-1} \Delta u + \nabla \cdot (u \otimes u) + \nabla p &= f \quad \text{in } \Omega \\
\nabla \cdot u &= 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega \\
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u &= 0 \quad \text{on } \partial \Omega \\
\int_{\Omega} p \, dx &= 0
\end{align*}
\]

where \( u \otimes v \) is the dyadic product, a \( d \times d \) tensor, with entries \((u \otimes v)_{i,j} := u_i v_j\). In the usual way, we define the filtered velocity as \( \overline{u} := g_\delta \ast u \), where \( g_\delta \) is a chosen filtering function (e.g. the sharp cut-off or Gaussian filters).

Convolving (1) with an appropriately chosen filtering function \( g_\delta \), we obtain the system

\[
\begin{align*}
-\text{Re}^{-1} \Delta \overline{u} + \nabla \cdot (\overline{u} \otimes \overline{u}) + \nabla \overline{p} + \nabla \cdot \mathcal{R}(u, \overline{u}) &= \overline{f} \quad \text{in } \Omega \\
\nabla \cdot \overline{u} &= 0 \quad \text{in } \Omega \\
\overline{u} &= 0 \quad \text{on } \partial \Omega \\
\int_{\Omega} \overline{p} \, dx &= 0
\end{align*}
\]
Fig. 5. Plots of the velocity field in Example 1 for $\alpha = 3.0$ and approximately 25,000 and 75,000 degrees of freedom, respectively.

where $\mathcal{R}(\mathbf{u}, \mathbf{u}) = \mathbf{u} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{u}$ is the Reynolds stress tensor.

The overall goal of large eddy simulation is to appropriately approximate the Reynolds stress tensor with a functional $S(\mathbf{u}, \mathbf{u})$ which depends only upon the filtered velocity $\mathbf{u}$. The Boussinesq hypothesis leads to the approximation

$$S(\mathbf{u}, \mathbf{u}) = -\nu_T (\nabla \mathbf{u}) \nabla \mathbf{u}, \quad \text{(plus term incorporated into } p)$$

also known as the “eddy viscosity” model. The simplest eddy viscosity model results from the assumption that $\nu_T$ is constant with respect to $\mathbf{u}$, i.e.

$$S_2(\mathbf{u}, \mathbf{u}) = -\nu \nabla \mathbf{u}.$$

However, more advanced eddy viscosity models may be studied, including Smagorinsky’s model [2]

$$S_3(\mathbf{u}, \mathbf{u}) = -\mu |\nabla \mathbf{u}| \nabla \mathbf{u},$$

and the one introduced by Ladyzhenskaya [14,15] and studied by Layton [4],

$$S_\alpha(\mathbf{u}, \mathbf{u}) = -\mu |\nabla \mathbf{u}|^{\alpha-2} \nabla \mathbf{u}, \quad (4)$$

which leads to a modified Navier–Stokes problem including a $\alpha$-Laplacian term. (We use $\alpha$ instead of $p$ to avoid confusion with the pressure term.) In fact, (4) was analyzed in detail by Ladyzhenskaya [14,15] and Du and Gunzburger [16,17].
Fig. 6. Plots of the velocity field in Example 1 for $\alpha = 3.5$ and approximately 25,000 and 75,000 degrees of freedom, respectively.

The main reason for implementing a Reynolds closure model (e.g. (4)) is that the system

$$
\begin{align*}
\mathcal{A}_\alpha(u) - Re^{-1} \Delta u + u \cdot \nabla u + \nabla p &= f & \text{in } \Omega \\
\nabla \cdot u &= 0 & \text{in } \Omega \\
\nabla u &= 0 & \text{on } \partial \Omega \\
\int_\Omega p dx &= 0
\end{align*}
$$

(5)

where, for example,

$$
\mathcal{A}_\alpha(u) := -\mu \nabla \cdot |\nabla u|^\alpha - 2 \nabla u
$$

(6)

possesses more stability than the usual Navier–Stokes equations (1).

In order to discuss the solution of (5), we introduce its corresponding variational form

$$
\mu(|\nabla u|^\alpha - 2 \nabla u, \nabla v) + Re^{-1}(\nabla u, \nabla v) + (u \cdot \nabla u, v) - (p, \nabla \cdot v) + (q, \nabla \cdot u) = (f, v),
$$

(7)

$\forall v \in (H^1_0(\Omega))^d$, $q \in L^2(\Omega)$, respectively.

As (7) is a nonlinear equation in $u$, we must apply a Newton’s method; that is, to solve the functional equation,

$$(\mathcal{G}(u), v) = 0, \quad \forall v \in X = (H^1_0(\Omega))^d.$$

fix $u^{(0)}$, and solve for $n = 1, 2, \ldots$,

$$(\mathcal{J}_u(\mathcal{G}(u^{(n-1)}), \mathcal{G}(u^{(n)} - u^{(n-1)}), v) = -(\mathcal{G}(u^{(n-1)}), v),$$

(8)

where $\mathcal{J}_u \mathcal{G}(w)$ denotes the Gateaux derivative of $\mathcal{G}$ in the direction of $u$ evaluated at $w$. 
From [18], we have that for
\[
(\mathcal{G}(u), v) := (|\nabla u|^{|2-2\alpha|} u, \nabla v),
\]
\[
(\mathcal{J}_h \mathcal{G}(w), v) = (|\nabla u|^{|2-2\alpha|} w, \nabla v) + (\alpha - 2)(|\nabla u|^{|2-2\alpha|} \nabla u, \nabla v).
\]
(9)

From (8), we can easily derive that the appropriate Newton iteration corresponding to (7) is given by
\[
\mu(|\nabla u^{(n-1)}|^{|2-2\alpha|} \nabla u^{(n)}, \nabla v) + \mu(\alpha - 2)(|\nabla u^{(n-1)}|^{|2-2\alpha|} \nabla u^{(n)}, \nabla v) + Re^{-1}(\nabla u^{(n)}, \nabla v) + (u^{(n-1)} \cdot \nabla u^{(n)}, v) + (u^{(n)} \cdot \nabla u^{(n-1)}, v) - (p^{(n)}, \nabla \cdot v) + (q, \nabla \cdot u^{(n)})
\]
\[
= \mu(\alpha - 2)(|\nabla u^{(n-1)}|^{|2-2\alpha|} \nabla u^{(n-1)}, \nabla v) + (u^{(n-1)} \cdot \nabla u^{(n-1)}, v) + (f, v), \quad \forall v \in (H_0^1(\Omega))^d, q \in L_0^2(\Omega).
\]

As usual, in order to approximate the solution of (7) we let \( S_h \) denote a family of partitions of \( \Omega \), with grid parameter \( h \). Associated with \( S_h \), define the finite dimensional subspace \( X_h \subset H_0^1(\Omega) \) to be the basis of piecewise polynomials of order \( m - 1 \), where \( m \geq 1 \in \mathbb{N} \). Also, associated with \( S_h \) we define the finite dimensional subspace \( Q_h \subset L_0^2(\Omega) \) to be any finite element approximation space such that the pair \( (X_h, Q_h) \) satisfies the discrete LBB (inf-sup) condition for the Stokes problem.

Therefore, the Galerkin approximation of (7) is any \((u_h, p_h) \in X_h \times Q_h\) solving
\[
\mu(|\nabla u_h|^{|2-2\alpha|} \nabla u_h, \nabla v_h) + Re^{-1}(\nabla u_h, \nabla v_h) + (u_h \cdot \nabla u_h, v_h) - (p_h, \nabla \cdot v_h) + (q_h, \nabla \cdot u_h) = (f, v_h),
\]
(10)

for all \((v_h, q_h) \in X_h \times Q_h\).

Another concern of implementing (10) as a Reynolds closure model is to establish an appropriate dependence of \( \mu \) on \( h, \mu = \mu(h) \). In [4], Layton addressed this issue, showing that for
\[
\mu(h) := \mu_0 |\ln(h)|^{\frac{(1-\gamma)(d-1)}{d}} h^\gamma,
\]
(11)
any $u_h$ solving (10) satisfies a max-norm estimate for $\|u_h\|_{L^\infty(\Omega)}$, where $\alpha$ is the order of the $\alpha$ Laplacian operator, $d$ is the dimension of $\Omega$, and $\sigma$ is a modeling parameter.

**Theorem 2.1** ([4]). Let $(u_h, p_h) \in X_h \times Q_h$ solve (10) for $f \in W^{-1,\alpha'}(\Omega)$. Then

$$\|u_h\|_{L^\infty(\Omega)} \leq C \mu(h)^{1/\alpha} |\ln(h)|^{1-\frac{1}{\alpha}} h^{\frac{d}{2}} \|f\|^{\frac{1}{\alpha'}}_{W^{-1,\alpha'}(\Omega)},$$

(12)

where $\alpha'$ satisfies $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$.

**Corollary 2.2.** For $\mu(h)$ satisfying (11) and $(u_h, p_h) \in X_h \times Q_h$ satisfying (10), we have the estimate

$$\|u_h\|_{L^\infty(\Omega)} \leq C h^{-r} \|f\|^{\alpha-1}_{W^{-1,\alpha'}(\Omega)},$$

where

$$\sigma \leq (\alpha - 1) \left( r + 1 - \frac{d}{2} \right).$$

**Remark 2.1.** In our computations, we take $\sigma := \alpha - 1$, and $r := \frac{d}{2}$, the values suggested in [4]. However, higher values of $\sigma$ may be computationally feasible for laminar or nearly-laminar flows.

**Remark 2.2.** As Layton suggests, we can compute values of $u_h$ solving (5) as close to solutions $u$ of (1) as we wish by taking values of $\alpha$ large. However, in our computational experiments, we see that large values of $\alpha$ decrease the convergence rate of the Newton iteration.
3. A posteriori estimates for NSE with subgridscale viscosity

The residual-based a posteriori error estimator for the general nonlinear equation

\[ F(u) = 0 \]  \hspace{1cm} (13)

given by Verfurth [9] assumes that if \( u_0 \) solves (13), then the Fréchet derivative of \( F \) at \( u_0 \), \( DF(u_0) \) is a linear homeomorphism. Following is the argument central to the analysis presented in [9].

Let \( X \) and \( Y \) be two Banach spaces with norms \( \| \cdot \|_X, \| \cdot \|_Y \), respectively. Set \( B(u, R) \) to be the ball of radius \( R \) about \( u \), \( \mathcal{L}(X, Y) \) to be the space of continuous linear maps from \( X \) to \( Y \), \( \text{Isom}(X, Y) \) the space of linear homeomorphisms from \( X \) to \( Y \), and let \( X', Y' \) denote the dual spaces of \( X \) and \( Y \), respectively. If \( F \) in (13) is a continuously differentiable function from \( X \) into \( Y' \), we have the following result.
Fig. 11. Plots of the velocity field in Example 2 with $Re = 300$, $\alpha = 2.5$ and approximately 25,000 and 75,000 degrees of freedom, respectively.

Fig. 12. Plots of the velocity field in Example 2 with $Re = 300$, $\alpha = 3.0$ and approximately 25,000 and 75,000 degrees of freedom, respectively.

**Theorem 3.1** ([9]). Let $u_0 \in X$ be a regular solution of (13), i.e. $DF(u_0) \in \text{Isom}(X, Y')$. Assume that $DF$ is Lipschitz continuous, i.e. there exists $R_0 > 0$ such that

$$\gamma := \sup_{u \in B(u_0, R_0)} \frac{\|DF(u) - DF(u_0)\| L(X, Y')}{\|u - u_0\| X} < \infty.$$ 

Then for all $u \in B(u_0, R)$,

$$\frac{1}{2}\|DF(u_0)\| L(X, Y')^{-1}\|F(u)\| Y' \leq \|u - u_0\| X \leq 2\|DF(u_0)^{-1}\| L(Y', X)\|F(u)\| Y',$$

where $R = \min\{R_0, \gamma^{-1}\|DF(u_0)^{-1}\|^{-1}, 2\gamma^{-1}\|DF(u_0)\|\}$.

Although we have no indication as to the relative magnitudes of $DF(u_0)$, $DF(u_0)^{-1}$ the assumption that each of these is bounded yields a (robust) residual based a posteriori error estimator for (13).

Now, if $X_h$ is a finite dimensional subspace of the space $X$ given above, we assume that we can find a unique $u_h \in X_h$ solving $F_h(u_h) = 0$. Assuming some approximation properties associated with the space $X_h \subset X$ and noting that we cannot compute the residual $F(u_h)$ in (14) exactly, we can show the (non-robust) a posteriori error estimate

$$\|u - u_h\| X \leq C\|F_h(u_h)\| Y_h',$$

where $Y_h'$ is an appropriately chosen finite dimensional subspace of $Y'$. 
Fig. 13. Plots of the velocity field in Example 2 with \( Re = 300, \alpha = 3.5 \) and approximately 25,000 and 75,000 degrees of freedom, respectively.

Fig. 14. Plots of the velocity field in Example 2 with \( Re = 300, \alpha = 4.0 \) and approximately 25,000 and 75,000 degrees of freedom, respectively.

Finally, we consider the pair \((u, p) \in (H^1_0(\Omega))^2 \times L^2_0(\Omega)\) as the unique solution of (7) and \((u_h, p_h) \in X_h \times Q_h\) as the solution of (10). Using the notation of Verfurth [9] we define the element error estimator as

\[
\eta_T^2 := h_T^2 \| - \text{Re}^{-1} \Delta u_h + u_h \cdot \nabla u_h + \nabla p_h - \pi_{0,T}f \|_{L^2(T)}^2 + \| \nabla \cdot u_h \|_{L^2(T)}^2 + \| \text{AV}_a(u_h) \nabla u_h \|_{L^2(T)}^2 + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_h, \Omega} h_E \| (\text{Re}^{-1} n_E \cdot \nabla u_h - p_h n_E) \|_{L^2(E)}^2,
\]

where \( T \) is some element in the set \( T_h \), \( E \) denotes the face of an element with outward normal \( n_E \), \( \mathcal{E}(T) \cap \mathcal{E}_h, \Omega \) denotes the set of all edges which are not also contained in \( \partial \Omega \), and \( \pi \) is the projection operator.

Then, we have [1],

\[
\| u - u_h \|_{H^1(T)} + \| p - p_h \|_{L^2(T)} \leq C \| D F(u)^{-1} \|_{\mathcal{L}(V', X)} \eta_T.
\]

We also define the quantities

\[
\eta_{\text{tot}} := \sum_{T \in T_h} \eta_T, \quad \eta_{\text{max}} := \max_{T \in T_h} \eta_T.
\]
4. Computational experiments

For our computational experiments, we apply the $\alpha$-Laplacian based subgrid scale model with artificial viscosity inserted at a rate of $h_T^{\alpha-1} |\nabla u|^{\alpha-2}$ and use the Taylor–Hood finite element pair of piecewise quadratic polynomials for the velocity, coupled with piecewise linear polynomials for the pressure. We apply the Newton iteration scheme in Section 2 with relative tolerance $10^{-8}$. It is also important to note that after a mesh refinement, the initial guess for the new Newton iteration is defined as the solution to the variational problem on the old mesh. For our mesh refinement, we follow the recursive routine given in [12]. For each iteration we calculate $\eta_T$ for each triangle in the computational domain, and mark each triangle for refinement for which $\eta_T \geq 0.5\eta_{\text{max}}$. If not enough triangles are marked, we relax the criterion by dividing by 1.1. We apply a conservative marking strategy, by which we mark at least 7.5% and at most 15% of the triangles. We have not implemented a de-refinement strategy in our computational routine. Beginning with a specified initial mesh, we continue until the maximum value of 75,000 degrees of freedom is reached. Although computationally speaking more degrees of freedom are feasible via parallel processing, the results from this study provide a valuable insight into the effect of the parameter $\alpha$ on the approximation of viscous incompressible flow problems.

Example 1 (Flow over a Backward Facing Step). In this example, we consider the problem of approximating the flow over a backward facing step at $Re = 1000$. The initial computational domain and mesh are illustrated in Fig. 1. The left
Fig. 17. Log–log plots of error vs. degrees of freedom for Example 2 with $\alpha = 4.0$ and $Re = 300$ and 2000, respectively. Note that for the first few iterations, the a posteriori errors are very close, but as more iterations are performed the error for the case 2000 begins to increase, although the flow field is becoming closer to the actual physical flow.

Fig. 18. Initial mesh and computational domain used for Example 3. (inflow) boundary condition is set to be parabolic ($u_1 = 16(x - 1/2)(1 - x)$, $u_2 = 0$) and the right (outflow) boundary condition is a pure-Neumann or “free” boundary condition. All other boundaries are assumed to be no-slip. We compare the $\alpha$-Laplacian subgridscale models quantitatively, by noting the differences in a posteriori error estimates and number of Newton iterations for approximately 75,000 degrees of freedom. Adaptive FEM computations were performed using $\alpha = 2.5, 3.0, 3.5, 4.0, 4.5$, terminating in the first iteration containing more than 75,000 degrees of freedom. We compare the models qualitatively by comparing the separation layer to a fine, uniform mesh with no subgridscale artificial viscosity term. Fig. 2 displays the velocity fields for this example with approximately 75,000
degrees of freedom and uniformly spaced vertices. Note that in this and all subsequent vector plots, the vectors are scaled to be of equal length in order to represent the direction of the flow. For the plot in Fig. 2, we note the shape of the separation as well as the existence of the corner vortex at the lower left corner of the region beyond the step.

First, we note from Table 1 that as the value of $\alpha$ increases the a posteriori error estimate decreases. The behavior of the a posteriori error as the mesh is refined can be seen in Fig. 3, a log–log plot of the error versus the number of degrees of freedom. Note that for the value of $\alpha = 2.5$, the error is actually increasing as we refine the mesh. Qualitative differences for the case of 75,000 degrees of freedom can be noted in the bottom panels of Figs. 4–8. All values of $\alpha$ seem to adequately model the separation except for $\alpha = 2.5$. However, if we plot the velocity field for a coarser mesh in our iterative scheme, 25,000 degrees of freedom, as in the top panels of Figs. 4–8, we see that the values of $\alpha = 4.0, 4.5$ adequately model the separation, and others do not. We note that the decrease in the a posteriori error for the NSE with no subgridscale modeling term results from the absence of the stabilization term, and that this example is unique in that the flow can be resolved by direct numerical simulation.

**Example 2 (Flow Past a Circular Cylinder).** In this example, we consider the problem of approximating the flow past a circular cylinder. The initial computational domain and mesh are illustrated in Fig. 9. Again, the left (inflow) boundary condition is set to be parabolic ($u_1 = 4x(1 - x), u_2 = 0$) and the right (outflow) boundary condition is a pure-Neumann or “free” boundary condition. All other boundaries are assumed to be no-slip. First, setting $Re = 300$, we expect to see separation in the flow resulting in two recirculating eddies. Again, we compare results for $\alpha = 2.5, 3.0, 3.5, 4.0, 4.5$, and qualitatively compare these values by the downstream distances to the cores of...
these two eddies. Also, we note that in this case, for a uniform grid with approximately 75,000 degrees of freedom, direct numerical simulation fails. In Table 2, we see again that as $\alpha$ increases the a posteriori error decreases, and the average number of Newton iterations per nonlinear solve increases. In Fig. 10, a log–log plot of the error versus the number of degrees of freedom illustrates this, and follows the same pattern as in Example 1. The bottom panels
of Figs. 11–15 display the flow field for approximately 75,000 degrees of freedom, while the top panels display the flow field for approximately 25,000 degrees of freedom. Again, we see the same pattern as in Example 1, that the recirculating eddies are placed further downstream for higher values of $\alpha$.

We also include a calculation for this example for $\alpha = 4.0$ and $Re = 2000$. For this case we expect a turbulent wake downstream from the cylinder as opposed to the pattern for lower Reynolds numbers. We hope that our method can capture at least part of this phenomenon. Fig. 16 displays plots of the flow field for this example with 25,000 and 75,000 degrees of freedom, respectively. The simulated flow falls short of the expected turbulent wake pattern, but at least captures the expected separation. Fig. 17 illustrates another important aspect of this example is that as the mesh is further refined, the a posteriori error increases, rather than decreases. We assume that one reason for this behavior is that the “higher” Reynolds number produces higher Péclet numbers $Re \cdot h_T$, which interferes with the efficacy of the a posteriori error estimator. This phenomenon is investigated for convection–diffusion equations in [19].

**Example 3 (Full Cavity Flow).** In this example, we consider the problem of approximating the flow over a cavity at Reynolds number $10^5/1.75$, as given in [1]. The computational domain and initial mesh are illustrated in Fig. 18.
Again, the left (inflow) boundary condition is set to be parabolic \((u_1 = 4(x - 5)(6 - x), u_2 = 0)\) and the right (outflow) boundary condition is a pure-Neumann or “free” boundary condition. All other boundaries are assumed to be no-slip. We note that in [1], parabolic outflow boundary conditions were imposed. Instead, we choose the “free” boundary condition at the outflow, as this is physically more realistic for a cavity flow problem. Also, we note that in [1], the authors chose \(\alpha = 3.0\) with leading coefficient \(\mu(h) = h^3\), which is computationally feasible, but does not follow the theory presented in [4]. For practical purposes, however, one may choose any power of \(h\) for which the Newton iterations converge, although there is at present no theoretical justification of this phenomenon.

In Table 3, we see a different pattern than in the previous examples. As \(\alpha\) increases the a posteriori error increases, and so does the average number of Newton iterations per nonlinear solve. In Fig. 19, we see that as the mesh is further refined, the a posteriori error increases, although the flow field seems to be getting closer to the actual flow. Again, we assume that the reason for this behavior is that the “higher” Reynolds number produces higher Péclet numbers \(Re \cdot h_T\), which interferes with the efficacy of the a posteriori error estimator. However, in view of Figs. 20–24, we again see that higher values of \(\alpha\) produce flows that intuitively resemble an expected physical flow. In fact, for \(\alpha = 4.0, 4.5\), the flows presented in Figs. 23 and 24 for 75,000 degrees of freedom include the vortex in the center of the cavity, the
two corner vortices at the bottom left and right of the cavity, as well as a separation layer at the top right of the cavity. In Fig. 25, we display a magnification of the top right portion of the flow for $\alpha = 4.0, 4.5$ at approximately 75,000 degrees of freedom.
5. Conclusions

In this numerical study, we have shown that adaptive mesh refinement combined with subgridscale models produces a physically realistic flow which is cheaper and more stable than either method alone. Also, we have seen that by varying the parameter \(\alpha\) in an \(\alpha\)-Laplacian subgridscale model, we can produce flows which are more physically realistic in exchange for a relatively small increase in the number of Newton iterations per nonlinear solve. Therefore, subgridscale models which use values of \(\alpha\) higher than 3 (Smagorinsky’s model) are preferable.

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Fig. 24. Flow field for Example 3 with \(\alpha = 4.5\) with 25,000 and 75,000 degrees of freedom.
Fig. 25. Partial enlargements of the flow field about the forward facing step in Example 3. Top: $\alpha = 4.0$; bottom: $\alpha = 4.5$.

References