A new type of singly-implicit Runge-Kutta method

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Abstract

Implicit Runge-Kutta methods are considered which combine the single-implicitness or diagonal-implicitness property with a zero first row in the coefficient matrix. Acceptable stability for stiff problems is retained by requiring the last stage of a step to be identical to the output value. This requirement, which corresponds to the FSAL property for explicit Runge-Kutta methods, allows the method to have one less SIRK stage to achieve a specific stage order. Examples are given of DIRK, SIRK as well as DESI methods modified in this way. Methods are also proposed which have less than the full stage-order compared with the overall order of the method.

Keywords: Singly-implicit Runge-Kutta methods; FSAL methods; DESI methods

1 Introduction

Because for many stiff problems, implicit Runge-Kutta methods are cheaper to implement if they have a DIRK \cite{?} or SIRK \cite{?, ?} structure, we will concentrate on this type of method. The new feature considered in the present paper is the use of a first stage identical to the input approximation. This means that the first row of the coefficient matrix will be exactly zero. To compensate for the decreased degree of the denominator of the stability function, we will need to lower of the degree of the numerator, and this can be achieved by requiring that the last stage is identical with the output value. This means that the last row of the coefficient matrix is identical with the vector of output value coefficients. Thus, the methods are like FSAL methods in that the computation of the first stage becomes unnecessary because it is identical with the input value. Since we are dealing with stiff problems, using the final stage derivative in the previous step is the only appropriate way to compute the derivative of the first stage of the method. This means that, unlike a FSAL explicit method, some special starting step is required before the first step of the method is carried out.

To obtain even better stability, and to obtain greater flexibility in the choice of the method, we will consider methods in which the stage order $q$ is less than the order $p$, and we also consider methods in which the number of stages $s$ is greater than necessary, as in DESI methods \cite{?, ?}, to obtain the required order and stage order.

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Because of the single-implicitness and the stage order conditions for the singly block of the FSAL methods, a relationship can be found between the abscissae for the singly block and the derivative of the Laguerre polynomial. In Section ??, we discuss this relationship. In Section ??, we construct some fourth order methods with stage order 2 and 3 respectively. Starting methods for the new methods are discussed in Section ???. For reasons of stability we will restrict these to DESI methods without the FSAL property. The final sections contain numerical comparisons of the new methods with existing alternatives (Section ???) and a discussion of the relative advantages of various schemes (Section ??).

2 The new methods

Let \((A, b, c)_s\) denote a method with stage number \(s\). Write \(p\) and \(q\) for the order and the stage order of the method respectively. Hence, the number of singly-implicit stages for this method is \(q - 1\). Because the first row of the coefficient matrix is zero, we will consider only \(q \geq 2\). Recalling that the abscissae of the singly-implicit block for the classical DESI methods must be chosen to be proportional to the zeros of the Laguerre polynomial (cf. ??), we note that the abscissae for the methods incorporating an initial derivative approximation have a similar property as follows.

**Theorem 1** For an \(s\)-stage method with \(\sigma(A) = \{0, \lambda\}\) stage order \(q\), and \(c_1 = 0\), the abscissae for the singly-implicit stages \(c_2, \ldots, c_q\) satisfy

\[
c_i = \lambda \xi_{i-1}, \quad i = 2, 3, \ldots, q,
\]

where \(\xi_1, \xi_2, \ldots, \xi_{q-1}\) are the zeros of \(L'_q(x) = 0\) for \(L_q\) the \(q\)-degree Laguerre polynomial.

**Proof:** Let \(A_q\) be the singly-implicit block of the method, \(c = [c_1, c_2, c_3, \ldots, c_q]^T\) and \(c^k\) denote the component-wise \(k\)-th power of \(c\). Since \(\sigma(A_q) = \{0, \lambda\}\) and the first row of \(A_q\) is the zero vector, the characteristic polynomial of \(A_q\) can be written as

\[
det(zI - A_q) = z(z - \lambda)^{q-1}.
\]

By the Cayley-Hamilton theorem, we have

\[
A_q(A_q - \lambda I_q)^{q-1}e = 0.
\]

By \(C(q)\), for \(k = 0, 1, 2, \ldots, q - 1\), we have

\[
A_qc^k = \frac{\xi^{k+1}}{(k+1)!},
\]

and this implies

\[
A_q^kc = \frac{\xi^{k+1}}{(k+1)!}, \quad k = 0, 1, \ldots, q - 1.
\]
Hence, it follows that

\[
0 = (A_q - \lambda I_q)^{-1} A_q c = (A_q - \lambda I_q)^{q-1} c = \sum_{k=0}^{q-1} \left( \frac{q-1}{k} \right)(-1)^k \lambda^k A_q^{q-k-1} c
\]

Multiply both sides of this equation by \(\lambda^{-q}\) and write \(q - k = l\). We then have

\[
0 = \sum_{k=0}^{q-1} \left( \frac{q-1}{k} \right)(-1)^k \lambda^k \frac{1}{(q-k)!} c^{q-k}.
\]

Hence it follows that each component of \(c\) satisfies the equation

\[
\sum_{l=1}^{q} (-1)^{q-l} \frac{q!}{(q-l)!l!} \frac{1}{(l-1)!} \left( \frac{c_i}{\lambda} \right)^l = 0
\]

which is equivalent to \(\left( \frac{c_i}{\lambda} \right) L' \left( \frac{c_i}{\lambda} \right) = 0\). \(\square\)

### 3 Constructions of the methods

We begin with the second order method with \(q = 2\). It is easy to see that the eigenvalue necessary for this two-stage method to be A-stable is \(\lambda = \frac{1}{2}\) and it is not possible to have L-stable methods. From theorem ?? and the \(C(2)\) stage order conditions, the corresponding tableau is

\[
\begin{array}{ccc}
0 & 0 & 0 \\
1 & \frac{1}{2} & \frac{1}{2} \\
1 & \frac{1}{2} & \frac{1}{2}
\end{array}
\]

as for the trapezoidal rule. An L-stable method can be obtained by adding one diagonally-implicit stage. In this case, the stability function for the method is the same as for the second order SIRK method. It is known that the method is A-stable if and only if \(\frac{1}{2}(2 - \sqrt{2}) \leq \lambda \leq \frac{1}{2}(2 + \sqrt{2})\). Furthermore, we have an L-stable method if \(\lambda = \frac{1}{2}(2 \pm \sqrt{2})\). To obtain smaller coefficients, we choose \(\lambda = \frac{1}{2}(2 - \sqrt{2})\). Hence \(c_2 = 2\lambda = 2 - \sqrt{2}\) (2 is the zero of \(L'_2(x) = 0\)) and the L-stable second order method takes the form:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
2 - \sqrt{2} & 1 - \frac{1}{2}\sqrt{2} & 1 - \frac{1}{2}\sqrt{2} & 0 \\
1 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 1 - \frac{1}{2}\sqrt{2} \\
1 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 1 - \frac{1}{2}\sqrt{2}
\end{array}
\]

3
As for the second order case, we cannot have L-stability for a method with \( p = 3, q = 3 \). We also note that to obtain A-stability, we need to choose \( \lambda = \frac{1}{6}(3 + \sqrt{3}) \) instead of \( \lambda = \frac{1}{6}(3 - \sqrt{3}) \), where we note that \( 3 \pm \sqrt{3} \) are the zeros of \( L_2'(x) \). In this case, we have \( c_2 = 1, c_3 = \lambda(3 + \sqrt{3}) = 2 + \sqrt{3} \).

By \( C(3) \), the tableau for this A-stable method can be expressed numerically as follows:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
1 & 0.455342 & 0.561004 & -0.016346 \\
3.73205 & -0.455342 & 3.171046 & 1.016346 \\
1 & 0.455342 & 0.561004 & -0.016346 \\
\end{array}
\]

It is possible to obtain L-stable methods with order 3, but only by adding a diagonally-implicit stage. If \( \xi_1, \xi_2, \xi_3 \) are the zeros in increasing order of \( L_3 \) and \( \eta_1, \eta_2 \) are the zeros of \( L_3' \), then the tableau for the new method has the form

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\lambda \eta_1 & a_{21} & a_{22} & a_{23} & 0 & 0 \\
\lambda \eta_2 & a_{31} & a_{32} & a_{33} & 0 & 0 \\
1 & b_1 & b_2 & b_3 & \lambda & 0 \\
1 & b_1 & b_2 & b_3 & \lambda & 0 \\
\end{array}
\]

where \( \lambda = \frac{1}{\xi_2} \). The abscissae have the numerical values, \( \{0, 0.552657, 2.06254, 1\} \) compared with \( \lambda \xi_1 = 0.181222, \lambda \xi_3 = 2.74158, \lambda \xi_2 = 1 \) for the standard SIRK method. Thus, just as for the second order L-stable case, the use of the FSAL feature has the double benefit of (1) reducing from 3 to 2 the number of \textit{singly implicit} (as distinct from diagonally implicit) stages and (2) reducing from 2.74158 to 2.06254 the size of the abscissae outside the current step. We note that when \( s = 5, p = 3, q = 3 \), \( \lambda \approx 0.22364780 \) and \( c_2 = 0.283574, c_3 = 1.05831 \).

It is also interesting to ask how many diagonally-implicit stages need to be added to obtain a \( p \)-th order method of the new type. Since \( q \geq 2 \), we consider the fourth order method and \( q = 2 \). By \( C(2) \), we have three order conditions need to be satisfied in order to obtain an order-4 method; these are

\[
 b^T c^2 = \frac{1}{3}, \quad b^T c^3 = \frac{1}{4}, \quad b^T Ae^2 = \frac{1}{12}.
\]

(1)

Clearly, we need at least three additional diagonally-implicit stages to fulfill this requirement. Furthermore, we need to take the stability into account. If we take \( s = 5 \), since the first row of the method is the zero row, the degree of the numerator of the stability function is 4. Thus, we can have only A-stability. For L-stability, one more stage is necessary. Therefore, we need 6 stages to obtain a 4-th order L-stable method with stage order 2. Similarly, we need 7 stages for a 5-th order L-stable method. In general, it can be concluded that the minimum stage number for constructing a \( p \)-th order L-stable method with stage order \( p - 2 \) is \( p + 2 \).
As for the families of singly-implicit methods, we can use the stability requirement to specify the non-zero eigenvalue $\lambda$. We then determine the abscissae and finally, we derive the coefficient matrix $A$ and the weight vector $b$ by using the order conditions.

We now consider the construction of a fourth-order, six-stage L-stable method satisfying $C(2)$. Since $a_{ij} = 0$ for $j = 1, 2, \ldots, s$, the denominator of the stability function $R(z)$ of the method is $(1 - \lambda z)^5$. $R(z)$ can be written as

$$R(z) = \frac{p(z)}{(1 - \lambda z)^5} = \exp(z) + O(z^5),$$

where $\deg(p(z)) = 4$. It follows that

$$p(z) = 1 + (1 - 5\lambda)z + \left(\frac{1}{2} - 5\lambda + 10\lambda^2\right)z^2 + \left(\frac{1}{6} - \frac{5}{2}\lambda + 10\lambda^2 - 10\lambda^3\right)z^3 + \left(\frac{1}{24} - \frac{5}{6}\lambda + 5\lambda^2 - 10\lambda^3 + 5\lambda^4\right)z^4.$$

The $E$-polynomial $E(y)$ is

$$E(y) = (1 + y^2\lambda^2)^5 - p(iy)p(-iy).$$

and it is found that

$$E(y) \geq 0 \text{ if and only if } \lambda \in (0.247995, 0.676042).$$

If we choose $\lambda$ to be $\frac{1}{4}$, then the abscissae for the singly-implicit stages can be determined by Theorem 3. The derivative of the second order Laguerre polynomial has zero 2, hence we have $c_1 = 0$, $c_2 = 2\lambda = 1$. By $C(2)$, we find $a_{21} = \frac{1}{4}$, $a_{22} = \frac{1}{4}$. The other abscissae are free parameters except $c_6 = 1$. If we choose $c_3 = \frac{1}{4}$, $c_4 = \frac{1}{2}$, $c_5 = \frac{3}{4}$, then we can use the order conditions to determine the other coefficients for the method.

Note that, using the $C(2)$ condition together with the additional order condition (??), there are still 3 free coefficients to be determined. A further relation is found from the L-stability requirement. This is equivalent to the conditions that the $b$-coefficients should satisfy so that the numerator of the stability function

$$R(z) = 1 + zb^T(I - zA)^{-1}e = \frac{p(z)}{(1 - \lambda z)^5}$$

to be of 4th degree. These conditions reduce to the requirement that the coefficient of $z^5$ in $p(z)$, which is a linear function of $b_i$, is equal to 0. Consequently, an additional linear equation for the specification of $b_i$ is available. In terms of $a_{43}$ and $a_{64}$, the remaining coefficients are
Ideally, we may use the two free parameters to construct an error estimator. For example, if we use the embedding technique, then we can choose \( a_{43}, a_{64} \) to make the error estimate as close to fifth order as possible. Let \( d^T = [d_1, d_2, \ldots, d_6] \) be the weights for the error estimator, in this case, we can only find the relation between \( a_{43} \) and \( a_{64} \) based on the condition \( d^T \text{diag}(c) A c^2 = \frac{1}{15} \); this is \( a_{43}a_{64} = \frac{1}{9} \). Therefore, if we let \( a_{64} = \frac{1}{2} \), then the tableau for the method including an error estimator is

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{16} & -\frac{1}{16} & \frac{1}{4} & 0 & 0 \\
\frac{1}{2} & -\frac{7}{36} & -\frac{4}{9} & \frac{8}{9} & \frac{1}{4} & 0 \\
\frac{3}{4} & -\frac{5}{48} & -\frac{257}{768} & \frac{5}{6} & \frac{27}{256} & \frac{1}{4} & 0 \\
1 & \frac{1}{4} & \frac{2}{3} & -\frac{1}{3} & \frac{1}{2} & -\frac{1}{3} & \frac{1}{4} \\
1 & \frac{1}{4} & \frac{2}{3} & -\frac{1}{3} & \frac{1}{2} & -\frac{1}{3} & \frac{1}{4} \\
\frac{7}{90} & \frac{3}{20} & \frac{16}{45} & -\frac{1}{60} & \frac{16}{45} & \frac{7}{90}
\end{array}
\]

(2)

It is also easy to construct a fourth order FSAL method satisfying \( C(3) \) with 6 stages. In this case, the first derivative of the third order Laguerre polynomial has two zeros, \( 3 - \sqrt{3} \) and \( 3 + \sqrt{3} \). Because the stability function of this method is the same as for method (??), we should choose \( \lambda = \frac{1}{2} \). By theorem ??, the abscissae for the singly-implicit block are given by \( c_1 = 0, c_2 = \lambda(3 - \sqrt{3}) = \frac{1}{2}(3 - \sqrt{3}), c_3 = \lambda(3 + \sqrt{3}) = \frac{1}{2}(3 + \sqrt{3}) \). If we choose \( c_3 = \frac{1}{2}, c_4 = \frac{3}{4}, c_5 = 1, \) then in a similar way, we can find the coefficients for this method by using the \( C(3) \) condition, the order condition \( b^TC^3 = \frac{1}{4} \) and the stability restriction. We note that there is a free parameter after we use the order conditions and the L-stability requirement for a fourth order FSAL method. We may use this free parameter to fulfill the fifth order condition (fourth order “bushy tree”). Because the method satisfies the \( C(3) \) condition, it is easier to find an asymptotic error estimator of order 4. As in the previous case, let \( d^T = [d_1, d_2, \ldots, d_6] \) be the weight vector for the error estimator and let \( e = [1, 1, 1, 1, 1, 1]^T \). Using the notation as in the proof of Theorem ??, if \( d \) satisfies
\[ d^T e = 1, \quad d^T c = \frac{1}{2}, \quad d^T c^2 = \frac{1}{3}, \]
\[ d^T c^3 = \frac{1}{4}, \quad d^T c^4 = \frac{1}{5}, \quad d^T Ac^3 = \frac{1}{20}. \]

then we can have a fourth order error estimator \((b^T - d^T)hF\), where \(b\) is the 6th row of \(A = A(6, :)\) and \(hF\) is the column vector composed of \(hf(Y_1), hf(Y_2), \ldots, hf(Y_6)\). Therefore, the coefficient matrix with error estimator can be found once the remaining free parameter is given. The following tableau is obtained with the choice of the free parameter \(a_{65} = \frac{4}{35}\), to satisfy the “fourth order bushy tree condition”.

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3 - \sqrt{3}}{4} & \frac{\sqrt{3}}{12} & \frac{6 - \sqrt{3}}{24} & \frac{12 - 7\sqrt{3}}{24} & 0 & 0 & 0 & 0 \\
\frac{3 + \sqrt{3}}{4} & -\frac{\sqrt{3}}{12} & \frac{12 + 7\sqrt{3}}{24} & \frac{6 + \sqrt{3}}{24} & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{7}{36} & \frac{1 + \sqrt{3}}{36} & \frac{1 - \sqrt{3}}{36} & \frac{1}{4} & 0 & 0 & 0 \\
\frac{3}{4} & -\frac{29}{128} & \frac{114 + 61\sqrt{3}}{128} & \frac{114 - 61\sqrt{3}}{128} & -\frac{135}{128} & \frac{1}{4} & 0 & 0 \\
\frac{1}{4} & \frac{7}{60} & \frac{11 + 9\sqrt{3}}{90} & \frac{11 - 9\sqrt{3}}{90} & \frac{3}{10} & \frac{4}{25} & \frac{1}{4} & 0 \\
1 & \frac{7}{60} & \frac{11 + 9\sqrt{3}}{90} & \frac{11 - 9\sqrt{3}}{90} & \frac{3}{10} & \frac{4}{25} & \frac{1}{4} & 0 \\
\frac{193}{1710} & \frac{124 + 92\sqrt{3}}{855} & \frac{124 - 92\sqrt{3}}{855} & \frac{32}{95} & \frac{128}{855} & \frac{41}{190} & & \\
\end{array}
\]

We note that method (??) has \(c_3 \approx 1.183\) greater than 1.

### 4 Starting methods

Because the first stage value of a FSAL method comes from the last stage of the previous step, we need a starting method for FSAL methods. The stage value of the last stage for a \(p\)-th order FSAL method is an approximation of order \(p\), and the stability function for a \(p\)-th order FSAL method (say with \(p + 1\) nonzero stages) has the denominator \((1 - \lambda z)^{p+1}\). Due to these two considerations, we consider to use the DESI type methods with lower stage order as the starting methods for the FSAL methods. If we want to have a \(p\)-th order method satisfying \(C(q)\) (\(q \leq p\)), unlike the classical DESI method, the number of the singly-implicit stages is only \(q\). The first row of the starting methods is not a zero vector, hence the minimum stage number we need is one stage less than the corresponding FSAL method. For example, for a fourth order and fifth order starting methods, we only need 5 and 6 stages respectively.

We now give a starting method for method (??) and method (??). Because the stability function for this 5-stage, order-4 method is the same as for the methods (??) and (??), we choose \(\lambda = \frac{1}{4}\) for this starting method. From [?] or [?], the first two abscissae of the method are given by \(c_1 = \lambda \eta_1\), \(c_2 = \lambda \eta_2\), where \(\eta_1, \eta_2\) are two zeros of the second Laguerre polynomial. Therefore we have

\[
c_1 = \lambda(2 - \sqrt{2}) = \frac{2 - \sqrt{2}}{4}, \quad c_2 = \lambda(2 + \sqrt{2}) = \frac{2 + \sqrt{2}}{4}.
\]
If we choose $c_3 = 0$, $c_4 = \frac{1}{2}$, $c_5 = 1$, then the coefficient matrix $A$ can be determined by the order conditions immediately. We write the corresponding tableau for this method as follows.

\[
\begin{array}{cccc|cccc}
2-\sqrt{2} & 4-\sqrt{2} & 4-3\sqrt{2} & 0 & 0 & 0 \\
2+\sqrt{2} & 4+3\sqrt{2} & 4+\sqrt{2} & 0 & 0 & 0 \\
0 & -1-\sqrt{2} & -1+\sqrt{2} & \frac{1}{4} & 0 & 0 \\
\frac{1}{2} & \frac{7(1+\sqrt{2})}{80} & \frac{7(1-\sqrt{2})}{80} & \frac{3}{40} & \frac{1}{4} & 0 \\
1 & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{4} & \frac{5}{6} & \frac{1}{4} \\
1 & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{4} & \frac{5}{6} & \frac{1}{4}
\end{array}
\]  

(4)

We note that method (??) has order 4, stage order 2 with 5 stages (including 2 singly-implicit stages and 3 diagonally-implicit stages). The fourth order FSAL method (??) has 1 singly-implicit stage and 4 diagonally-implicit stages. It is interesting to see if there is any advantage in the “FSAL” idea by comparing these two methods. Furthermore, it is also easy to construct a method to be the same as the method (??) but with stage order 3. In other words, this L-stable method proposed has fourth order, five stages and satisfies $C(3)$. The tableau of this method is given numerically as follows.

\[
\begin{array}{cccc|cccc}
0.1039 & 0.1196 & -0.0177 & 0.0019 & 0 & 0 \\
0.5735 & 0.3294 & 0.2539 & -0.0097 & 0 & 0 \\
1.5724 & 0.0885 & 1.1074 & 0.3764 & 0 & 0 \\
0 & -0.3269 & 0.0871 & -0.0101 & 0.25 & 0 \\
1 & 0.6506 & 0.3673 & -0.0180 & -0.25 & 0.25 \\
1 & 0.6506 & 0.3673 & -0.0180 & -0.25 & 0.25
\end{array}
\]  

(5)

Similarly, we also want to examine the difference of the numerical behaviours between the FSAL method (??) and (??).

5 Numerical experiments

In this section, we present some numerical results for the Robertson problem

\[
\begin{align*}
y'_1 &= -0.04y_1 + 10^4y_2y_3, & y_1(0) &= 1, \\
y'_2 &= 0.04y_1 - 10^4y_2y_3 - 3 \times 10^7y_2^2, & y_2(0) &= 0, \\
y'_3 &= 3 \times 10^7y_2^2, & y_3(0) &= 0, \\
x &\in [0, 10^10],
\end{align*}
\]  

(6)
and the Van der Pol oscillator

\[
\begin{align*}
    y'_1(x) &= y_2(x), & y_1(0) &= 2, \\
    y'_2(x) &= 10^6(1 - y_1(x)^2)y_2(x) - y_1(x), & y_2(0) &= 0,
\end{align*}
\]

\[x \in [0, 2].\] (7)

For \( q = 2 \), we compare the methods (??) (START24) and FSAL (??) for these problems. For \( q = 3 \), we compare methods (??) (START34) and FSAL (??). For two FSAL methods, we use method (??) as the starting methods. All experiments have been done by using variable step size. We note that all these methods have 5 stages and that the FSAL methods have one less singly-implicit stage than the corresponding starting method. The flops/error diagrams are shown in the Figure ??, ??, ??, ??, ??.. As we can see, for both test problems, two FSAL methods are more efficient than their corresponding starting methods. The differences are larger in the case of \( q = 3 \) (especially for the Van der Pol problem). We note that one of the abscissae of the method (??) is \( c_3 \approx 1.5724 \).

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**Figure 1**: Work/precision diagram for the Van der Pol problem (??) using methods (??) and (??)

**Figure 2**: Work/precision diagram for the Robertson problem (??) using methods (??) and (??)

### 6 Conclusions

In this paper, we have shown that the derivative of the first internal stage for a singly-implicit Runge-Kutta procedure in solving the stiff differential equations can be obtained from the previous step without any loss of performance. Furthermore, there may even be computational advantages because of the reduction in the number of singly-implicit stages. When more additional diagonally-implicit stages are added, there are more free parameters which can be used for estimating the error. It even becomes possible to estimate the error for possible higher order methods; this can be used for variable order purpose.
Figure 3: Work/precision diagram for the
Van der Pol problem (??) using methods (??)
and (??)

Figure 4: Work/precision diagram for the
Robertson problem (??) using methods (??)
and (??)

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