Some problems and results related to nil and graded-nilpotent rings

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1 Introduction

A nilpotent ring is a ring $R$ such that $R^n = \{0\}$ for some natural number $n > 1$, where $R^n$ denotes the ideal of $R$ consisting of all sums and differences of products consisting of $n$ factors from $R$, and a nilpotent algebra is defined similarly. The degree of nilpotency (or class) of a nilpotent ring $R$ is the smallest such exponent $n$. For example, the family 

$$\left\{ \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} : q \in \mathbb{Q} \right\}$$

forms a nilpotent subring of $M_2(\mathbb{Q})$, and the degree of nilpotency of this subring is 2. Observe that if a ring $R$ is nilpotent then $M_i(R)$ is nilpotent for all $i \in \mathbb{N}$ [5].

A locally nilpotent ring is a ring $R$ such that every finitely generated subring of $R$ is nilpotent, and a locally nilpotent algebra is defined similarly. For example, the ring generated by the family \( \{x_i\}_{i \in \mathbb{N}} \) whereby $x_j x_i = 0$ if $j > i$ for all indices $i, j \in \mathbb{N}$ is locally nilpotent [24]. Every nilpotent ring is locally nilpotent. However, the converse does not hold. For example, letting $A_\mathbb{k}$ denote the commutative algebra over a field $\mathbb{k}$ such that the family of symbols $U = \{u_r\}_{r \in (0,1)}$ indexed by the open interval $(0,1)$ is a basis for $A_\mathbb{k}$, with multiplication defined so that $u_x u_y = u_{x+y}$ if $x+y < 1$ and $u_x u_y = 0$ otherwise, $A_\mathbb{k}$ is locally nilpotent but not nilpotent [12]. If a ring $R$ is locally nilpotent, then $M_i(R)$ is locally nilpotent for all $i \in \mathbb{N}$ [5].

A nilpotent element in a ring $R$ is an element $r \in R$ such that $r^n r = 0$ for some $n_r \in \mathbb{N}$. The smallest such $n_r \in \mathbb{N}$ is referred to as the nilpotency index of $r \in R$. A nil ring (or nilring) is a ring $R$ such that every element of $R$ is nilpotent, and a nil algebra (or nilalgebra) is defined similarly. For example, the matrix ring $\left\{ \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} : q \in \mathbb{Q} \right\}$ given above is a nilring, and the nilpotency index of each nonzero element in this matrix ring is 2. If there exists a fixed natural number $n$ such that $r^n = 0$ for each element $r$ in a nilring $R$, then the least such exponent $n$ is called the nil exponent (or nil exponent) of $R$. As indicated in [12], if $R$ is nilpotent (resp. locally nilpotent, nil) then:

1. Every subring of $R$ is nilpotent (resp. locally nilpotent, nil),

2. Every homomorphic image of $R$ is nilpotent (resp. locally nilpotent, nil),

3. If $R$ is an ideal of the ring $R'$ and $R'/R$ is nilpotent (resp. locally nilpotent, nil), then so is $R'$.

Observe that every locally nilpotent ring is nil. The difficult problem [12, 22, 25] of determining whether or not there exists a nil ring which is not locally nilpotent was considered by Jakob Levitzki as early as 1945. In 1964, Golod solved this problem using the Golod-Shafarevich construction, by constructing the first example of a nil ring which is not locally nilpotent. For decades, the construction used by Golod was essentially the only known construction of a nil ring which is not locally nilpotent [22, 25, 36]. The Golod-Shafarevich
construction, which gives a sufficient condition for an algebra defined by generators and relations to be infinite-dimensional, will be discussed in Section 3.

A ring $R$ is Jacobson radical (or simply radical) if for every $r \in R$ there exists an element $r' \in R$ such that $r + r' + rr' = 0$ [34]. In other words, $R$ forms a group, called the adjoint group $R^\circ$ of $R$, under the binary relation $\circ$ on $R$ whereby $a \circ b = a + b + ab$ for $a, b \in R$ [4]. The term Jacobson radical is also used to refer to a certain set associated with a ring, but we will not use this term in this sense. Every nil ring is Jacobson radical [4, 34, 35]. To briefly show this, letting $R$ be a nil ring, letting $r \in R$ be arbitrary, and letting $n_r \in \mathbb{N}$ be the nilpotency index of $r$, we have that $r \circ (-r + r^2 - \cdots \pm r^{n_r-1}) = 0$. However it is not the case that every Jacobson radical ring is a nil ring. For example, the subring $R = \left\{ xy : x, y \in \mathbb{Z}, p \mid x, p \nmid y \right\}$ of $\mathbb{Q}$ is a Jacobson radical ring which is not a nil ring, where $p$ is a fixed prime [5, 35]. We will now briefly show this. Let $x$ and $y$ be integers such that $p \mid x$ and $p \nmid y$, and consider the expression $x + \frac{x}{x+y} + \left( \frac{x}{x+y} \right) \left( \frac{-x}{x+y} \right) = 0$ we have that $R$ is Jacobson radical. However, the element $\frac{p}{p+1}$ in $R$ is not nilpotent, so $R$ is not a nilring. A Jacobson radical ring $R$ is nil if and only if every subring of $R$ is radical [5]. If a ring $R$ is radical, then $M_i(R)$ is radical for all $i \in \mathbb{N}$ [5].

Our above discussion is summarized in the below chain of implications; as discussed above, for each such implication, the converse is not true.

$$\text{nilpotent} \implies \text{locally nilpotent} \implies \text{nil} \implies \text{Jacobson radical} \quad (1)$$

There are many very well-known theorems which prove “partial converses” with respect to the above chain of implications. For example, there are well-known theorems which state that nil implies nilpotent given certain additional conditions [19]. In [9] Cherlin proved that every $\aleph_0$-categorical nil ring is nilpotent. The well-known Nagata-Higman theorem [9] states that if $R$ is a nil ring of nilexponent $n \in \mathbb{N}$ such that $R$ is either of characteristic 0 or of characteristic $p > n$, then $R$ is nilpotent. It is well known that nil algebras of bounded index are locally nilpotent [21]. Nil rings satisfying the ascending chain condition on left annihilators are locally nilpotent, and nil rings which satisfy a polynomial identity are locally nilpotent [19]. Every Noetherian nil algebra is nilpotent and every Artinian nil algebra is nilpotent [25].

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1 Letting $\kappa$ be a cardinal, a first-order structure $\mathcal{M}$ in a countable language is said to be $\kappa$-categorical if there is a unique (up to isomorphism) model of $\text{Th}(\mathcal{M})$ of cardinality $\kappa$.
Interestingly, of the classes of rings indicated in (1), the only class which is not known to be preserved under the process of taking finite matrix rings is the class of nil rings [28]. In fact, the problem of determining whether or not $M_2(R)$ is nil for a nil ring $R$ remains open [28, 34]. It is well known that this problem is equivalent to the famous Köthe conjecture, which is perhaps the most important unsolved conjecture concerning nil rings [34]. The Köthe conjecture states that: if a ring $R$ has no nonzero nil ideals, then $R$ has no nonzero nil one-sided ideals. Also note that the problem of determining whether or not $M_2(R)$ is nil for a nil ring $R$ is equivalent to the problem of determining whether or not $M_n(R)$ is nil for a nil ring $R$ for all indices $n \in \mathbb{N}$ [28]. Although it is unknown whether or not the property of being nil is not preserved under the process of taking finite matrix rings, the property of being a nil algebra over an uncountable field is preserved under the process of taking finite matrix rings, and more generally, the property of being an algebraic algebra over an uncountable field is preserved under the process of taking finite matrix rings [29].

An algebraic algebra is an algebra $A$ over a field $\mathbb{k}$ such that each element $a \in A$ is algebraic over $\mathbb{k}$.

A graded ring is a ring $R$ that is a direct sum

$$R = \bigoplus_{i \in \mathbb{N}_0} R_i$$

doing abelian groups such that $R_i R_j \subseteq R_{i+j}$ for all indices $i, j \in \mathbb{N}_0$. For an index $i \in \mathbb{N}_0$, elements of the graded component $R_i$ are referred to as the homogeneous elements of degree $i$ [18]. A graded $\mathbb{k}$-algebra is a $\mathbb{k}$-algebra that is a direct sum

$$A = \bigoplus_{i \in \mathbb{N}_0} A_i$$

doing $\mathbb{k}$-vector spaces such that $A_i A_j \subseteq A_{i+j}$ for all indices $i, j \in \mathbb{N}_0$. A graded ring $R = \bigoplus_{i \in \mathbb{N}_0} R_i$ is said to be positively graded if $R_0$ is trivial, and a graded $\mathbb{k}$-algebra $A = \bigoplus_{i \in \mathbb{N}_0} A_i$ is said to be connected if $A_0 = \mathbb{k}$. A graded ring $R$ is graded-nilpotent if every subring of $R$ which is generated by homogeneous elements of the same degree is nilpotent [7, 17]. A graded-nilpotent algebra may be defined similarly.

The concept of a graded ideal will be very important in our proof of the Golod-Shafarevich theorem. The concept of a graded ideal will also be important in Section 4 in our discussion of graded monomial algebras. Given a graded ring $R = \bigoplus_{i \in \mathbb{N}_0} R_i$ and a (proper) ideal $I$ of $R$, the ideal $I$ is said to be graded (or homogeneous) if the following equivalent conditions hold [27]:

1. If $f \in I$, then every homogeneous component of $f \in R$ is in $I$,
2. $I = \bigoplus_{i \in \mathbb{N}_0} I_i$ where $I_i = R_i \cap I$ for all indices $i$,
3. The ideal generated by all homogeneous elements in $I$ is equal to $I$,
4. $I$ is homogeneously generated.
whether or not a monomial algebra defined by an automatic word is graded-nilpotent. This ideal is referred to as the augmentation ideal of \( R = \bigoplus_{i \in \mathbb{N}_0} R_i \). This ideal is preserved under graded homomorphic images, and we will show that if nilpotency is preserved under the process of taking finite matrix rings, graded-nilpotency is preserved under graded homomorphic images, and we will show that if \( R \) is a graded-nilpotent ring, and \( R \) is a graded ideal in a graded ring \( S \), then \( S \) is graded-nilpotent. Section 3 is purely expository: in this section, we will describe the Golod-Shafarevish construction, and Golod’s construction of a finitely generated nil algebra which is not nilpotent. In Section 4, we will explore the problem of how one can decide whether or not a monomial algebra defined by an automatic word is graded-nilpotent.

\[
R/I = \bigoplus_{i \in \mathbb{N}_0} (R/I)_i = \bigoplus_{i \in \mathbb{N}_0} (R_i/I_i).
\] (2)

Given an arbitrary ring \( k \), and given an arbitrary family \( \{x_i\}_{i \in I} \) of indeterminates, letting \( I \) be an index set, as in [23] we define the free \( k \)-algebra

\[
k\langle x_i : i \in I \rangle
\]
generated by \( \{x_i\}_{i \in I} \) to be the \( k \)-algebra consisting of polynomials in the noncommuting variables in \( \{x_i\}_{i \in I} \) with coefficients from \( k \), such that the coefficients commute with each variable in \( \{x_i\}_{i \in I} \). The word “free” in the term free \( k \)-algebra refers to the following universal property. Given an arbitrary ring homomorphism \( \phi : k \rightarrow k' \) and a family \( \{a_i\}_{i \in I} \) consisting of elements in \( k' \) indexed by \( I \) each element of which commutes with each element of \( \phi(k) \), then there exists a unique morphism \( \psi : R \rightarrow k' \) such that the restriction of \( \psi \) to \( k \) is equal to \( \phi \) and \( \psi(x_i) = a_i \) for all indices \( i \in I \) [23]. Equivalently, an arbitrary mapping \( f : \{x_i\}_{i \in I} \rightarrow \mathcal{R} \) into a \( k \)-algebra \( \mathcal{R} \) such that the image of \( \{x_i\}_{i \in I} \) centralizes \( k \) can be extended to a unique \( k \)-algebra homomorphism of \( k\langle x_i : i \in I \rangle \) into \( \mathcal{R} \) [10]. Observe that it is also natural to define free \( k \)-algebras as in terms of monoids. Given a monoid \( M \), let \( k[M] \) denote the monoid algebra of \( M \) over \( k \). Letting \( M \) be the free monoid \( \{x_i\}_{i \in I}^* \), then \( k[M] \) is precisely the polynomial ring \( k\langle x_i : i \in I \rangle \). Observe that \( k\langle x, y \rangle \) contains isomorphic copies of \( k\langle x_0, x_1, \ldots, x_n \rangle \) for all \( n \in \mathbb{N}_0 \), and contains an isomorphic copy of \( k\langle x_0, x_1, \ldots \rangle \) [23]. Given a family \( F = \{f_j\}_{j \in J} \) contained in the free \( k \)-algebra \( k\langle x_i : i \in I \rangle \) where \( J \) is an index set, letting \( \langle F \rangle \) denote the ideal generated by \( F \), the quotient algebra

\[
k\langle x_i : i \in I \rangle / \langle F \rangle
\]
is referred to as the algebra generated over \( k \) by \( \{x_i\}_{i \in I} \) with relations \( F \) [23] and is a central object of study in our paper.

There are many open problems concerning nil rings. In this paper we will explore problems related to nil and graded-nilpotent rings, such as the Kurosh problem of determining whether or not there exists a nil ring which is not locally nilpotent. In Section 2, we will prove some basic properties concerning graded-nilpotency; in particular, we will show that graded-nilpotency is preserved under the process of taking finite matrix rings, graded-nilpotency is preserved under graded homomorphic images, and we will show that if \( R \) is a graded-nilpotent ring, and \( R \) is a graded ideal in a graded ring \( S \), and if \( S/R \) is graded-nilpotent, then \( S \) is graded-nilpotent. Section 3 is purely expository: in this section, we will describe the Golod-Shafarevish construction, and Golod’s construction of a finitely generated nil algebra which is not nilpotent. In Section 4, we will explore the problem of how one can decide whether or not a monomial algebra defined by an automatic word is graded-nilpotent.
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2 Graded-Nilpotent Rings

Numerous open problems concerning graded-nilpotent rings were recently solved in Iterative algebras by Jason Bell and Blake Madill [7]. For example, in [7] it is shown that there exists a graded-nilpotent algebra which is not Jacobson radical. In this section, we will prove some basic properties concerning the class of graded-nilpotent rings.

Since nilpotency, local nilpotency, the property of being a Jacobson radical ring, and the property of being an algebraic algebra over an uncountable field are all properties that are preserved under the process of taking finite matrix rings, it seems natural to ask: is graded-nilpotency preserved under the process of taking finite matrix rings?

Let \( R = \bigoplus_{i \in \mathbb{N}_0} R_i \) be a graded ring. Letting \( n \in \mathbb{N} \) be arbitrary, the finite matrix ring \( M_n(R) \) inherits the grading from \( R \) canonically in the following sense. For all indices \( i \in \mathbb{N}_0 \), the \( i \)th graded component \( (M_n(R))_i \) of \( M_n(R) \) consists of all matrices \( M \) in \( M_n(R) \) such that each entry of \( M \) is in \( R_i \). Observe that \( (M_n(R))_i(M_n(R))_j \subseteq (M_n(R))_{i+j} \) for arbitrary indices \( i, j \in \mathbb{N}_0 \).

**Theorem 1.** Graded-nilpotency is preserved under the process of taking finite matrix rings.

**Proof.** Let \( R = \bigoplus_{i \in \mathbb{N}_0} R_i \) be an arbitrary graded-nilpotent ring, and let \( n \in \mathbb{N} \) be arbitrary. Consider the canonical grading

\[
M_n(R) = \bigoplus_{i \in \mathbb{N}_0} (M_n(R))_i
\]

of the matrix ring \( M_n(R) \) inherited from the graded ring \( R \). We will now show that the graded ring \( M_n(R) = \bigoplus_{i \in \mathbb{N}_0} (M_n(R))_i \), is graded-nilpotent. We must show that every subring of \( M_n(R) \) which is generated by homogeneous elements of the same degree is nilpotent. So let \( \{M_k\}_{k \in K} \) be a family of matrices such that each matrix \( M_k \) in this set is in the graded component \( (M_n(R))_m \), where \( m \) is a fixed index in \( \mathbb{N}_0 \), and \( K \) is some index set. So, this family is an arbitrary set of homogeneous elements of the same degree. Now let \( R \) denote the subring of \( R \) generated by \( \{M_k\}_{k \in K} \). Write \( M_k = ((m_{k,i,j})_{1 \leq i,j \leq n}) \) for \( k \in K \). Now consider the set

\[
S = \{(m_{k,i,j}) : k \in K, 1 \leq i, j \leq n\}
\]

consisting of the entries of the matrices in the family \( \{M_k\}_{k \in K} \). Since each matrix in this family is of degree \( m \in \mathbb{N}_0 \), each element in \( S \) is of degree \( m \). Now let \( \mathcal{S} \) denote the subring of \( R \) generated by \( S \). Since \( R \) is graded-nilpotent, we know that \( \mathcal{S} \) is nilpotent. So let \( \ell \in \mathbb{N} \) be such that \( \mathcal{S}^\ell \) is trivial. Now consider the expression \( R^\ell \). Each element in \( R^\ell \) is a sum/difference of products consisting of \( \ell \in \mathbb{N} \) matrices in \( R \). Since \( \mathcal{S} \) is generated by \( \{M_k\}_{k \in K} \), each entry in each element in \( \mathcal{S} \) is in the ring \( \mathcal{S} \). So each entry in a product of \( \ell \in \mathbb{N} \) matrices in \( \{M_k\}_{k \in K} \) is in \( \mathcal{S}^\ell = \{0\} \), and thus \( R^\ell \) is trivial as desired. \( \square \)
As indicated in Section 1, an arbitrary nilpotent (resp. locally nilpotent, nil) ring \( R \) satisfies the following: subrings of \( R \) are nilpotent (resp. locally nilpotent, nil), homomorphic images of \( R \) are nilpotent (resp. locally nilpotent, nil), and if \( R \) is an ideal in a ring \( R' \), and \( R'/R \) is nilpotent (resp. locally nilpotent, nil), then \( R' \) is nilpotent (resp. locally nilpotent, nil). So it seems natural to consider whether or not the class of graded-nilpotent rings is closed under such processes.

Given a graded ring \( R = \bigoplus_{i \in \mathbb{N}_0} R_i \), a subring \( S \) of \( R \) inherits the grading of \( R \) if for each element \( s \in S \), all the homogeneous components of \( s \in R \) are in \( S \). So graded-nilpotency is not in general preserved under taking subrings in the sense that a subring of a graded ring \( R \) does not necessarily inherit the grading of \( R \). However, a subring \( S \) of a graded-nilpotent ring \( R = \bigoplus_{i \in \mathbb{N}_0} R_i \) which inherits the grading of \( R \) is also graded-nilpotent.

Given graded rings \( R = \bigoplus_{i \in \mathbb{N}_0} R_i \) and \( S = \bigoplus_{i \in \mathbb{N}_0} S_i \), a ring homomorphism \( f : R \to S \) is said to be graded if \( f(R_i) \subseteq S_i \) for all indices \( i \in \mathbb{N}_0 \). So graded-nilpotency is not in general preserved under homomorphic images in the sense that the homomorphic image of a graded ring \( R \) does not necessarily inherit the grading of \( R \). This leads us to the following question:

**Question 1.** Is graded-nilpotency preserved under graded homomorphic images?

Let \( R = \bigoplus_{i \in \mathbb{N}_0} R_i \) and \( S = \bigoplus_{i \in \mathbb{N}_0} S_i \) be graded rings, and suppose that \( R \) is graded-nilpotent. Suppose that \( f : R \to S \) is a graded ring homomorphism. Consider the expression \( \phi(R) \). Let \( I \) be an index set, and let \( \{r_i\}_{i \in I} \) be a family contained in \( R \) such that \( \{f(r_i)\}_{i \in I} \) is a set of homogeneous elements in \( S \) which are all of positive degree \( n \in \mathbb{N} \). Let \( S \) denote the subring of \( S \) generated by \( \{f(r_i)\}_{i \in I} \subseteq S \). So it remains to consider whether or not \( S \) is nilpotent.

By definition of a direct sum, since \( S = \bigoplus_{i \in \mathbb{N}_0} S_i \), we have that \( S_i \cap S_j = \{0\} \) for distinct indices \( i, j \in \mathbb{N}_0 \). Since \( f(r_i) \) is homogeneous of fixed positive degree \( n \in \mathbb{N} \) for all \( i \in I \), it must be the case that \( r_i \) is homogeneous of degree \( n \) for all \( i \), because otherwise, if \( r_i \) were of degree \( m \neq n \) for some index \( i \in I \), with \( r_i \in R_m \), we would have that \( f(r_i) \in S_m \), but since \( f(r_i) \) is homogeneous of degree \( n \in \mathbb{N}_0 \), we would have that \( f(r_i) \in S_m \cap S_n = \{0\} \), and we would thus have that \( f(r_i) = 0 \), contradicting that \( f(r_i) \) is of positive degree. So since \( r_i \) is of degree \( n \) for all indices \( i \in I \), we have that \( S \) is nilpotent since \( R \) is graded-nilpotent and \( f \) is a morphism, thus answering Question 1 in the affirmative.

Again let \( R = \bigoplus_{i \in \mathbb{N}_0} R_i \) be a graded-nilpotent ring, and suppose that \( R \) is a graded ideal in a graded ring \( S = \bigoplus_{i \in \mathbb{N}_0} S_i \). Consider the canonical grading of \( S/R \):

\[
S/R = \bigoplus_{i \in \mathbb{N}_0} (S_i/R_i).
\]

It seems natural to ask:

**Question 2.** If \( S/R \) is graded-nilpotent, does it follow that \( S \) is graded-nilpotent?

Suppose that

\[
S/R = \bigoplus_{i \in \mathbb{N}_0} (S_i/R_i)
\]
is graded-nilpotent, letting $R$ and $S$ be as given above. Now let $I$ be an index set, and let the family $\{s_i\}_{i \in I}$ be contained in $S_n$ for some positive index $n \in \mathbb{N}$. Let $\mathcal{S}$ denote the subring of $S$ generated by the family $\{s_i\}_{i \in I}$. To answer Question 2 in the affirmative, it remains to prove that $\mathcal{S}$ is nilpotent. We know that ring generated by the family

$$\{s_i + R_n\}_{i \in I}$$

is nilpotent. We may thus deduce that there exists some natural number $m_1$ such that:

$$\mathcal{S}^{m_1} \subseteq R_n.$$

But since $R = \bigoplus_{i \in \mathbb{N}_0} R_i$ is graded nilpotent, we have that there exists some natural number $m_2$ such that:

$$R_n^{m_2} = \{0\}.$$

We thus have that

$$\mathcal{S}^{m_1 m_2} \subseteq R_n^{m_2} = \{0\}$$

thus proving that $\mathcal{S}$ is nilpotent as desired. We conclude this section with the following open problem considered by Jason Bell:

**Question 3.** Do graded-nilpotent algebras always have rational Hilbert series? [6]
3 The Golod-Shafarevich Construction

As discussed in the previous section, it is not the case that every nil ring is a nilpotent ring. So it seems natural to ask: What nil rings are nilpotent? This leads us to one of the most famous problems in ring theory, which was solved in 1964 by Golod: Is every finitely generated nil \( k \)-algebra finite-dimensional, and hence nilpotent? Equivalently, is every nil \( k \)-algebra locally nilpotent? This problem is a special case of the following famous problem, which was considered by Kurosh as early as 1940, and was independently formulated by Levitzki:

**Kurosh-Levitzki Problem**: Is every finitely generated algebraic \( k \)-algebra finite dimensional?

Note that the following two problems which are related to Kurosh’s Problem (KP) remain open [37]: Is every finitely presented algebraic algebra finite dimensional? Is every finitely presented nil algebra finite dimensional? Kurosh’s Problem holds for finitely generated algebraic algebras of bounded degree: in particular, by Kaplansky’s theorem, given a finitely generated algebra \( A \) over a field, if there exists a fixed natural number \( n \in \mathbb{N} \) such that \( a^n = 0 \) for all \( a \in A \), then \( A \) is finite-dimensional [32]. The Kurosh-Levitzki Problem was solved in the negative by the following seminal theorem, which was proven using the Golod-Shafarevich construction:

**Golod’s theorem** [16]: For an arbitrary field \( k \) there exists an infinite-dimensional finitely generated (graded) nil \( k \)-algebra \( A \), such that \( A \) is not locally nilpotent, and thus non-nilpotent.

Informally, the Golod-Shafarevich theorem states that if the number of relations of a given degree which define an algebra \( A \) is “small enough”, then \( A \) is infinite-dimensional. Intuitively, the key idea behind the negative solution of the Kurosh problem using the Golod-Shafarevich construction is to construct an algebra whose relations are sufficiently “sparse” for the algebra to be infinite-dimensional but which is nevertheless nil; i.e., an algebra \( A \) with enough relations so that \( A \) is nil but not with so many that \( A \) is finite-dimensional [6, 29, 30]. The main idea behind the technique used by Golod and Shafarevich arose from the study of Galois groups of algebraic extensions of \( \mathbb{Q} \) [32].

3.1 Some Preliminaries on Formal Power Series

Letting \( R \) be a commutative ring, the ring of formal power series in the variable \( t \) over \( R \) is denoted by \( R[[t]] \). If \( R \) is a commutative ordered ring, letting \( \leq \) denote the underlying total order of \( R \), letting

\[
\sum_{i \in \mathbb{N}_0} a_i t^i, \sum_{i \in \mathbb{N}_0} b_i t^i \in R[[t]]
\]

9
be arbitrary, define the partial order $\leq$ on $R[[t]]$ so that
\[ \sum_{i \in \mathbb{N}_0} a_i t^i \leq \sum_{i \in \mathbb{N}_0} b_i t^i \]
if and only if
\[ (a_i)_{i \in \mathbb{N}_0} \leq (b_i)_{i \in \mathbb{N}_0} \]
where the order relation $\leq$ in this latter inequality denotes the product order (coordinatewise order) on the cartesian product $R^\omega$. An element
\[ \sum_{i=0}^{\infty} r_i t^i \]
in $R[[t]]$ has an inverse in $R[[t]]$ if and only if $r_0$ is invertible in $R$. The coefficients of
\[ (\sum_{i=0}^{\infty} r_i t^i)^{-1} = \sum_{i=0}^{\infty} s_i t^i \]
are given recursively as follows: $s_0 = r_0^{-1}$, and $s_i = -r_0^{-1} \sum_{j=1}^{i} r_j s_{i-j}$ for all indices $i \in \mathbb{N}$.

Let $A$ be a connected graded algebra over a field $\mathbb{k}$, such that $A$ is generated by a finite set of elements of positive degree, writing $A = \bigoplus_{i \in \mathbb{N}_0} A_i$. Then the Hilbert function
\[ H_A : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \]
maps $i \in \mathbb{N}_0$ to $\dim_\mathbb{k} A_i$ for all indices $i$, and the Hilbert series $F_A(t)$ is defined as follows:
\[ F_A(t) = \sum_{i \in \mathbb{N}_0} H_A(i) t^i. \]

Although our presentation of the proof of the Golod-Shafarevich theorem does not explicitly use the theory of Hilbert series, it is convenient to use notation and terminology commonly used in the theory of Hilbert series.

### 3.2 Proof of the Golod-Shafarevich Theorem

Let $\mathbb{k}$ be an arbitrary field and let $d \in \mathbb{N}$. Write $T = \mathbb{k}\langle x_1, x_2, \cdots, x_d \rangle$. Here, $T$ has a very natural structure as a connected graded $\mathbb{k}$-algebra. The canonical grading of the free algebra $T$ is described below. Write $T_0 = \mathbb{k}$. For $i \in \mathbb{N}$, let $T_i$ denote the $\mathbb{k}$-vector spanned by the set consisting of the $d^i$ products consisting of exactly $i$ not necessarily distinct elements from $\{x_1, x_2, \cdots, x_d\}$. We thus have that
\[ T = \bigoplus_{i \in \mathbb{N}_0} T_i \]
with $T_i T_j \subseteq T_{i+j}$ for arbitrary indices $i, j \in \mathbb{N}_0$. As a vector space, $T_i$ is of dimension $d^i$ for $i \in \mathbb{N}$. Let $\{f_i\}_{i \in \mathbb{N}}$ be a family consisting of homogeneous elements of $T$, with $f_i \in T_n$. 

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for all indices $i \in \mathbb{N}$, and $2 \leq n_1 \leq n_2 \leq \cdots$. Also let $r_i$ be the number of elements in the family $\{n_j\}_{j \in \mathbb{N}}$ which are equal to $i$. Let $\mathcal{U} = \langle f_1, f_2, \cdots \rangle$ denote the (two-sided) ideal of $T$ generated by the family $\{f_i\}_{i \in \mathbb{N}}$.

Now, let $A$ denote the ring generated over $k$ by $\{x_1, x_2, \cdots, x_d\}$ with relations $\{f_i\}_{i \in \mathbb{N}}$. That is, $A = T/\mathcal{U}$. Since $\mathcal{U}$ is homogeneously generated, $\mathcal{U}$ is a graded ideal (see Section 1): writing $\mathcal{U}_i = \mathcal{U} \cap T_i \leq T_i$, we have $\mathcal{U} = \bigoplus_{i \in \mathbb{N}_0} \mathcal{U}_i$. Now consider the canonical grading of the quotient ring $T/\mathcal{U}$, writing $A_i$ in place of $T_i/\mathcal{U}_i$:

$$A = \bigoplus_{i \in \mathbb{N}_0} A_i.$$ 

Observe that since $T_i$ is finite-dimensional for all indices $i$, $A_i$ is finite dimensional for all indices $i$, and observe that since $T$ is generated by a finite set of elements of positive degree, $A$ is also generated by a finite set of elements of positive degree. The following famous theorem gives a sufficient condition for $A$ to be infinite dimensional over $k$ [18].

The Golod-Shafarevich theorem:

1. For $n \in \mathbb{N}$, $\mathcal{H}_A(n) \geq d\mathcal{H}_A(n - 1) - \sum_{n_i \leq n} \mathcal{H}_A(n - n_i)$.

2. If $r_i \leq \left(\frac{d-1}{2}\right)^2$ for each index $i$, then $A$ is infinite dimensional over $k$.

We will now prove the above theorem based on a proof given in Herstein’s classic Noncommutative rings [18]. The basic strategy used in the proof of the Golod-Shafarevich theorem is the construction of linear mappings $\phi$ and $\psi$ such that the sequence

$$\bigoplus_{n_i \leq n} A_{n-n_i} \xrightarrow{\phi} A_{n-1} \oplus A_{n-1} \oplus \cdots \oplus A_{n-1} \xrightarrow{\psi} A_n \to 0$$

is exact. The first inequality given in the above formulation of the Golod-Shafarevich theorem follows from the fact that the above sequence is exact, as will be discussed later (see Lemma 5 below). The majority of our proof consists of proving that the mappings $\phi$ and $\psi$ are well-defined, and proving that the above sequence is exact. To define $\phi$ and $\psi$, as in [18], we begin by defining a non-exact sequence

$$\bigoplus_{n_i \leq n} T_{n-n_i} \xrightarrow{\Phi} T_{n-1} \oplus T_{n-1} \oplus \cdots \oplus T_{n-1} \xrightarrow{\Psi} T_n \to 0$$

where $\Phi$ and $\Psi$ are linear mappings defined below. Define $\Psi$ so that

$$\Psi(t_1, t_2, \cdots, t_d) = \sum_{i=1}^d t_i x_i$$

for arbitrary $t_1, t_2, \cdots, t_d \in T_{n-1}$. Now, let

$$(s_{n-n_1}, s_{n-n_2}, \cdots, s_{n-n_k}, 0, 0, \cdots)$$
be an arbitrary tuple in the direct sum $\bigoplus_{n_i \leq n} T_{n-n_i}$, with $s_{n-n_i} \in T_{n-n_i}$ for all indices $j$. We thus have that $s_{n-n_i}$ is of degree $n-n_i$ for all indices $j$. Since $f_i$ is homogeneous of degree $n_i$ for all indices $i$, we have that $s_{n-n_i} f_i$ is homogeneous of degree $n$ for all indices $j \in \{1, 2, \cdots, k\}$. Therefore, the finite sum $\sum_{n_i \leq n} s_{n-n_i} f_i$ is in $T_n$ and we thus have that there are unique elements $u_1, u_2, \cdots, u_d \in T_{n-1}$ such that:

$$\sum_{n_i \leq n} s_{n-n_i} f_i = \sum_{i=1}^d u_i x_i.$$ 

Now define $\Phi$ as follows:

$$\Phi(s_{n-n_1}, s_{n-n_2}, \cdots, s_{n-n_k}, 0, 0, \cdots) = (u_1, u_2, \cdots, u_d).$$

We have that $\Phi$ and $\Psi$ are both linear and well-defined. Now define

$$\psi : A_{n-1} \oplus A_{n-1} \oplus \cdots \oplus A_{n-1} \rightarrow A_n$$

so that given an arbitrary sequence

$$(t_1 + \mathcal{U}_{n-1}, t_2 + \mathcal{U}_{n-1}, \cdots, t_d + \mathcal{U}_{n-1})$$

where $t_1, t_2, \cdots, t_d \in T_{n-1}$,

$$\psi(t_1 + \mathcal{U}_{n-1}, t_2 + \mathcal{U}_{n-1}, \cdots, t_d + \mathcal{U}_{n-1}) = \left(\sum_{i=1}^d t_i x_i\right) + \mathcal{U}_{n}.$$

Given an arbitrary tuple

$$(s_{n-n_1} + \mathcal{U}_{n-n_1}, s_{n-n_2} + \mathcal{U}_{n-n_2}, \cdots, s_{n-n_k} + \mathcal{U}_{n-n_k}, 0 + \mathcal{U}_{n-n_{ik+1}}, 0 + \mathcal{U}_{n-n_{ik+2}}, \cdots)$$

in $\bigoplus_{n_i \leq n} A_{n-n_i}$, where $s_{n-n_i} \in T_{n-n_i}$ for all indices $j$, let $u_1, u_2, \cdots, u_d$ be the unique elements in $T_{n-1}$ such that:

$$\sum_{n_i \leq n} s_{n-n_i} f_i = \sum_{i=1}^d u_i x_i.$$ 

Now define the mapping

$$\phi : \bigoplus_{n_i \leq n} A_{n-n_i} \rightarrow A_{n-1} \oplus A_{n-1} \oplus \cdots \oplus A_{n-1}$$

as follows:

$$\phi(s_{n-n_1} + \mathcal{U}_{n-n_1}, s_{n-n_2} + \mathcal{U}_{n-n_2}, \cdots, s_{n-n_k} + \mathcal{U}_{n-n_k}, 0 + \mathcal{U}_{n-n_{ik+1}}, 0 + \mathcal{U}_{n-n_{ik+2}}, \cdots)$$

$$= (u_1 + \mathcal{U}_{n-1}, u_2 + \mathcal{U}_{n-1}, \cdots, u_d + \mathcal{U}_{n-1}) \in A_{n-1} \oplus A_{n-1} \oplus \cdots \oplus A_{n-1}.$$
Lemma 1. The mapping $\psi$ is well-defined.

Proof. Let $\tau_1, \tau_2, \cdots, \tau_d \in \mathcal{U}_{n-1}$ be arbitrary. Since $\mathcal{U}$ is a (two-sided) ideal, we have that:

$$
\sum_{i=1}^{d} \tau_i x_i \in \mathcal{U}.
$$

(3)

Furthermore, given the canonical grading of $T$, we have that:

$$
\sum_{i=1}^{d} \tau_i x_i \in T_n.
$$

(4)

From (3) and (4) together, we have that:

$$
\sum_{i=1}^{d} \tau_i x_i \in \mathcal{U}_n.
$$

(5)

We will now show that $\psi$ as given above is well-defined in the sense that the expression $\psi(\mathcal{F})$ for a tuple $\mathcal{F}$ in the domain of $\psi$ consisting of cosets does not depend on one’s choice of representatives of such cosets. So, let $t_1, t_2, \cdots, t_d \in T_{n-1}$, and let $t'_1, t'_2, \cdots, t'_d \in T_{n-1}$, and suppose that:

$$(t_1 + \mathcal{U}_{n-1}, t_2 + \mathcal{U}_{n-1}, \cdots, t_d + \mathcal{U}_{n-1}) = (t'_1 + \mathcal{U}_{n-1}, t'_2 + \mathcal{U}_{n-1}, \cdots, t'_d + \mathcal{U}_{n-1}).$$

Equivalently,

$$
((t_1 - t'_1) + \mathcal{U}_{n-1}, (t_2 - t'_2) + \mathcal{U}_{n-1}, \cdots, (t_d - t'_d) + \mathcal{U}_{n-1})
$$

$$
= (0 + \mathcal{U}_{n-1}, 0 + \mathcal{U}_{n-1}, \cdots, 0 + \mathcal{U}_{n-1}).
$$

Write $\tau_i = t_i - t'_i \in \mathcal{U}_{n-1}$ for all indices $i$. By the definition of $\psi$, we have that:

$$
\psi(\tau_1 + \mathcal{U}_{n-1}, \tau_2 + \mathcal{U}_{n-1}, \cdots, \tau_d + \mathcal{U}_{n-1}) = \left( \sum_{i=1}^{d} \tau_i x_i \right) + \mathcal{U}_n.
$$

But from (5), we have that

$$
\psi(\tau_1 + \mathcal{U}_{n-1}, \tau_2 + \mathcal{U}_{n-1}, \cdots, \tau_d + \mathcal{U}_{n-1}) = 0 + \mathcal{U}_n
$$

and thus by linearity of $\psi$ we have that $\psi$ is well-defined. \qed

Lemma 2. The mapping $\phi$ is well-defined.
Proof. Let \( \sigma_{n-n_j} \) be an element in \( \mathcal{U}_{n-n_j} \) for all indices \( j \) in \( \{1, 2, \ldots, k\} \). Let \( \mu_1, \mu_2, \ldots, \mu_d \) be the unique elements in \( T_{n-1} \) defined by:

\[
\sum_{n_i \leq n} \sigma_{n-n_i} f_i = \sum_{i=1}^{d} \mu_i x_i.
\]  

(6)

We will now show that \( \mu_1, \mu_2, \ldots, \mu_d \in \mathcal{U}_{n-1} = \mathcal{U} \cap T_{n-1} \). Let \( n_i \leq n \), and consider the homogeneous element \( f_i \in \mathcal{U} \). Write: \( f_i = \sum_{j=1}^{d} g_{ij} x_j \). Therefore,

\[
\sigma_{n-n_i} f_i = \sum_{j=1}^{d} s_{n-n_i} g_{ij} x_j.
\]

(7)

From (6) and (7) we have that \( \mu_j = \sigma_{n-n_i} g_{ij} \). Since \( \sigma_{n-n_i} \in \mathcal{U}_{n-n_i} \), we have that \( \sigma_{n-n_i} \in \mathcal{U} \). Therefore, \( \mu_j = \sigma_{n-n_i} g_{ij} \) is in the ideal \( \mathcal{U} \). In particular, since \( \mu_j \in T_{n-1} \), we have that \( \mu_j \in \mathcal{U}_{n-1} \).

We will now show that \( \phi \) is well-defined in the sense that given a tuple \( \mathcal{T} \) in the domain of \( \phi \) consisting of cosets, the expression \( \phi(\mathcal{T}) \) does not depend on the representatives of the entries of \( \mathcal{T} \). Let \( s_{n-n_i}, s_{n-n_i}, \ldots, s_{n-n_i} \) be as given above, and let \( u_1, u_2, \ldots, u_d \) be as given above. Now let

\[
(s'_{n-n_i} + \mathcal{U}_{n-n_i}, s'_{n-n_i} + \mathcal{U}_{n-n_i}, \ldots, s'_{n-n_i} + \mathcal{U}_{n-n_i}, 0 + \mathcal{U}_{n-n_i}, 0 + \mathcal{U}_{n-n_i}, \ldots)
\]

be another tuple in the domain \( \bigoplus_{n_i \leq n} A_{n-n_i} \) of \( \phi \), where \( s'_{n-n_i} \in T_{n-n_j} \) for all indices \( j \), let \( u'_1, u'_2, \ldots, u'_d \) be the unique elements in \( T_{n-1} \) such that:

\[
\sum_{n_i \leq n} s'_{n-n_i} f_i = \sum_{i=1}^{d} u'_i x_i.
\]

Furthermore, suppose that:

\[
(s_{n-n_i} + \mathcal{U}_{n-n_i}, s_{n-n_i} + \mathcal{U}_{n-n_i}, \ldots, s_{n-n_i} + \mathcal{U}_{n-n_i}, 0 + \mathcal{U}_{n-n_i}, 0 + \mathcal{U}_{n-n_i}, \ldots)
\]

\[
= (s'_{n-n_i} + \mathcal{U}_{n-n_i}, s'_{n-n_i} + \mathcal{U}_{n-n_i}, \ldots, s'_{n-n_i} + \mathcal{U}_{n-n_i}, 0 + \mathcal{U}_{n-n_i}, 0 + \mathcal{U}_{n-n_i}, \ldots).
\]

Writing \( \sigma_{n-n_j} = s_{n-n_j} - s'_{n-n_j} \) for all indices \( j \),

\[
(\sigma_{n-n_i} + \mathcal{U}_{n-n_i}, \sigma_{n-n_i} + \mathcal{U}_{n-n_i}, \ldots, \sigma_{n-n_i} + \mathcal{U}_{n-n_i}, 0 + \mathcal{U}_{n-n_i}, 0 + \mathcal{U}_{n-n_i}, \ldots)
\]

\[
= (0 + \mathcal{U}_{n-n_i}, 0 + \mathcal{U}_{n-n_i}, \ldots, 0 + \mathcal{U}_{n-n_i}, \ldots).
\]

That is, \( \sigma_{n-n_j} \in \mathcal{U}_{n-n_j} \) for all indices \( j \). Letting \( \mu_1, \mu_2, \ldots, \mu_d \in T_{n-1} \) be as given above, as shown above we have that \( \mu_1, \mu_2, \ldots, \mu_d \in \mathcal{U}_{n-1} \). Therefore,

\[
\phi(\sigma_{n-n_i} + \mathcal{U}_{n-n_i}, \sigma_{n-n_i} + \mathcal{U}_{n-n_i}, \ldots, \sigma_{n-n_i} + \mathcal{U}_{n-n_i}, 0 + \mathcal{U}_{n-n_i}, 0 + \mathcal{U}_{n-n_i}, \ldots)
\]
equals
\[ (\mu_1 + \mathcal{U}_{n-1}, \mu_2 + \mathcal{U}_{n-1}, \ldots, \mu_d + \mathcal{U}_{n-1}) = (0 + \mathcal{U}_{n-1}, 0 + \mathcal{U}_{n-1}, \ldots, 0 + \mathcal{U}_{n-1}) \]
thus effectively completing the proof by linearity of \( \phi \).

Now, to show that the sequence
\[ \bigoplus_{n_i \leq n} A_{n-n_i} \xrightarrow{\phi} A_{n-1} \oplus A_{n-1} \oplus \cdots \oplus A_{n-1} \xrightarrow{\psi} A_n \rightarrow 0 \]
is exact, we begin with the following lemma.

**Lemma 3.** The composition \( \psi \circ \phi \) is trivial.

**Proof.** Let \( s_{n-n_i} \in T_{n-n_i} \) for all indices \( j \) in \( \{1, 2, \ldots, k\} \), so that
\[
(s_{n-n_{i_1}}, s_{n-n_{i_2}}, \ldots, s_{n-n_{i_k}}, 0, 0, \ldots) \in \bigoplus_{n_i \leq n} T_{n-n_i}
\]
is an arbitrary element in the domain of \( \Phi \). We thus have that
\[
\Phi(s_{n-n_{i_1}}, s_{n-n_{i_2}}, \ldots, s_{n-n_{i_k}}, 0, 0, \ldots) = (u_1, u_2, \ldots, u_d) \in T_{n-1} \oplus T_{n-1} \oplus \cdots \oplus T_{n-1}
\]
where \( u_1, u_2, \ldots, u_d \in T_{n-1} \) are the unique elements in \( T_{n-1} \) such that \( \sum_{n_i \leq n} s_{n-n_i} f_i = \sum_{i=1}^d u_i x_i \). Now, by definition of the mapping
\[
\Psi : T_{n-1} \oplus T_{n-1} \oplus \cdots \oplus T_{n-1} \rightarrow T_n
\]
defined above, we have that:
\[
\Psi(\Phi(s_{n-n_{i_1}}, s_{n-n_{i_2}}, \ldots, s_{n-n_{i_k}}, 0, 0, \ldots)) = \Psi(u_1, u_2, \ldots, u_d)
\]
\[= \sum_{i=1}^d u_i x_i \]
\[= \sum_{n_i \leq n} s_{n-n_i} f_i. \]

Since the expressions of the form \( f_i \) in the sum \( \sum_{n_i \leq n} s_{n-n_i} f_i \) are in \( \mathcal{U} \), we thus have that \( \sum_{n_i \leq n} s_{n-n_i} f_i \in \mathcal{U} \), thus proving that \( \psi \circ \phi \) maps the domain of \( \Phi \) into \( \mathcal{U} \), which in turn shows that \( \psi \circ \phi \) maps the domain of \( \phi \) to 0. \( \square \)
As an immediate consequence of Lemma 3 we have that \( \text{im}(\phi) \subseteq \ker(\psi) \). To briefly show this, let \( T \in \bigoplus_{n_1 \leq n} A_{n-n_1} \) be an arbitrary tuple in the domain of \( \phi \), so that \( \phi(T) \) is an arbitrary element in \( \text{im}(\phi) \). By Lemma 3, we have that \( \psi(\phi(T)) = 0 \), thus proving that \( \phi(T) \) is in the kernel of \( \psi \). We will now prove the reverse inclusion.

**Lemma 4.** The image of \( \phi \) contains \( \ker(\psi) \).

**Proof.** Our strategy is to prove that for arbitrary \( t_1, t_2, \cdots, t_d \in T_{n-1} \) if \( \Psi(t_1, t_2, \cdots, t_d) = \sum_{i=1}^{d} t_i x_i \in \mathcal{U} \) then there exist corresponding elements \( u_1, u_2, \cdots, u_d \in T_{n-1} \) such that \( t_i - u_i \in \mathcal{U} \) for all indices \( i \), and such that

\[
\sum_{i=1}^{d} u_i x_i = \sum_{n \leq n} s_{n-1} f_i
\]

for some \( s_{n-1} \in T_{n-n_1}, s_{n-2} \in T_{n-n_2} \), etc. Let \( * \) denote this statement. Observe that \( \Psi(t_1, t_2, \cdots, t_d) \) is in \( \mathcal{U} \) if and only if the following equality holds:

\[
\psi(t_1 + \mathcal{U}_{n-1}, t_2 + \mathcal{U}_{n-1}, \cdots, t_d + \mathcal{U}_{n-1}) = 0 + \mathcal{U}.
\]

Therefore, the tuple

\[
(t_1 + \mathcal{U}_{n-1}, t_2 + \mathcal{U}_{n-1}, \cdots, t_d + \mathcal{U}_{n-1})
\]

is an arbitrary element in \( \ker(\psi) \). So \( * \) implies that this element in \( \ker(\psi) \) is equal to the image of some element under \( \phi \). In other words, \( * \) implies that \( \ker(\psi) \subseteq \text{im}(\phi) \) and it thus suffices to prove \( * \). So let \( t_1, t_2, \cdots, t_d \in T_{n-1} \) be arbitrary, and suppose that:

\[
\Psi(t_1, t_2, \cdots, t_d) = \sum_{i=1}^{d} t_i x_i \in \mathcal{U}.
\]

\( \mathcal{U} \) is a two-sided ideal, and \( \mathcal{U} \) is generated by the (homogeneous) family \( \{f_i\}_{i \in \mathbb{N}} \), and \( \sum_{i=1}^{d} t_i x_i \in \mathcal{U} \). We may thus deduce that there exist homogeneous elements \( a_{k,q}, b_{k,q}, \) and \( c_q \) such that

\[
\sum_{i=1}^{d} t_i x_i = \sum_{k \in K} a_{k,q} f_k b_{k,q} + \sum_{q \in Q} c_q f_q \in \{f_1, f_2, \cdots\}
\]

for some appropriate index sets \( K \) and \( Q \subseteq \mathbb{N} \) such that the degree of \( b_{k,q} \) is at least 1 for all indices \( k \) and \( q \). Observe that \( t_1, t_2, \cdots, t_d \in T_{n-1} \) and consider the degree of each side of the above equality. We may assume without loss of generality that \( a_{k,q} f_k b_{k,q} \in T_n \) and \( c_q f_q \in T_n \) for all indices \( k \in K \) and \( q \in Q \). Since the degree of \( b_{k,q} \) is at least 1 for all indices \( k \) and \( q \), we may write:

\[
b_{k,q} = \sum_{m=1}^{d} d_{k,q,m} x_m.
\]
Rewriting the expression $\sum_{k \in K} a_{k,q} f_k b_{k,q}$ using the equality (8), we have

$$\sum_{k \in K} a_{k,q} f_k b_{k,q} = \sum_{k \in K} a_{k,q} f_k d_{k,q,m} x_m$$

and thus writing $d_m = \sum_{k \in K} a_{k,q} f_k d_{k,q,m}$ for indices $m$ in $\{1, 2, \ldots, d\}$ we have that:

$$\sum_{k \in K} a_{k,q} f_k b_{k,q} = \sum_{m=1}^d d_m x_m.$$

Now observe that $d_m \in \mathcal{U}$ for all indices $m$, because $f_q \in \mathcal{U}$ for all indices $q \in Q$ and $\mathcal{U} = \langle f_1, f_2, \cdots \rangle$. Now, writing $\sum_{q \in Q} c_q f_q = \sum_{i=1}^d u_i x_i$ we thus have that $\sum_{i=1}^d t_i x_i = \sum_{i=1}^d d_i x_i + \sum_{i=1}^d u_i x_i$ and we thus have that $t_i - u_i = d_i \in \mathcal{U}$ for all indices $i$ as desired. Since $c_q f_q$ is in $T_n$ for all indices $q \in Q \subseteq \mathbb{N}$, we may thus rewrite the sum $\sum_{q \in Q} c_q f_q$ as $\sum_{n_i \leq n} s_{n-n_i} f_i$ where $s_{n-n_j} \in T_{n-n_j}$ for all indices $j$, thus proving $(\ast)$. \hfill \square

We have thus far shown that the sequence

$$\bigoplus_{n_i \leq n} A_{n-n_i} \xrightarrow{\phi} A_{n-1} \oplus A_{n-1} \oplus \cdots \oplus A_{n-1} \xrightarrow{\psi} A_n \to 0$$

is exact at

$$A_{n-1} \oplus A_{n-1} \oplus \cdots \oplus A_{n-1}$$

with $\text{im}(\phi) = \ker(\psi)$. The mapping

$$\Psi : \underbrace{T_{n-1} \oplus T_{n-1} \oplus \cdots \oplus T_{n-1}}_d \to T_n$$

as given above is surjective, which shows that the above sequence is exact at $A_n$, thus proving that the above sequence is exact as desired. The following lemma proves the first part of the Golod-Shafarevich theorem.

**Lemma 5.** For $n \in \mathbb{N}$, $\mathcal{H}_A(n) \geq d \mathcal{H}_A(n-1) - \sum_{n_i \leq n} \mathcal{H}_A(n-n_i)$.

**Proof.** The dimension of $A_n$ over $k$ is denoted by $\mathcal{H}_A(n)$. Since $A_{n-1}$ is finite-dimensional, by the rank-nullity theorem we have that

$$\dim \left( \underbrace{A_{n-1} \oplus A_{n-1} \oplus \cdots \oplus A_{n-1}}_d \right) = \dim \left( \text{im}(\psi) \right) + \dim \left( \ker(\psi) \right).$$

Therefore,
\[ d\mathcal{H}_A(n - 1) = \dim \left( \text{im}(\psi) \right) + \dim \left( \ker(\psi) \right). \]

Since \( \psi \) is surjective, we thus have that:

\[ d\mathcal{H}_A(n - 1) = \mathcal{H}_A(n) + \dim \left( \ker(\psi) \right). \] (9)

By the above lemmas we have that the sequence

\[ \bigoplus_{n_i \leq n} A_{n-n_i} \xrightarrow{\phi} A_{n-1} \oplus A_{n-1} \oplus \cdots \oplus A_{n-1} \xrightarrow{\psi} A_n \to 0 \]

is exact. We thus have that \( \ker(\psi) \) is the image of \( \bigoplus_{n_i \leq n} A_{n-n_i} \) under the linear mapping \( \phi \). By the first isomorphism theorem, we have that:

\[ \dim \left( \bigoplus_{n_i \leq n} A_{n-n_i} / \ker(\phi) \right) = \dim \left( \ker(\psi) \right) \leq \dim \left( \bigoplus_{n_i \leq n} A_{n-n_i} \right) = \sum_{n_i \leq n} \mathcal{H}_A(n-n_i). \] (10)

From (9) and (10) together, we have that:

\[ d\mathcal{H}_A(n - 1) \leq \mathcal{H}_A(n) + \sum_{n_i \leq n} \mathcal{H}_A(n-n_i) \]

as desired. \( \square \)

We now conclude our proof of the Golod-Shafarevich theorem with the following lemma.

**Lemma 6.** If for each index \( i \), \( r_i \leq \left( \frac{d-1}{2} \right)^2 \), then \( A \) is infinite dimensional over \( k \).

**Proof.** By Lemma 5, we have the following inequality:

\[ \sum_{n=1}^{\infty} \mathcal{H}_A(n) t^n \geq \sum_{n=1}^{\infty} d\mathcal{H}_A(n - 1) t^n - \sum_{n=1}^{\infty} \sum_{n_i \leq n} \mathcal{H}_A(n-n_i) t^n. \]

The key idea behind our proof of Lemma 6 is is to rewrite the double sum

\[ \sum_{n=1}^{\infty} \sum_{n_i \leq n} \mathcal{H}_A(n-n_i) t^n \]

using expressions of the form \( r_i \). Since \( r_i \) denotes the number of elements in the family \( \{n_j\}_{j \in \mathbb{N}} \) which are equal to \( i \), we may rewrite this double sum as follows:
\[
\sum_{n=1}^{\infty} \sum_{n_i \leq n} \mathcal{H}_A(n - n_i) t^n = \sum_{n \in \mathbb{N}} \sum_{0 \leq n - n_i} \mathcal{H}_A(n - n_i) t^n \\
= \sum_{m \in \mathbb{N}_0} \sum_{i \in \mathbb{N}} \mathcal{H}_A(m) t^{n_i + m} \\
= \left( \sum_{i \in \mathbb{N}} t^{n_i} \right) \left( \sum_{m \in \mathbb{N}_0} \mathcal{H}_A(m) t^m \right) \\
= \left( \sum_{i=2}^{\infty} r_i t^i \right) \left( \sum_{m \in \mathbb{N}_0} \mathcal{H}_A(m) t^m \right).
\]

The expression \( \sum_{m \in \mathbb{N}_0} \mathcal{H}_A(m) t^m \) is precisely the Hilbert series \( \mathcal{F}_A(t) \). We thus have that:

\[
\mathcal{F}_A(t) - 1 \geq dt \mathcal{F}_A(t) - \left( \sum_{i=2}^{\infty} r_i t^i \right) \mathcal{F}_A(t).
\]

Therefore,

\[
\mathcal{F}_A(t) \left( 1 - dt + \sum_{i=2}^{\infty} r_i t^i \right) \geq 1.
\]

Now consider the following expression:

\[
q(t) = \left( 1 - dt + \sum_{i=2}^{\infty} r_i t^i \right)^{-1} = \sum_{k \geq 0} q_k t^k.
\]

Based on a proof given in [13], we will now show that if \( q_k \geq 0 \) for all indices \( k \), then \( q(t) \) cannot be a polynomial. Suppose that \( q_k \) is nonnegative for all \( k \geq 0 \), and by way of contradiction, suppose that \( q(t) \) is a polynomial, with

\[
q(t) = q_0 + q_1 t + \cdots + q_n t^n
\]

where \( q_n \neq 0 \). Since

\[
q(t) \left( 1 - dt + \sum_{i=2}^{\infty} r_i t^i \right) = 1
\]

we thus have that:

\[
q(t) \left( 1 + \sum_{i=2}^{\infty} r_i t^i \right) = 1 + dt q(t).
\]
Comparing the coefficients of $t^{n+2}$ on both sides of (11), we have that:

$$q_0r_{n+2} + q_1r_{n+1} + \cdots + q_n r_2 = 0.$$ 

Since $q_i, r_i \geq 0$ for all indices $i$, and since $q_n > 0$, we have that $r_2 = 0$ [13]. A similar argument shows that $r_3 = 0$, and $r_4 = 0$, and more generally $r_j = 0$ for all indices $j$, thus proving that

$$\frac{1}{1 - dt} = 1 + dt + d^2 t^2 + \cdots$$

thus contradicting that $q(t)$ is a polynomial.

We will now show that if

$$r_i \leq \left(\frac{d-1}{2}\right)^2$$

for all indices $i$ then $q_k \geq 0$ for all indices $k$. Suppose that $r_i \leq \left(\frac{d-1}{2}\right)^2$ for all indices $i$. Following [13], write $s_i$ in place of $\left(\frac{d-1}{2}\right)^2$. Again following [13], write:

$$p(t) = \frac{1}{1 - dt + \sum_{n \geq 2} s_n t^n} = \sum_{n \geq 0} p_n t^n.$$ 

Evaluating $p(t)$, we have:

$$p(t) = \frac{1 - t}{(1 - (\frac{d+1}{2}) t)^2}$$

In general, for a scalar $\alpha$,

$$\frac{1 - t}{(1 - \alpha t)^2} = \sum_{n=0}^{\infty} (\alpha^n (n+1) - \alpha^{n-1} n) t^n.$$

Writing $\alpha$ in place of $\frac{d+1}{2}$, we thus have that:

$$p(t) = \frac{1}{1 - dt + \sum_{n \geq 2} s_n t^n} = \sum_{n \geq 0} p_n t^n = \sum_{n=0}^{\infty} (\alpha^n (n+1) - \alpha^{n-1} n) t^n.$$

Now, since $d \geq 1$, we have that $\alpha \geq 1$, and thus the expression $\alpha^n (n+1) - \alpha^{n-1} n$ is nonnegative for all indices $n \in \mathbb{N}_0$. So all the coefficients of the series

$$p(t) = \frac{1}{1 - dt + \sum_{n \geq 2} s_n t^n} = \sum_{n \geq 0} p_n t^n$$

are nonnegative and by assumption, $r_n \leq s_n = \left(\frac{d-1}{2}\right)^2$ for all indices $n \geq 2$. Again following [13], write:

$$u(t) = 1 - dt + r(t) = 1 - dt + \sum_{n \geq 2} r_n t^n.$$
\[ s(t) = 1 - dt + \sum_{n \geq 2} s_n t^n. \]

We thus have that \( p(t) = \frac{1}{s(t)} \) and \( q(t) = \frac{1}{u(t)} \). By assumption that \( r_n \leq s_n = \left( \frac{d-1}{2} \right)^2 \) for all indices \( n \geq 2 \), we thus have that \( s(t) \geq u(t) \). We are using the order relation \( \leq \) to denote the partial order on formal power series described in the previous section. We also have that \( p(t) \geq 0 \). Our strategy is to show that \( q(t) \geq p(t) \). So it suffices to show that \( \frac{1}{u(t)} \geq \frac{1}{s(t)} \).

Now observe that the formal power series \( v(t) = s(t) - u(t) \) satisfies \( v(t) \geq 0 \) and has no constant term. Again following [13], rewrite the formal power series \( u(t) \) as follows:

\[
\begin{align*}
 u(t) &= s(t) - v(t) \\
 &= s(t) - s(t)v(t) \frac{1}{s(t)} \\
 &= s(t) \left( 1 - \frac{v(t)}{s(t)} \right).
\end{align*}
\]

We thus have that
\[
\frac{1}{u(t)} = \frac{1}{s(t)} \cdot \frac{1}{1 - \frac{v(t)}{s(t)}},
\]
and we thus have that:
\[
q(t) = p(t) \cdot \frac{1}{1 - \frac{v(t)}{s(t)}}.
\]

Since \( v(t) \) has no constant term, \( \frac{v(t)}{s(t)} \) has no constant term, and we thus have that:
\[
\frac{1}{1 - \frac{v(t)}{s(t)}} = 1 + \frac{v(t)}{s(t)} + \left( \frac{v(t)}{s(t)} \right)^2 + \cdots.
\]

From the inequality \( v(t) \geq 0 \) and the inequality \( \frac{1}{s(t)} \geq 0 \) we have that \( \frac{v(t)}{s(t)} \geq 0 \). We thus have that
\[
\frac{1}{1 - \frac{v(t)}{s(t)}} = 1 + \frac{v(t)}{s(t)} + \left( \frac{v(t)}{s(t)} \right)^2 + \cdots \geq 1
\]
and we thus have that \( q(t) \geq p(t) \) as desired. We have shown that if
\[
 r_i \leq \left( \frac{d-1}{2} \right)^2
\]
for all indices \( i \) then \( q_k \geq 0 \) for all indices \( k \), and we have shown that if \( q_k \geq 0 \) for all indices \( k \), then
\[
q(t) = \left( 1 - dt + \sum_{i=2}^{\infty} r_i t^i \right)^{-1} = \sum_{k \geq 0} q_k t^k
\]
cannot be a polynomial. Since
\[ F_A(t) \left( 1 - dt + \sum_{i=2}^{\infty} r_i t^i \right) \geq 1, \]
we thus have that: \( F_A(t) \geq q(t) \). So if
\[ r_i \leq \left( \frac{d-1}{2} \right)^2 \]
for all indices \( i \) then \( q_k \geq 0 \) for all indices \( k \) and since \( q(t) \) cannot be a polynomial, infinitely many of the coefficients of \( q(t) \) must be positive. But then \( F_A(t) \geq q(t) \), infinitely many of the coefficients of \( F_A(t) \) must be positive. But then \( H_A(m) \) must be positive for infinitely many \( m \in \mathbb{N}_0 \). That is, \( \dim_k(A_m) \) must be positive for infinitely many \( m \in \mathbb{N}_0 \). But then \( A \) is infinite dimensional over \( k \) as desired.

### 3.3 A Finitely Generated Nil Algebra which is Not Nilpotent

Following [18], we will now settle the Kurosh problem in the negative using the Golod-Shafarevich construction. Let \( k \) be an arbitrary countable field. Consider the free algebra \( T = k \langle x_1, x_2, x_3 \rangle \), letting \( x_1, x_2 \) and \( x_3 \) be noncommuting indeterminates. The algebra \( T \) is a connected graded \( k \)-algebra under the canonical grading of \( T \), with
\[ T = \bigoplus_{i \in \mathbb{N}_0} T_i \]
and \( T_0 = k \) and \( T_i \) homogeneous of degree \( i \) for \( i \in \mathbb{N} \). Now consider the augmentation ideal
\[ T' = \bigoplus_{i \in \mathbb{N}} T_i \]
of \( T \). Observe that \( T' \) is countable. Let
\[ T' = \{ s_i \}_{i \in \mathbb{N}} \]
be an arbitrary enumeration of \( T' \). Let \( m_1 \) be a natural number such that \( m_1 \geq 2 \), and let
\[ s_1^{m_1} = s_{1,2} + s_{1,3} + \cdots + s_{1,k_1} \]
where \( s_{1,j} \in T_j \) for \( j = 2, 3, \cdots \). Now let \( m_2 \in \mathbb{N} \) be such that \( m_2 > k_1 \), and let
\[ s_2^{m_2} = s_{2,k_1+1} + s_{2,k_1+2} + \cdots + s_{2,k_2} \]
where \( s_{2,k_1+j} \in T_{k_1+j} \) for all indices \( j \). Continuing in this manner, letting \( m_i > k_{i-1} \) for all indices \( i \), let \( \mathcal{W} \) be the ideal of \( T \) generated by the set of all elements of the form \( s_{i,j} \). Adopting notation from the previous subsection, the integers of the form \( r_k \) are all at most
1. Again using notation from the previous subsection, we have that $d = 3$ in this case, and we thus have that

$$r_i \leq \left( \frac{d - 1}{2} \right)^2$$

for all indices $i$. By the Golod-Shafarevich theorem, we thus have that $T/\mathcal{U}$ is infinite dimensional. Since $\mathcal{U}$ is contained in $T^\prime$, we thus have that $T^\prime/\mathcal{U}$ is infinite dimensional. However, $T^\prime/\mathcal{U}$ is a nil algebra. We thus have that $T^\prime/\mathcal{U}$ is a nil algebra that is generated by three elements, thus answering the Kurosh problem in the negative.
4 Decidability and Graded-Nilpotency of Monomial Algebras

A complete, formal definition of the term *decidable* (*recursively solvable*) is beyond the scope of this paper. We begin this concluding section with a brief overview of this term. For an introduction to recursive function theory see [11]. Informally, if there exists an “effective” algorithm which is used to numerically compute a function \( f \), then \( f \) is said to be *computable* [11]. More formally, a *computable* function is a numerical function which is \( URM \)-computable, i.e. computable by an *unlimited register machine* [11, 33]. Given an \( n \)-ary predicate \( M(x) \) of natural numbers, the predicate \( M(x) \) is *decidable* if the characteristic function

\[
c_M(x) = \begin{cases} 
1 & \text{if } M(x) \text{ holds}, \\
0 & \text{otherwise}
\end{cases}
\]

is computable [11].

Given a ring-theoretic property such as the property of being graded-nilpotent, it is natural to consider if it is decidable whether a class of rings satisfies this property. For example, the following problem remains open: is it decidable whether a finite ring is of finite representation type\(^2\) [26]? A natural way of approaching the problem of determining whether or not it is decidable whether a ring is graded-nilpotent is to restrict one’s attention to rings defined using automatic sequences, as indicated in Question 4 below. There are many problems involving automatic sequences that are known to be decidable [8, 15]. For example, it is decidable whether a given \( k \)-automatic sequence is ultimately periodic, as was proven in 1986 by Honkala in [20]. A *monomial algebra* may be defined as an algebra generated over a field by \( \{x_1, x_2, \cdots, x_d\} \) for some \( d \in \mathbb{N} \) with relations consisting of a set of monomials in the free monoid \( \{x_1, x_2, \cdots, x_d\}^* \).

**Question 4.** *Given an automatic word, and given a monomial algebra formed from this word, can we somehow decide if this algebra is graded-nilpotent, just from the automatic word [6]?*

In Section 4.1, we will review some basic terminology and results related to automaticity. Finally, Section 4.2 is devoted to exploring Question 4.

4.1 Deterministic Finite Automatons

Conventional definitions of the term *automatic sequence* are usually formulated using finite-state machines (see Definition 2 below). Alternatively, an automatic sequence may be defined as follows (see [2]).

\(^2\)Given an algebra \( A \), a representation of \( A \) is a module for \( A \). A module \( M \) is said to be indecomposable if \( M \) is nontrivial and \( M \) cannot be written as a direct sum of two nontrivial submodules. An algebra \( A \) with only finitely many non-isomorphic indecomposable modules is said to have *finite representation type*. 
Definition 1. Let \( f \) be a map from \( \mathbb{N}_0 \) to \( \Delta \), where \( \Delta \) is a finite set, and let \( k \in \mathbb{N} \). For \( j \in \{0, 1, \cdots, k - 1\} \) define \( e_j : \mathbb{N}_0 \to \mathbb{N}_0 \) so that \( e_j(n) = kn + j \) for all \( n \in \mathbb{N}_0 \). Let \( \mathcal{S} \) be the semigroup generated by the set of all mappings of the form \( e_j \) under composition. Then \( f \) is a k-automatic sequence (or a k-automatic map) if \( \{f \circ e\}_{e \in \mathcal{S}} \) is finite.

Remark 1. Observe that the underlying binary operation of the semigroup \( \mathcal{S} \) is not commutative. In fact, two generators \( e_i \) and \( e_j \) of this semigroup commute iff \( i = j \). Also observe that there is no identity element in the semigroup \( \mathcal{S} \). It can be shown by induction that if

\[
e_{j_1} \circ e_{j_2} \circ \cdots \circ e_{j_r} = e_{k_1} \circ e_{k_2} \circ \cdots \circ e_{k_r}
\]

then \( j_1 = k_1, j_2 = k_2, \cdots, j_r = k_r \), with \( r_1 = r_2 \). This property of \( \mathcal{S} \) will later be used to prove that a certain output function is well-defined.

A deterministic finite automaton (DFA) or deterministic finite state machine (DFSM) is an ordered 5-tuple

\[
\Gamma = (Q, \Sigma, \delta, q_0, F)
\]

where \( Q \) is a finite set consisting of elements called states, the elements in the finite set \( \Sigma \) are called input symbols, \( \delta : Q \times \Sigma \to Q \) is called a transition function, \( q_0 \in Q \) is called the initial state, and the elements in \( F \subseteq Q \) are called final or accepting states.

Observe that \( \delta \) extends in a natural way to the domain \( Q \times \Sigma^* \), with \( \delta(q, s_1s_0) = \delta(q, s_1) \delta(q, s_0) \) [31] where \( \Sigma^* \) denotes the free monoid on \( \Sigma \). That is, given some finite input word consisting of letters in \( \Sigma \), a DFA reads the letters in this word one at a time from left to right. If the input character is \( c \) and the DFA is in a state \( q_i \in Q \) for some index \( i \), then the DFA changes the current state to \( \delta(Q, c) \) (see [14], p. 29).

Definition 2. A sequence \( f : \mathbb{N}_0 \to \Delta \), where \( \Delta \) is a finite set, is k-automatic if there exists a DFA \( \Gamma = (Q, \Sigma = \{0, 1, \cdots, k - 1\}, \delta, q_0, F) \) and an output function \( O : Q \to \Delta \) such that for all \( n \in \mathbb{N}_0 \), \( f(n) = O(\delta(q_0, \omega_n)) \), where \( \omega_n \) is the unique element of \( \Sigma^* \) such that \([\omega_n]_k = n\), i.e. \( \omega_n \) is the base-k expansion of \( n \) (see [31]).

The following definition is standard.

Definition 3. Let \( f : \mathbb{N}_0 \to \Delta \), where \( \Delta \) is a finite set. For \( i \in \mathbb{N}_0 \) and \( j \in \{0, 1, \cdots, k^i - 1\} \) where \( k \in \mathbb{N} \) define \( f_{i,j} : \mathbb{N}_0 \to \Delta \) so that \( f_{i,j}(n) = f(k^i n + j) \) for all \( n \in \mathbb{N}_0 \). Then the k-kernel of \( f : \mathbb{N}_0 \to \Delta \) is the set of all distinct maps of the form \( f_{i,j} : \mathbb{N}_0 \to \Delta \) together with \( f \) (see [31]).

Remark 2. A k-automatic sequence is sometimes defined as a sequence \( (f_n)_{n \in \mathbb{N}_0} \) such that the codomain of \( (f_n)_{n \in \mathbb{N}_0} \) is finite and the k-kernel of \( (f_n)_{n \in \mathbb{N}_0} \) is finite. See [3] for example.

Theorem 2. Let \( \Delta \) be a finite set and let \( k \geq 2 \) be a natural number, and let \( f : \mathbb{N}_0 \to \Delta \). Then \( \{f \circ e\}_{e \in \mathcal{S}} \cup \{f\} \) is the k-kernel of \( f \).
Proof. Consider the semigroup $\mathcal{S}$. Each element in this semigroup is of the form

$$e_{i_j} \circ e_{i_{j-1}} \circ \cdots \circ e_{i_1}$$

where $j \in \mathbb{N}$ and the set of all indices in the above product is a subset of $\{0, 1, \ldots, k - 1\}$. Consider the expression

$$e_{i_j} \circ e_{i_{j-1}} \circ \cdots \circ e_{i_1}(n)$$

where $n \in \mathbb{N}_0$. Compute the expressions $e_{i_1}(n)$, $e_{i_2}(e_{i_1}(n))$, and $e_{i_3}(e_{i_2}(e_{i_1}(n)))$ as follows:

$$e_{i_1}(n) = kn + i_1,$n

$$e_{i_2}(e_{i_1}(n)) = e_{i_2}(kn + i_1) = k(kn + i_1) + i_2 = k^2n + i_1k + i_2,$n

$$e_{i_3}(e_{i_2}(e_{i_1}(n))) = e_{i_3}(k^2n + i_1k + i_2) = k(k^2n + i_1k + i_2) + i_3 = k^3n + i_1k^2 + i_2k + i_3.$n

Since $i_1 < k$, $i_2 < k$, and $i_3 < k$, we have that $k^3n + i_1k^2 + i_2k + i_3$ is of the form $k^3n + k_3$ where $k_3 < k^3$. A simple inductive argument shows that

$$e_{i_j} \circ e_{i_{j-1}} \circ \cdots \circ e_{i_1}(n)$$

is of the form $k^{i_j}n + k_{i_j}$ where $k_{i_j} < k^{i_j}$. Writing

$$e = e_{i_j} \circ e_{i_{j-1}} \circ \cdots \circ e_{i_1}$$

we have that the composition $f \circ e$ maps $n \in \mathbb{N}_0$ to $f(k^{i_j}n + k_{i_j})$ to where $k_{i_j} < k^{i_j}$. So $f \circ e$ is in the $k$-kernel of $f$. So $\{f \circ e\}_{e \in \mathcal{S}}$ is contained in the $k$-kernel of $f$. Therefore, since $f$ is in the $k$-kernel of $f$, $\{f \circ e\}_{e \in \mathcal{S}} \cup \{f\}$ is contained in the $k$-kernel of $f$. Conversely, let $f_{i,j}: \mathbb{N}_0 \rightarrow \Delta$ be an arbitrary sequence in the $k$-kernel of $f$. Now, $f_{0,0} = f \in \{f \circ e\}_{e \in \mathcal{S}} \cup \{f\}$. So $f_{i,j}(n) = f(k^jn + j)$ for arbitrary $n \in \mathbb{N}_0$, with $j < p^i$. Now consider the base-$k$ expansion of $j < k^i$. Suppose that:

$$j = \ell_1k^{i-1} + \ell_2k^{i-2} + \cdots + \ell_i.$$

Now given the computations

$$e_{i_1}(n) = kn + i_1,$n

$$e_{i_2}(e_{i_1}(n)) = k^2n + i_1k + i_2,$n

$$e_{i_3}(e_{i_2}(e_{i_1}(n))) = k^3n + i_1k^2 + i_2k + i_3,$n

given above, we have that:

$$e_{\ell_i}(\cdots(e_{\ell_2(e_{\ell_1}(n))})) = k^jn + \ell_1k^{i-1} + \ell_2k^{i-2} + \cdots + \ell_i.$$

Writing $e = e_{\ell_i} \circ e_{\ell_{i-1}} \circ \cdots \circ e_{\ell_1}$, we thus have that:
\[ e(n) = k^n + j. \]

Therefore, \( f \circ e = f_{i,j}. \) So each element in the \( k \)-kernel of \( f \) is in \( \{ f \circ e \}_{e \in \mathcal{S}} \cup \{ f \}. \) So by mutual inclusion, we have that \( \{ f \circ e \}_{e \in \mathcal{S}} \cup \{ f \} \) equals the \( k \)-kernel of \( f. \)

So based on Definition 1, given a finite set \( \Delta \) and a natural number \( k \geq 2, \) a map \( f : \mathbb{N}_0 \to \Delta \) is \( k \)-automatic iff the \( k \)-kernel of \( f \) is finite. To briefly show this, based on Definition 1, \( f : \mathbb{N}_0 \to \Delta \) is \( k \)-automatic iff \( \{ f \circ e \}_{e \in \mathcal{S}} \) is finite. By Theorem 2, \( \{ f \circ e \}_{e \in \mathcal{S}} \cup \{ f \} \) equals the \( k \)-kernel of \( f. \) So \( f : \mathbb{N}_0 \to \Delta \) is \( k \)-automatic iff the \( k \)-kernel of \( f \) is finite.

**Corollary 1.** If \( g(n) \) is in the \( k \)-kernel of \( f \) then the \( k \)-kernel of \( g(n) \) is contained in the \( k \)-kernel of \( f(n). \)

**Proof.** Suppose that \( g(n) \) is in the \( k \)-kernel of \( f. \) By Theorem 2, the \( k \)-kernel of \( f \) is \( \{ f \circ e \}_{e \in \mathcal{S}} \cup \{ f \}. \) So either \( g = f \) or \( g \) is of the form \( f \circ \eta \) for some \( \eta \in \mathcal{S}. \) Now consider the \( k \)-kernel of \( g. \) By Theorem 2, the \( k \)-kernel of \( g \) is \( \{ g \circ e \}_{e \in \mathcal{S}} \cup \{ g \}. \) If \( g = f, \) then the \( k \)-kernel of \( g(n) \) equals the \( k \)-kernel of \( f(n). \) So let \( g = f \circ \eta. \) We thus have that the \( k \)-kernel of \( g \) is \( \{ f \circ (\eta \circ e) \}_{e \in \mathcal{S}} \cup \{ g \}. \) Now, \( \{ \eta \circ e \}_{e \in \mathcal{S}} \subseteq \mathcal{S} \) for fixed \( \eta. \) So since \( \{ \eta \circ e \}_{e \in \mathcal{S}} \subseteq \mathcal{S} \) we thus have that \( \{ f \circ (\eta \circ e) \}_{e \in \mathcal{S}} \subseteq \{ f \circ e \}_{e \in \mathcal{S}} \) and \( \{ f \circ (\eta \circ e) \}_{e \in \mathcal{S}} \cup \{ g \} \subseteq \{ f \circ e \}_{e \in \mathcal{S}} \cup \{ f \}. \) That is, the \( k \)-kernel of \( g \) is contained in the \( k \)-kernel of \( f. \)

We will now show that a sequence \( f \) satisfies the conditions given in Definition 2 if and only if the \( k \)-kernel of \( f \) is finite. Our proof of the below theorem is inspired by a similar proof given in [31].

**Theorem 3.** Let \( f : \mathbb{N}_0 \to \Delta \) be a sequence, where \( \Delta \) is a finite set. Then there exists a DFA \( \Gamma = (Q, \Sigma = \{ 0, 1, \ldots, k - 1 \}, \delta, q_0, F) \) and an output function \( O : Q \to \Delta \) such that for all \( n \in \mathbb{N}_0, \) \( f(n) = O(\delta(q_0, \omega_n)) \) iff the \( k \)-kernel of \( f \) is finite.

**Proof.** (\( \leftarrow \rightarrow \)) Suppose that the \( k \)-kernel of \( f \) is finite. Let \( Q \) denote the \( k \)-kernel of \( f. \) The sequence \( (f_n)_{n \in \mathbb{N}_0} \) is in the \( k \)-kernel of \( f. \) Write \( q_0 = (f_n)_{n \in \mathbb{N}_0}. \) Now, by Theorem 2, we know that \( Q = \{ f \circ e \}_{e \in \mathcal{S}}. \) Write \( \Sigma = \{ 0, 1, \ldots, k - 1 \}. \) Now, define the transition function

\[ \delta : Q \times \Sigma \to Q \]

so that given an arbitrary integer \( j \in \{ 0, 1, \ldots, k - 1 \} = \Sigma \) and an arbitrary element of the form \( f \circ e \) in the \( k \)-kernel \( Q, \) where \( e \in \mathcal{S}, \)

\[ \delta(f \circ e, j) = f \circ (e_j \circ e) \in Q. \]

Also, let \( \delta(f, j) = f \circ e_j. \) Letting \( F \subseteq Q \) be arbitrary, we have thus constructed a DFA \( \Gamma = (Q, \Sigma = \{ 0, 1, \ldots, k - 1 \}, \delta, q_0, F). \) Now let \( n \in \mathbb{N}_0. \) Consider the base-\( k \) expansion of \( n. \) Suppose that \( n = a_0k^m + a_1k^{m-1} + \cdots + a_{m-1}k + a_m, \) where \( a_0, a_1, \ldots, a_m \in \{ 0, 1, \ldots, k - 1 \} = \Sigma. \) Consider the input word \( a_0a_1 \cdots a_m \in \Sigma^*. \) Since \( \delta \) satisfies \( \delta(q, a_0) = \delta(q, a_0), a_1) \), we may compute the expressions \( \delta(f, a_m), \delta(f, a_{m-1}a_m), \) and \( \delta(f, a_{m-2}a_{m-1}a_m) \) as follows.
\[
\delta(f, a_m) = f \circ e_{a_m} \in Q,
\]

\[
\delta(f, a_{m-1}a_m) = \delta(\delta(f, a_m), a_{m-1}) = \delta(f \circ e_{a_m}, a_{m-1}) = f \circ e_{a_{m-1}} \circ e_{a_m},
\]

\[
\begin{align*}
\delta(f, a_{m-2}a_{m-1}a_m) &= \delta(\delta(f, a_{m-1}a_m), a_{m-2}) \\
&= \delta(f \circ e_{a_{m-1}} \circ e_{a_m}, a_{m-2}) \\
&= f \circ e_{a_{m-2}} \circ e_{a_{m-1}} \circ e_{a_m}.
\end{align*}
\]

From the above computations, we have that:

\[
\delta(f, a_0a_1 \cdots a_m) = f \circ e_{a_0} \circ e_{a_1} \circ \cdots \circ e_{a_m} \in Q.
\]

Now, define the output function \( \mathcal{O} : Q \to \Delta \) as follows. Let \( \mathcal{O}(f) = f_0 \), and for \( j_1, j_2, \ldots, j_r \in \{0, 1, \ldots, k-1\} \) where \( r \in \mathbb{N}_0 \), let:

\[
\mathcal{O}(f \circ e_{j_1} \circ e_{j_2} \cdots \circ e_{j_r}) = f([j_1j_2 \cdots j_r]) \in \Delta
\]

where \([j_1j_2 \cdots j_r]\) denotes the integer the base-\(k\) expansion of which is \( j_1j_2 \cdots j_r \). Given Remark 1, the function \( \mathcal{O} \) is well-defined.

That is, given elements \( \alpha = f \circ e_{j_1} \circ \cdots \circ e_{j_{r_1}} \) and \( \beta = f \circ e_{k_1} \circ \cdots \circ e_{k_{r_2}} \) in the \( k \)-kernel of \( f \), if \( \alpha = \beta \), then \( r_1 = r_2 \), and \( e_{j_1} = e_{k_1}, \ldots, e_{j_{r_1}} = e_{k_{r_2}} \), so \( \mathcal{O}(\alpha) = \mathcal{O}(\beta) \) since \([j_1j_2 \cdots j_{r_1}] = [k_1k_2 \cdots k_{r_2}]\). Thus, \( \mathcal{O} \) is defined so that given an input word \( a_0a_1 \cdots a_m \in \Sigma^* \),

\[
\mathcal{O}(\delta(q_0, a_0a_1 \cdots a_m)) = \mathcal{O}(\delta(f, a_0a_1 \cdots a_m)) = \mathcal{O}(f \circ e_{a_0} \circ e_{a_1} \circ \cdots \circ e_{a_m}) = f([a_0a_1 \cdots a_m]).
\]

So we have constructed a DFA \( \Gamma = (Q, \Sigma = \{0, 1, \ldots, p-1\}, \delta, q_0, F) \) and an output function \( \mathcal{O} : Q \to \Delta \) such that for all \( n \in \mathbb{N}_0 \), \( f(n) = \mathcal{O}(\delta(q_0, \omega_n)) \).

(\(\Rightarrow\)) Now suppose that there exists a DFA \( \Gamma_0 = (Q, \Sigma = \{0, 1, \ldots, k-1\}, \delta, q_0, F) \) and an output function \( \mathcal{O} : Q \to \Delta \) such that for all \( n \in \mathbb{N}_0 \), \( f(n) = \mathcal{O}(\delta(q_0, \omega_n)) \). Let \((f_{n+k_0})_{n \geq 0}\) be a sequence in the \( k \)-kernel of \( f \). Let \( d_{e-1} \cdots d_1d_0 \) be the base-\(k\) expansion of \( j \). Write \( q_{e,j} = \delta(q_0, d_{e-1} \cdots d_1d_0) \) in \( Q \). Now consider the DFA

\[\Gamma_{e,j} = (Q, \Sigma = \{0, 1, \ldots, k-1\}, \delta, q_{e,j} = \delta(q_0, d_{e-1} \cdots d_1d_0), F)\]

and observe that since \( q_{e,j} \) is the initial state of \( \Gamma_{e,j} \), given the input word \( n = [a_0a_1 \cdots a_m] \), and since \( \delta \) satisfies \( \delta(q, a_1a_0) = \delta(\delta(q, a_0), a_1) \), we have that:

\[
\delta(q_{e,j}, a_n) = \delta(\delta(q_0, d_{e-1} \cdots d_1d_0), F), a_n) = \delta(q_0, a_n d_{e-1} \cdots d_1d_0),
\]

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\[ \delta(q_{e,j}, a_{n-1}a_n) = \delta(\delta(q_0, d_{e-1} \cdots d_1d_0), F), a_{n-1}a_n) = \delta(q_0, a_{n-1}a_n d_{e-1} \cdots d_1d_0). \]

Therefore,

\[ \delta(q_{e,j}, a_1a_2 \cdots a_{n-1}a_n) = \delta(\delta(q_0, d_{e-1} \cdots d_1d_0), a_1a_2 \cdots a_{n-1}a_n) = \delta(q_0, a_1a_2 \cdots a_{n-1}a_n d_{e-1} \cdots d_1d_0). \]

Therefore,

\[ O(\delta(q_{e,j}, a_1a_2 \cdots a_{n-1}a_n))) = O(\delta(q_0, a_1a_2 \cdots a_{n-1}a_n d_{e-1} \cdots d_1d_0)) = f_{k^n+j}. \]

So the output function \( O \) evaluated at the output of the DFA

\[ \Gamma_{e,j} = (Q, \Sigma = \{0, 1, \ldots, k - 1\}, \delta, q_{e,j} = \delta(q_0, d_{e-1} \cdots d_1d_0), F) \]

given the input word \( a_1a_2 \cdots a_{n-1}a_n \) is \( f_{k^n+j} \). We thus have an injection from the \( k \)-kernel of \( f \) to the finite set \( Q \) of all states. So the finiteness of the \( k \)-kernel of \( f \) follows from the finiteness of the set of all states of \( \Gamma \).

We may thus deduce that Definition 1 and Definition 2 are equivalent. A sequence \( f \) satisfies the conditions given in Definition 1 iff its \( k \)-kernel is finite by Theorem 2 as outlined above, and a sequence \( f \) satisfies the conditions given in Definition 2 iff its \( k \)-kernel is finite by Theorem 3. So Definition 1 and Definition 2 are equivalent.

### 4.2 Monomial Algebras Defined Using Automatic Sequences

The following well-known theorem from [1], [8] and [15] is very useful for proving results regarding decidability and automaticity:

**Theorem 4.** If we can express a property of a \( k \)-automatic \( x \) using quantifiers, logical operations, integer variables, the operations of addition, subtraction, indexing into \( x \), and comparison of integers or elements of \( x \), then this property is decidable. Furthermore, it is decidable if this property holds infinitely often.

Let \( w = w_1w_2 \cdots \) be a fixed infinite word over the alphabet \( \{x_1, x_2, \cdots, x_d\} \) endowed with a degree mapping

\[ \deg : \{x_1, x_2, \cdots, x_d\} \rightarrow \mathbb{N} \]

and suppose that \( w \), regarded as a sequence, is a \( k \)-automatic sequence. Since \( w \) is automatic, the degree sequence

\[ d = (\deg(w_1), \deg(w_2), \cdots) = (d_i)_{i \in \mathbb{N}} \]

...
is also \(k\)-automatic sequence. Now consider the polynomial \(k\)-algebra
\[ T = k\langle x_1, x_2, \cdots, x_d \rangle \]
and consider the grading induced by the degree mapping indicated above: write \(T_0 = k\), and for an arbitrary index \(i \in \mathbb{N}\), let \(T_i\) denote the \(k\)-vector space spanned by the (finite) set consisting of all monomials in \(\{x_1, x_2, \cdots, x_d\}\) of degree \(i\). We thus have that \(T\) is a connected graded \(k\)-algebra, with:
\[ T = \bigoplus_{i \in \mathbb{N}_0} T_i. \]

Let \(I\) be the ideal of \(T\) generated by the set consisting of all nonempty monomials in \(\{x_1, x_2, \cdots, x_d\}\) which do not appear as a subword of \(w\). Now observe that \(I\) is homogeneously generated with respect to the grading on \(T\) given above. We thus have that \(I\) is a graded ideal of \(T\). Writing \(I_i = I \cap T_i\) for arbitrary \(i \in \mathbb{N}_0\), we thus have that
\[ I = \bigoplus_{i \in \mathbb{N}_0} I_i \]
and, writing \(A_i\) in place of \(T_i/I_i\) for arbitrary \(i \in \mathbb{N}_0\), we have that:
\[ A = T/I = \bigoplus_{i \in \mathbb{N}_0} A_i. \quad (12) \]

Every element in \(I\) has constant term equal to 0. Therefore:
\[ I_0 = I \cap T_0 = I \cap k = \{0\} \]
and we thus have that
\[ A_0 = T_0/I_0 = k/\{0\} \cong k \]
thus proving that \(A\) as given in (12) is a connected graded \(k\)-algebra. Now define
\[ B = B(w) = \bigoplus_{i \in \mathbb{N}} B_i \]
where \(B_i = A_i\) for arbitrary \(i \in \mathbb{N}\). So \(B \subseteq A\) is the augmentation ideal (see Section 1) of \(A\), regarded as a non-connected graded \(k\)-algebra, the grading of which is inherited from \(A\).

We assumed that \(w\), regarded as a sequence, is a \(k\)-automatic sequence. We are interested in deciding whether or not this automatic sequence is such that the corresponding algebra \(B\) is graded-nilpotent. So it seems natural to begin by formulating the property of being graded-nilpotent using first-order logic. \(B\) is not graded-nilpotent if and only if:
\[ \exists N_1 \in \mathbb{N} \\forall N_2 \in \mathbb{N} \ B_{N_1}^{N_2} \neq \{0\}. \]

The above statement is equivalent to:
\[ \exists N_1 \in \mathbb{N} \forall N_2 \in \mathbb{N} \text{ there exist words } v_1, v_2, \ldots, v_{N_2} \text{ in } \{x_1, x_2, \ldots, x_d\}^* \text{ such that: } v_1 v_2 \cdots v_{N_2} \text{ is a subword of } w \text{ and } \deg(v_1) = \deg(v_2) = \cdots = \deg(v_{N_2}) = N_1. \]

Equivalently,

\[ P(d) \equiv \exists N_1 \in \mathbb{N} \forall N_2 \in \mathbb{N} \exists i_1 \in \mathbb{N} \exists i_2 \in \mathbb{N} \cdots \exists i_{N_2} \in \mathbb{N} \exists i_{N_2+1} \in \mathbb{N} \]

\[ \sum_{i=i_1}^{i_2} d_i = N_1, \sum_{i=i_2+1}^{i_3} d_i = N_1, \cdots, \sum_{i=i_{N_2+1}}^{i_{N_2+1}} d_i = N_1. \]

Certainly, \( P(d) \) is a property of the \( k \)-automatic sequence \( d = (d_i)_{i \in \mathbb{N}} \) which can be expressed using quantifiers, logical operations, integer variables, the operations of addition, subtraction, indexing into \( d \), and comparison of integers or elements of \( d \) (see Theorem 4). So by Theorem 4, the property \( P(d) \) is decidable, thus answering Question 4 in the affirmative.

Now observe that the number of quantifiers over variables of the form \( i_j \) for \( j \in \{1, 2, \cdots, N_2 + 1\} \) depends on a non-free variable. It is often difficult or “unpleasant” to work with a decidable property whereby the size of the corresponding automata “depends... on the number of quantifiers needed to state the logical expression characterizing the property being checked” [15].

To construct an algorithm to determine whether or not \( B \) is graded-nilpotent using the statement \( P(d) \), it seems natural to use the techniques given in [1], [8] and [15]. In particular, following a strategy outlined in [15], one may begin by creating an NFA \( M \) that accepts the following language, letting \((N_1)_k\) denote the reversed representation of \( N_1 \) in base \( k \) (see [15]).

\[ \left\{ (N_1)_k : \exists N_2 \in \mathbb{N} \forall i_1 \in \mathbb{N} \forall i_2 \in \mathbb{N} \cdots \forall i_{N_2} \in \mathbb{N} \forall i_{N_2+1} \in \mathbb{N} \right. \]

\[ \sum_{i=i_1}^{i_2} d_i \neq N_1, \sum_{i=i_2+1}^{i_3} d_i \neq N_1, \cdots, \sum_{i=i_{N_2+1}}^{i_{N_2+1}} d_i \neq N_1 \left\}. \right. \]

We presently leave it as an open problem to construct such an algorithm. Such an algorithm would be very useful for proving properties of graded-nilpotent rings.
References


