Rebalancing Strategies for Multi-Period Asset Allocation

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Adding alternative investments to a portfolio of traditional assets can improve performance for long-term investors, such as pension plans, university endowments, and high net worth individuals and family offices, who together comprise the majority of investments in this area. Some investors, including Harvard, Yale, and Princeton Universities, have achieved eye-popping returns over the past decade by employing venture capital, hedge funds, and private equity to a much greater degree than most investors.

Despite their obvious benefits, alternative investments are still underrepresented in a typical investment portfolio. One of the reasons, we argue, is the difficulty of rebalancing a portfolio that includes alternative investments.

This article presents a practical framework for analyzing a number of real-world issues arising in portfolio management.[1] A policy optimization model is proposed within the context of multi-period asset allocation for large wealthy investors. For simplicity, we have assumed that alternative investments are held through tax-exempt or tax-deferred locations, but the model could easily be modified to introduce taxes into the process.

We give a short discussion on the topic of alternative investment for the portfolio of institutional investors. Some pros and cons of alternative investments are listed below.

PROS

Institutional investors generally expect to earn higher returns than those available in similar long-only public equity strategies. The alternative asset manager (e.g., venture capitalist, buyout specialist, etc.) produces an important component of those investment returns. Private markets are not efficiently priced so a skillful manager can earn positive excess returns. Furthermore, the manager often participates in the governance and strategic management of its investee companies, which could contribute to the creation of additional value.

Even if the manager is not exceptional, higher returns should be expected for accepting long-term illiquidity associated with most alternative assets.

Many alternative asset strategies involve the use of financial leverage. This can be a positive source of return for the institution that cannot usually borrow against its portfolio for investment purposes. By investing in strategies that use financial leverage, the portfolio can be "levered up" to a risk level that suits the tolerances of the institution.

Diversification is available through the use of a fund-of-funds approach, using multiple managers in multiple strategies. Rebalancing is more easily accomplished when the alternative asset cash flows are spread across many different investments.

Hedge fund managers are able to use leverage, niche market expertise, liquidity provision, cross-market and cross-border arbitrage, and relative value strategies and timing to extract alpha from various markets, at times with impressive results.

CONS

Identifying and selecting superior alternative asset managers is difficult and time consuming. Institutions must meet and analyze each manager separately with sparse, non-standardized information. Even when the process is completed, institutions may find they cannot get access to the managers of their choice!

Most alternative asset funds require long-term commitments, often ten years or longer. Though the investor is likely to be compensated for illiquidity through returns, rebalancing of assets is hindered. By not taking advantage of a rebalancing opportunity, the investor is not able to fully use the benefits of diversification of the asset category and may be missing out on increased portfolio returns.

Even short-term strategies pursued by hedge funds run the risk of market advantages narrowing or closing, sometimes leading to spectacular losses as in Long Term Capital Management.

Analyzing alternative asset performance has been more like art than science. It is extremely difficult to determine the market value of private investments and, thus, constructing a benchmark and measuring performance against the benchmark is a daunting task.

Alternative asset returns and variance are best analyzed in a multi-year framework. However, many institutions use single-period models to assist in asset allocation decisions. A multi-year decision model may have to be used for a more accurate perspective.

The next section describes a framework for optimizing a portfolio of assets (including alternatives) over a long multi-period horizon. We show that multi-period asset allocation can give rise to higher returns than a single-period model by taking advantage of the market's inherent volatility. Alternative investments are desirable in this context. However, transaction costs and limitations will reduce the gain if naïve rebalancing rules are employed. Section 4 takes up the issue of rebalancing a portfolio in the face of transaction costs. We show that intelligent rebalancing rules can substantially reduce the impact of transaction costs. Some directions for future research are mentioned in Section 5.

MULTI-PERIOD ASSET ALLOCATION

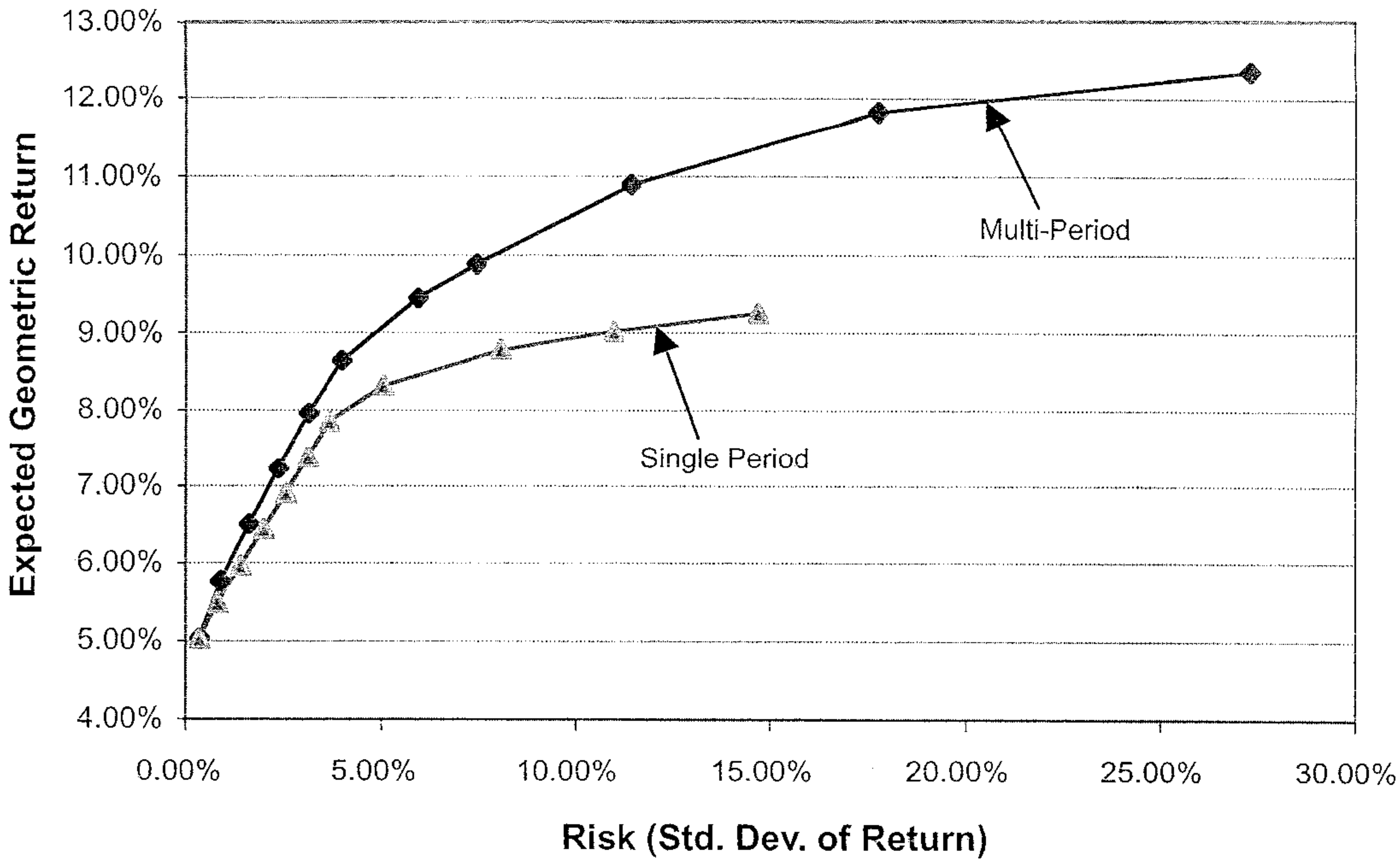
This section presents a systematic approach for modeling asset allocation decisions over a long-term horizon. In particular, we are interested in testing rebalancing rules in the face of illiquidity and substantial transaction costs, such as arising from alternative investments. A wealthy individual will be the focus for the study. See Reichenstein [2001] and Brunel [2001] for a discussion of the asset allocation process.

First, we compare the multi-stage asset allocation model with the traditional single-period Markowitz model. The standard output of a portfolio model consists of an efficient frontier, plotting expected reward against risk (e.g., standard deviation, downside risk, expected policyholder deficit). We will display the same output[2]—efficient frontier. However, the focus is investor's wealth at the end of the 10-year planning horizon after intermediate decisions and cash flows have occurred. The single-period efficient frontier requires a set of mean returns and covariances for the chosen asset categories (Appendix A). These numbers are derived from a combination of historical data, expert opinion, current market conditions, as well as by understanding the modeling implications of the estimates.

By solving the standard quadratic program in a sequential fashion with the data listed in Appendix A, we can compute the single-period efficient frontier shown in Exhibit 1 (with accompanying asset proportions). At the bottom left-hand side on the efficient frontier, cash is presented as the safe asset (first row in Appendix B, i.e., 99.6% allocation in cash). On the other hand, the top right side depicts the most aggressive portfolio, typically a single asset with the highest expected returns (last row in Appendix B). In our example, private equity is presented at the 100% proportion (with expected geometric return = 9.26% and standard deviation = 14.68%). Nothing is new here.

The problem becomes more complex when the investor wants to analyze his financial position over time. For instance, someone approaching retirement may wish to compute the chances that he has set aside adequate funds. What is the probability that the investor will have $1 million after 10 years of future savings and investment income? A single-period model will be unable to answer this question. Rather, we turn to multi-period optimization models; see Ziemba and Mulvey [1998] for a survey of asset and liability management systems. As presented in that book, there are two practical approaches for the modeling framework: policy optimization (Mulvey et al.

EXHIBIT 1
Efficient Frontiers for Single-Period and Multi-Period Solutions



[1997]) and multi-stage stochastic programming (Carino et al. [1994]; Consigli and Dempster [1998]; Kouwenberg and Zenios [2001]). To simplify the presentation, we will focus on policy optimization for asset allocation.

There are three basic elements of any multi-period asset allocation: 1) a system of stochastic equations for generating scenarios (the projection system); 2) a decision simulator; and 3) an optimization module for finding dominating portfolios. The scenario generator follows a factor approach, in which a sequence of economic factors drives the asset returns. See Mulvey [1996 and 2001] for further details regarding the scenario generation process. The output is a set of plausible scenarios for the asset returns over the long-term planning horizon.

To fix the decision process, we employ the fixed-mix policy rule. Here the goal is to rebalance the portfolio to a target mix at the beginning of each time period. For instance, we might set the total equity to 60% of the investor's wealth. The fixed-mix decision rule has several desirable characteristics, including stability and simplicity; it serves as a benchmark for more complex strategies. Letting x_j be the target proportion for asset j in asset set J, we can set up the multi-period optimization model with two objectives—expected reward and risk of the total asset wealth at the end of the planning horizon:

Multi-Period Asset Allocation:[3]

$$\text{Maximize } \{(1 - \lambda) * \text{Return}(\text{wealth}_t^s) - \lambda * \text{Risk}(\text{wealth}_t^s)\}$$

$$\text{Subject to } \sum_j x_{j,t}^s = \text{wealth}_t^s$$

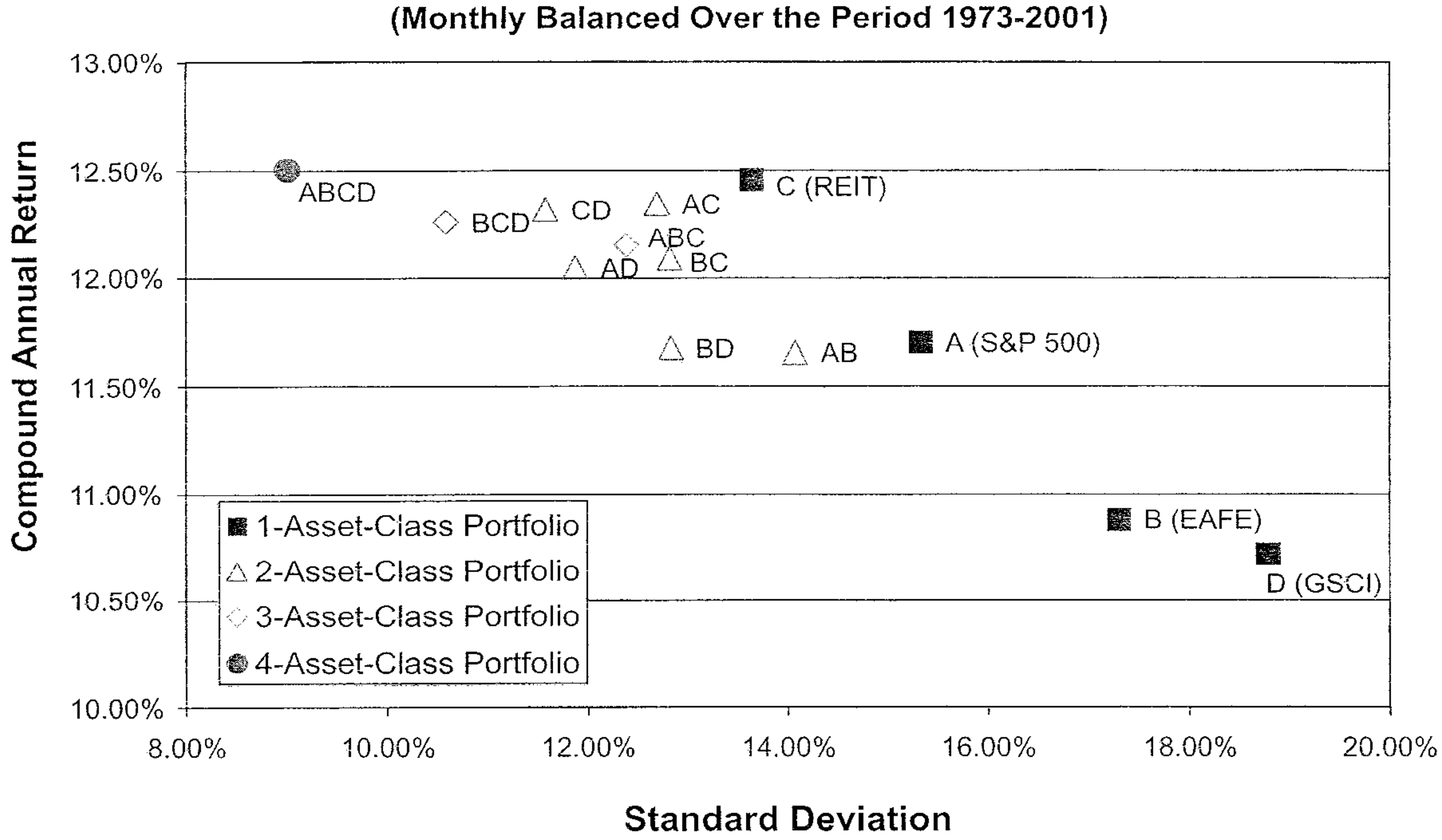
This optimization model is complicated due to the presence of nonconvexity, thus requiring sophisticated optimization algorithms. See Ziemba and Mulvey [1998] for a discussion of solving nonconvex models.

We solve the problem for our target investor. Exhibit 1 displays the results. There are several noteworthy features to the multi-period solution. First, the top right hand side of the efficient frontier will generally consist of multiple assets. In our case, the optimal solution consists of five assets—5.4% emerging market equity, 38.8% equity sector A, 37.8% equity sector B, 14.6% private equity, and 3.4% hedge funds (last row in Appendix C).

Importantly, the geometric returns for this mix are superior to *any* individual asset returns—over 12% versus 9.26% for the highest returning single asset (private equity in Appendix B). This characteristic is a result of a phenomenon known as volatility pumping, which distin-

EXHIBIT 2

The Rewards of Multiple-Asset-Class Investing (1973–2001)



guishes the risk tolerant side of the multi-period efficient frontier. For a theoretical analysis of volatility pumping, see Fernholtz and Shay [1982]. Remember that the highest point on the efficient frontier for a single-period model will be a single asset, unless the investor places constraints on certain asset mixes. (See Swenson [2000] for a pertinent, informal discussion of the idea that diversification can be developed by means of a collection of high return assets, including alternative investments.)

From a practical perspective for individual investors, the multi-period model offers numerous advantages. First, investors can readily include their future cash flows at each period, such as paying pension liabilities, making contributions and savings, as well as paying taxes and addressing transaction costs. The area of asset and liability management for individual investors becomes more accessible with the development of a multi-period framework.

To reinforce the volatility-pumping notion, we illustrate the historical returns of four primary asset categories—S&P 500, EAFE, REITs, and the Goldman Sachs Commodity Index (GSCI) over the period 1973 to 2001. Exhibit 2 shows the geometric returns and volatility of each of these assets. Note that the combination S&P 500, EAFE, and REITs (ABC) provides improvements over the single assets. But when the GSCI is added, the combination of four assets (ABCD) gives much lower volatility without sacrificing expected returns! The GSCI index has the lowest expected return and the highest volatility; it would never occupy the efficient frontier of a single-period model. Yet its addition helps the overall portfolio within a dynamic investment framework.

As good as volatility pumping can be, we must address transaction costs since the rebalancing of a portfolio each period (be it months, quarters, or years) will take a toll on the overall returns. The next section takes up this implementation issue.

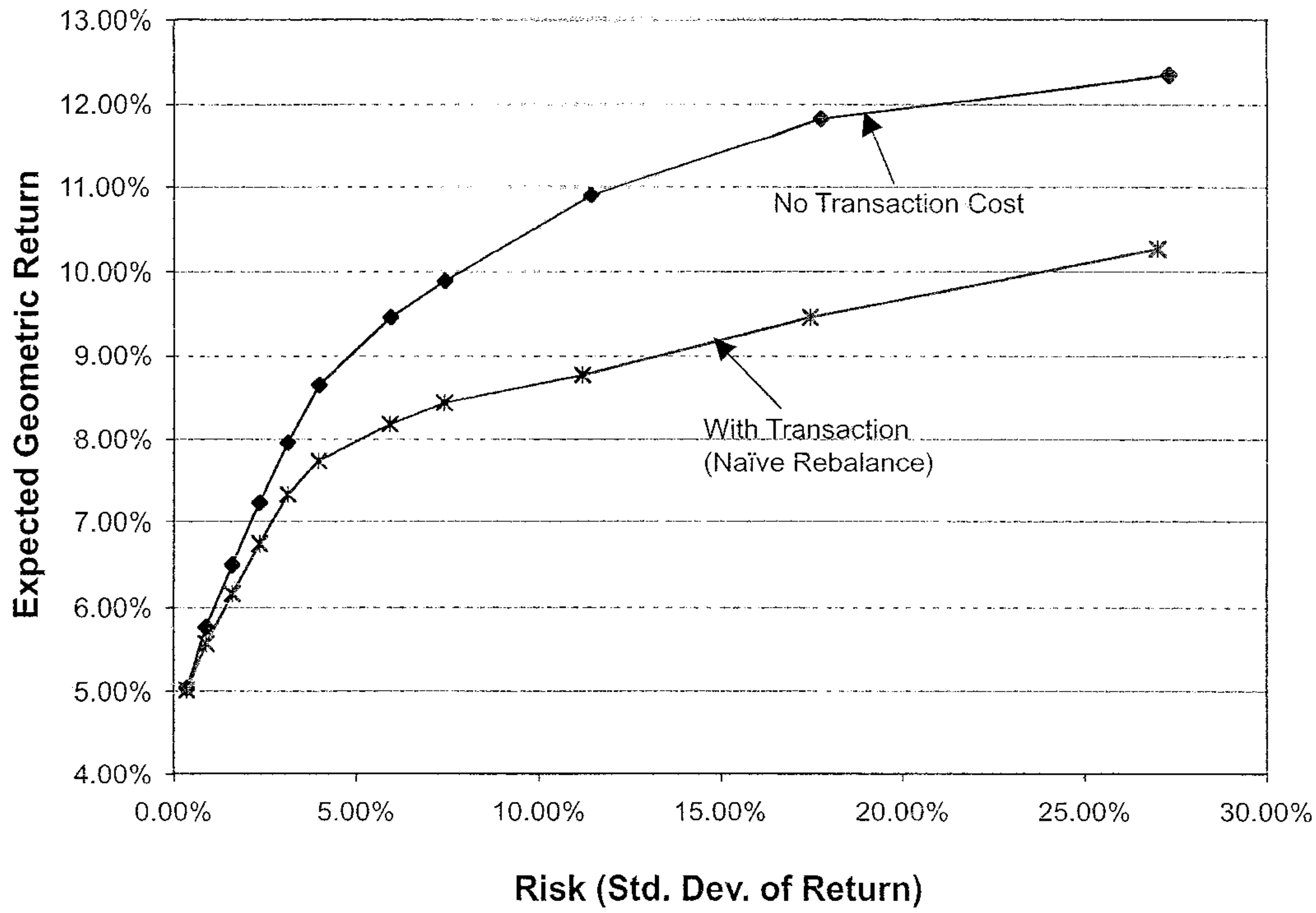
TRANSACTION AND MARKET IMPACT COSTS

This section considers the problem of transaction and market impact costs of rebalancing portfolios within the dynamic asset allocation framework.

As a first step, we combine a number of rebalancing issues under the term *transaction costs*. For example, the investor might have embedded capital gains in a taxable

EXHIBIT 3

Impact of Transaction Costs on Multi-Period Efficient Frontier



account, and selling these assets would trigger large tax payments. In an institutional setting, a pension plan may cause the market price to move when it tries to rebalance an asset category. In the alternative investment arena, there are numerous stipulations regarding the purchase or sale of the investor's assets. Venture capital firms and increasingly hedge funds, for example, require the investor to remain fully invested for a number of years, and may also expect further contributions in certain cases. There are formal constraints on the money flows.

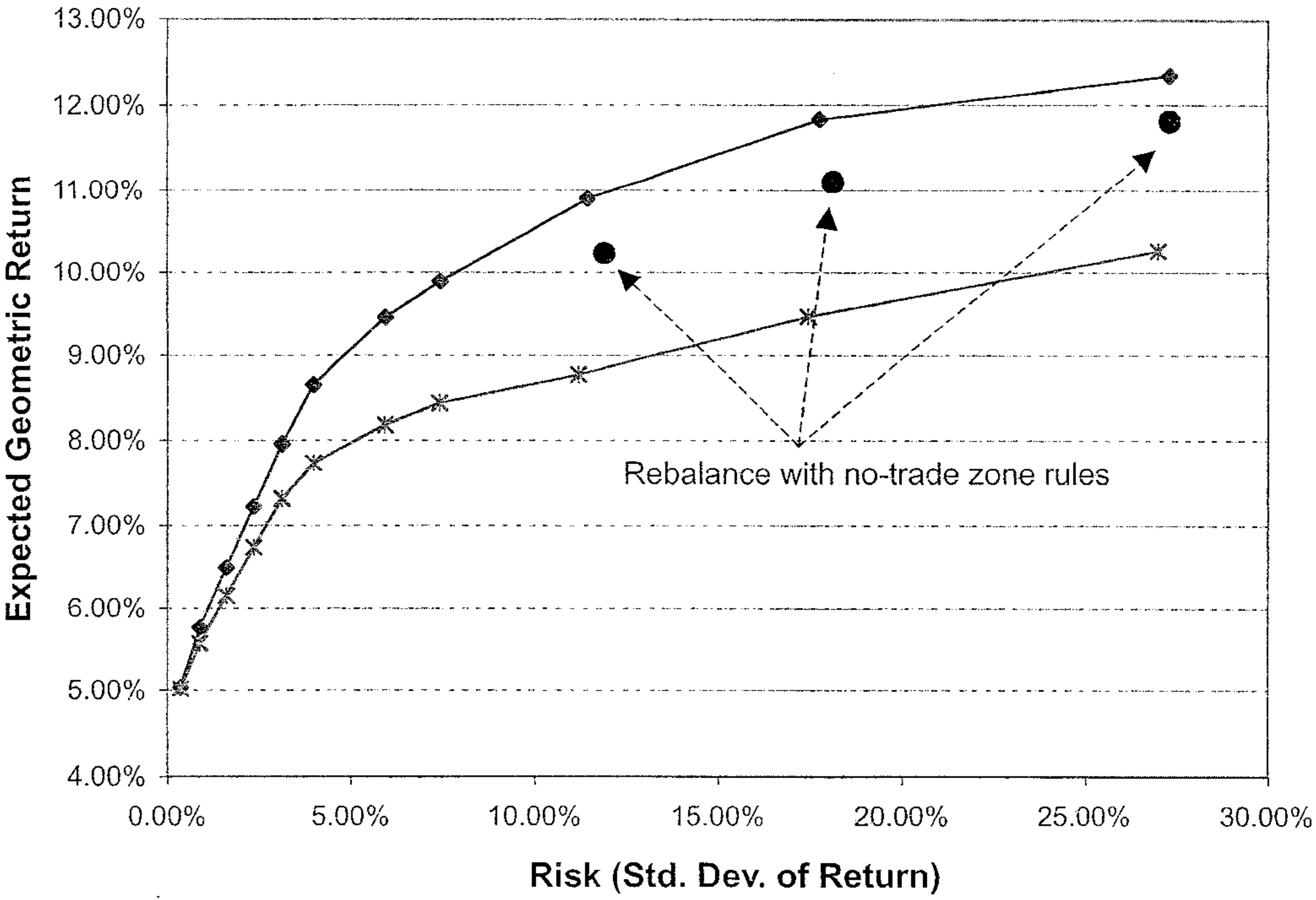
We can include many of these considerations within the multi-period framework. It is a matter of adjusting the cash flows conditional on the individual scenarios. A strong equity market scenario, for example, will likely show large gains accruing to the venture capital industry. Conversely, a recession scenario will cause pain for the high-tech venture firms. Further research is needed to refine the relationships between alternative scenarios and investment returns to the point that they can be modeled with confidence.

Nevertheless, we can make progress on the transaction cost issue. For simplicity, we assume that all transaction costs are calculated as a linear (or piecewise linear) function of purchases and sales. Appendix A depicts transaction costs estimates for the 15 asset categories. Let's look at the impact of these transaction costs on the investor's wealth at the end of the 10-year period. Exhibit 3 shows the impact on overall returns assuming that the investor rebalances his portfolio to the suggested targets at the beginning of each month and using the lower end of the range of transaction costs defined in Appendix A. (Kipcak [2001] obtained similar results for the other levels of transaction costs and re-balancing periods). In this example, the investor returns to the target proportions each time period. The impact of transaction costs is significant on the high end of the frontier (over 100 basis points), since the recommended assets possess relatively high transaction costs relative to the other assets.

In papers dealing with transaction costs, the form of the optimal policy is to establish a no-trade zone around each of the asset targets (proportions). For instance, the goal may be to maintain equities at 60% of wealth. However, we will not take voluntary actions unless the actual proportion falls outside the interval, say 57% to 63%. We call this interval the no-trade zone. By testing various rules, Kipcak [2001] found that acceptable results are possible

EXHIBIT 4

Rebalancing with Intelligent Rules—No-Trade Zone Rules



by setting the width of the zone as a function of the volatility and the transaction cost for each asset. For example, assets with high volatilities should be assigned larger zones than assets with low volatilities. A linear formula proved to be sound. Accordingly, the parameters are determined by means of an optimization model.

As Exhibit 4 shows, the use of the no-trade zone policy rule can lead to much improved expected returns. The rebalancing decisions have a significant impact on portfolio performance and can generate results approaching the original efficient frontier (ignoring transaction costs).

In this example, we were able to substantially reduce the impact of transaction costs by setting no-trade zones around each asset. The theory of stochastic control supports this contention. The multi-period asset allocation model allows the investor to evaluate his intuition with a set of reasonable assumptions.

CONCLUSIONS AND DIRECTIONS FOR FUTURE RESEARCH

This article describes a practical method for addressing transaction and market impact costs within a multi-period asset allocation framework. The basic idea is to form no-trade zones for each asset, as a function of that asset's volatility and transaction costs. This approach can achieve good performance, at least in terms of idealized but realistic examples. The solution to a multi-period optimization is much different from the solution derived from a traditional single-period Markowitz model, especially for risk-tolerant investors. For long-term investors, the goal should be to achieve the highest return possible, while diversifying through combinations of high-return assets. Partially as a result of volatility pumping, the top of a multi-period efficient frontier consists of multiple assets.

Private investments and hedge fund strategies promise to fill the role of the aforementioned assets. Unfortunately, these assets are notoriously difficult to rebalance. The no-trade zone concept provides an initial approach for addressing the illiquidity of alternative assets. Research is needed to refine the implementation tactics. Investors may be required to hold onto an asset category, such as venture capital, during sharp downturns. Surrogate investments may be able to hedge some of the rebalancing gains, when relatively high correlations exist. Basis risks may arise unexpectedly when markets become volatile and unstable. To this end, a multi-period investment model addresses the portfolio rebalancing issue in a systematic, scenario-dependent fashion.

APPENDIX A

Asset Class and Their Annualized Expected Returns, Risk (Measured via Standard Deviation), and Transaction Costs

Asset Class	Annual Expected Geo Return	Standard Deviation of Return	Low Transaction Costs Rate	Moderate Transaction Costs Rate	High Transaction Costs Rate
Cash	5.01%	1.75%	0.00%	0.00%	0.00%
US High Quality Bonds	7.03%	4.40%	0.01%	0.50%	1.00%
International Bonds	6.86%	6.87%	0.01%	1.00%	2.00%
Emerging Market Bonds	8.52%	16.96%	0.01%	3.00%	6.00%
Government Bonds	6.71%	3.83%	0.01%	0.50%	1.00%
US High Yield Bonds	8.28%	8.72%	0.03%	3.00%	6.00%
US Equity	8.96%	19.12%	0.50%	1.00%	2.00%
European Equity	8.92%	12.35%	0.50%	2.00%	4.00%
Far East Equity	9.11%	20.39%	0.50%	3.00%	6.00%
Emerging Market Equity	8.99%	24.80%	0.50%	3.00%	6.00%
Equity Sector A	8.95%	44.48%	3.00%	3.00%	6.00%
Equity Sector B	9.08%	38.85%	3.00%	3.00%	6.00%
Private Equity	9.26%	14.07%	5.00%	15.00%	30.00%
Real Estate	7.78%	15.34%	5.00%	5.00%	10.00%
Hedge Funds	8.33%	7.57%	10.00%	10.00%	20.00%

APPENDIX B

Efficient Mixes of Single-Period Strategy

Efficient Mix	Cash	US High Quality Bonds	International Bonds	Emerging Market Bonds	Government Bonds	US High Yield Bonds	US Equity	European Equity	Far East Equity	Emerging Market Equity	Equity Sector A	Equity Sector B	Private Equity	Real Estate	Hedge Funds	**Mean Geometric Return**	**Standard Deviation**
1	99.6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.0	0.2	**5.02**	**0.35**
2	82.4	3.5	3.6	0.0	0.0	1.5	0.0	0.0	0.0	0.0	0.0	0.1	0.7	1.1	7.0	**5.49**	**0.81**
3	64.9	7.4	7.3	0.0	0.0	3.2	0.0	0.0	0.0	0.0	0.0	0.3	0.9	2.6	13.5	**5.97**	**1.4**
4	47.9	10.4	10.6	0.0	0.0	5.4	0.0	0.0	0.0	0.0	0.0	0.4	1.1	4.2	20.0	**6.44**	**1.99**
5	30.2	14.3	14.7	0.0	0.0	6.6	0.0	0.0	0.0	0.0	0.0	0.6	1.5	5.9	26.2	**6.91**	**2.56**
6	12.7	18.2	18.3	0.0	0.0	8.2	0.0	0.0	0.0	0.0	0.0	0.8	1.5	7.4	32.9	**7.38**	**3.1**
7	0.0	15.8	18.3	0.0	0.0	12.4	0.0	0.0	0.0	0.0	0.0	1.0	3.1	7.7	41.6	**7.85**	**3.67**
8	0.0	0.0	4.7	0.0	0.0	19.0	0.0	0.0	0.0	0.0	0.0	1.6	9.3	5.7	59.7	**8.32**	**5.04**
9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	3.4	46.5	0.0	50.1	**8.79**	**8.06**
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	4.1	71.3	0.0	24.6	**9.02**	**10.97**
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	100.0	0.0	0.0	**9.26**	**14.68**

The last two columns were used to generate the efficient frontier curve on Exhibit 1.

APPENDIX C

Efficient Mixes of Multi-Period Strategy

Efficient Mix	Cash	US High Quality Bonds	International Bonds	Emerging Market Bonds	Government Bonds	US High Yield Bonds	US Equity	European Equity	Far East Equity	Emerging Market Equity	Equity Sector A	Equity Sector B	Private Equity	Real Estate	Hedge Funds	**Mean Geometric Return**	**Standard Deviation**
1	99.4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.3	0.0	0.3	**5.03**	**0.35**
2	77.8	3.4	4.7	0.0	0.0	1.3	0.0	0.0	0.0	0.0	0.0	0.9	1.4	1.7	8.9	**5.76**	**0.89**
3	55.7	7.2	9.3	0.0	0.0	3.0	0.0	0.0	0.0	0.0	0.0	1.8	1.9	3.9	17.2	**6.49**	**1.62**
4	33.5	11.9	13.4	0.0	0.0	4.5	0.0	0.0	0.0	0.0	0.0	2.7	2.5	6.0	25.5	**7.22**	**2.37**
5	11.4	16.4	17.4	0.0	0.0	6.1	0.0	0.0	0.0	0.0	0.0	3.6	3.2	8.3	33.7	**7.96**	**3.13**
6	0.0	16.2	14.1	0.0	0.0	7.1	0.0	0.0	0.0	0.3	0.0	6.0	6.5	9.5	40.2	**8.65**	**3.99**
7	0.0	2.4	2.0	0.0	6.0	5.0	0.0	0.0	0.0	3.3	1.3	9.6	9.7	8.3	52.5	**9.45**	**5.94**
8	0.0	0.0	0.0	0.4	2.6	5.2	0.6	0.7	0.7	4.2	3.1	11.0	10.2	8.5	52.8	**9.89**	**7.44**
9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	2.5	10.7	18.4	19.7	0.0	48.8	**10.9**	**11.43**
10	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	4.7	22.8	26.0	12.7	0.1	33.6	**11.83**	**17.76**
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	5.4	38.8	37.8	14.6	0.0	3.4	**12.34**	**27.31**

The last two columns were used to generate the efficient frontier curve on Exhibit 1.

ENDNOTES

[1]Thanks are in order to Ezra Zask (visiting lecturer at Princeton University) and Husnu Kipcak (MSE student at Princeton University).

[2]Numerous issues arise regarding the investor's preferences in a temporal setting. For instance, there are several approaches for measuring risk by means of volatility. Typically, it is best to display a number of alternative dimensions to the investor.

[3]The decision variables are distinguished by asset category j, at each time period t, for each scenario s.

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