FOURIER-LIKE SOLUTION OF THE DIRICHLET PROBLEM FOR THE LAPLACE EQUATION IN K-TYPE GIELIS DOMAINS

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Abstract

The interior and exterior Dirichlet problems for the Laplace equation in $k$-type Gielis domains are analytically addressed by using a suitable Fourier-like technique. A dedicated numerical procedure based on the computer-aided algebra

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tool Mathematica® is developed in order to validate the proposed approach. In this way, highly accurate approximations of the solution, featuring properties similar to the classical ones, are obtained. Computed results are found to be in good agreement with theoretical findings on Fourier series expansion presented by Carleson.

1. Introduction

Many boundary-value problems (BVPs) arising in mathematical physics and electromagnetics are related to the Laplacian operator [1, 7]. Among them, it is worth mentioning those relevant to the Laplace and Poisson equations, the wave and Helmholtz equations, as well as the Schrödinger equation. However, most of the mentioned differential problems can be solved in explicit way only in canonical domains with special shape or symmetries [7].

Different approaches based on boundary layer techniques, integral equation methods, conformal mappings, and least squares procedures, have been proposed in the scientific literature to overcome this limitation [6, 8, 10, 11]. In this paper, an original analytical approach, tracing back to the classical Fourier projection method [2, 9], is developed for solving the Dirichlet problem for the Laplace equation in complex two-dimensional domains, whose boundaries are described by a $k$-type Gielis equation providing a unified description of a wide variety of natural-shaped and abstract curves [4, 5]. Regular functions are considered for the boundary values, although the presented theory can be easily extended by assuming weakened hypotheses.

In order to assess the relevant technique, a suitable numerical procedure based on the computer-aided algebra tool Mathematica® has been developed. By using such procedure, a point-wise convergence of the Fourier-like series representation of the solution has been observed, with Gibbs-like phenomena potentially occurring in the singular points of the boundary. The obtained numerical results are in good agreement with the theoretical findings by Carleson [3].
2. The Laplacian in Stretched Polar Co-ordinates

Let us consider in the Euclidean plane $\mathbb{E}^2$, the classical polar co-ordinate system

\[
\begin{aligned}
x &= \rho \cos \vartheta, \\
y &= \rho \sin \vartheta,
\end{aligned}
\]  

(2.1)

and the starlike domain $\mathcal{D}$, whose boundary $\partial \mathcal{D}$ is described by the equation

\[
\rho = R(\vartheta),
\]  

(2.2)

where $R(\vartheta)$ is a $C^2$ function in $[0, 2\pi]$ such that $m_R = \min_{\vartheta \in [0, 2\pi]} R(\vartheta) > 0$, and $M_R = \max_{\vartheta \in [0, 2\pi]} R(\vartheta) < 1$. Therefore, upon introducing the stretched radius

\[
r = \rho / R(\vartheta),
\]  

(2.3)

it is straightforward to see that $\mathcal{D}$ satisfies the inequalities $0 \leq \vartheta \leq 2\pi$ and $0 \leq r \leq 1$.

Let us consider a $C^2(\mathcal{D})$ function $v(x, y) = u(\rho \cos \vartheta, \rho \sin \vartheta) = u(\rho, \vartheta)$, and the relevant Laplace operator

\[
\Delta u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \vartheta} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \vartheta^2}.
\]  

(2.4)

Upon setting

\[
U(r, \vartheta) = u(rR(\vartheta), \vartheta),
\]  

(2.5)

one can readily find

\[
\frac{\partial u}{\partial \rho} = \frac{1}{R} \frac{\partial U}{\partial r},
\]  

(2.6)

\[
\frac{\partial^2 u}{\partial \rho^2} = \frac{1}{R^2} \frac{\partial^2 U}{\partial r^2}.
\]  

(2.7)
Hence, substituting Equations (2.6)-(2.9) into Equation (2.4) yields the following expression of the Laplacian in the stretched co-ordinate system $r, \vartheta$:

$$
\Delta u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \vartheta^2} = \frac{1}{R^2} \left( 1 + \frac{R^2}{R^2} \right) \frac{\partial^2 U}{\partial r^2} + \frac{1}{rR^2} \left( 1 + \frac{2R^2}{R^2} \right) \frac{\partial U}{\partial r} - \frac{2R}{R^3} \frac{\partial^2 U}{\partial r \partial \vartheta} + \frac{1}{r^2 R^2} \frac{\partial^2 U}{\partial \vartheta^2}. 
$$

(2.10)

For $r = \rho$ and $R(\vartheta) = 1$, the Equation (2.4) is immediately recovered.

3. The Dirichlet Problem for the Laplace Equation

Let us consider the interior Dirichlet problem for the Laplace equation in a starlike domain $D$, whose boundary is described by the polar equation $(\rho = R(\vartheta));$

$$
\begin{cases}
\Delta v(x, y) = 0, & (x, y) \in D, \\
v(x, y) = f(x, y), & (x, y) \in \partial D.
\end{cases}
$$

(3.1)

We prove the following theorem:

**Theorem 3.1.** Let

$$
f(R(\vartheta) \cos \vartheta, R(\vartheta) \sin \vartheta) = F(\vartheta) = \sum_{m=0}^{+\infty} (\alpha_m \cos m\vartheta + \beta_m \sin m\vartheta),
$$

(3.2)

where

$$
\begin{bmatrix}
\alpha_m \\
\beta_m
\end{bmatrix} = \frac{\varepsilon_m}{2\pi} \int_0^{2\pi} \begin{bmatrix} \cos m\vartheta \\ \sin m\vartheta \end{bmatrix} F(\vartheta) d\vartheta,
$$

(3.3)

$\varepsilon_m$ being the usual Neumann’s symbol. Then, the interior boundary-value problem for the Laplace equation (3.1) admits a classical solution.
such that the following Fourier-like series expansion holds:

\[
v(rR(\vartheta) \cos \vartheta, rR(\vartheta) \sin \vartheta) = U(r, \vartheta) = \sum_{m=0}^{\infty} [rR(\vartheta)]^m (A_m \cos m\vartheta + B_m \sin m\vartheta).
\]

The coefficients \( A_m \) and \( B_m \) in (3.5) are determined by solving the infinite linear system

\[
\sum_{m=0}^{\infty} \begin{bmatrix} X^+_{n,m} & Y^+_{n,m} \\ X^-_{n,m} & Y^-_{n,m} \end{bmatrix} \begin{bmatrix} A_m \\ B_m \end{bmatrix} = \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix},
\]

where

\[
X^\pm_{n,m} = \frac{\varpi_n}{2\pi} \int_0^{2\pi} R(\vartheta)^m \cos m\vartheta \left\{ \begin{array}{c} \cos n\vartheta \\ \sin n\vartheta \end{array} \right\} d\vartheta,
\]

\[
Y^\pm_{n,m} = \frac{\varpi_n}{2\pi} \int_0^{2\pi} R(\vartheta)^m \sin m\vartheta \left\{ \begin{array}{c} \cos n\vartheta \\ \sin n\vartheta \end{array} \right\} d\vartheta,
\]

with \( m, n \in \mathbb{N}_0 \).

**Proof.** In the stretched co-ordinates system \( r, \vartheta \), the domain \( \mathcal{D} \) is transformed into the unit circle. Therefore, we can use the usual eigenfunction method \([7]\), and separation of variables (with respect to \( \rho, \vartheta \)) to solve (3.1). So, elementary solutions of the problem can be searched in the form

\[
u(\rho, \vartheta) = U(\rho / R(\vartheta), \vartheta) = P(\rho)\Theta(\vartheta).
\]

By substituting into the Laplace equation, we easily find out that the functions \( P(\cdot), \Theta(\cdot) \) must satisfy the ordinary differential equations

\[
\frac{d^2\Theta(\vartheta)}{d\vartheta^2} + \mu^2\Theta(\vartheta) = 0,
\]

\[
\frac{d^2P(\rho)}{d\rho^2} + \frac{n^2}{\rho^2}P(\rho) = 0.
\]
\[ \rho^2 \frac{d^2 P(\rho)}{d\rho^2} + \rho \frac{dP(\rho)}{d\rho} - \mu^2 P(\rho) = 0, \quad (3.11) \]

respectively.

The parameter \( \mu \) is a separation constant, whose choice is governed by the physical requirement that at any fixed point in \( \mathbb{E}^2 \), the scalar field \( u(\rho, \vartheta) \) must be single-valued. By setting \( \mu = m \in \mathbb{N}_0 \), we find

\[ \Theta(\vartheta) = a_m \cos m \vartheta + b_m \sin m \vartheta, \quad (3.12) \]

where \( a_m, b_m \in \mathbb{R} \) denote arbitrary constants. The radial function \( P(\cdot) \) satisfying (3.11) can be readily expressed as follows:

\[ P(\rho) = c_m \rho^m + d_m \rho^{-m} \quad (c_m, d_m \in \mathbb{R}). \quad (3.14) \]

As usual, we have to assume \( d_m = 0 \) for the boundedness of the solution. Therefore, the general solution of the interior Dirichlet problem (3.1) can be searched in the form

\[ u(\rho, \vartheta) = \sum_{m=0}^{+\infty} \rho^m (A_m \cos m \vartheta + B_m \sin m \vartheta). \quad (3.15) \]

Finally, imposing the boundary condition

\[ F(\vartheta) = U(1, \vartheta) = u(R(\vartheta), \vartheta) \]

\[ = \sum_{m=0}^{+\infty} R(\vartheta)^m (A_m \cos m \vartheta + B_m \sin m \vartheta), \quad (3.16) \]

and using the Fourier's projection method, the system of linear equations (3.6)-(3.8) easily follows.
Remark 3.1. Let us consider the associated interior Dirichlet problem for the Laplace equation on the unit circle with boundary values \( F(\vartheta) \) given by (3.2). The solution of such problem can be readily expressed as

\[
U(r, \vartheta) = \sum_{m=0}^{+\infty} r^m (\alpha_m \cos m\vartheta + \beta_m \sin m\vartheta).
\] (3.17)

By virtue of the maximum principle, the assumption \( 0 < m_R \leq R(\vartheta) \leq M_R < 1 \) implies that the solution of the problem (3.1) is dominated by (3.17). As a consequence, we have

\[
\left| \sum_{m=0}^{+\infty} R(\vartheta)^m (A_m \cos m\vartheta + B_m \sin m\vartheta) \right| \leq \left| \sum_{m=0}^{+\infty} (\alpha_m \cos m\vartheta + \beta_m \sin m\vartheta) \right|,
\] (3.18)

and by using the linearity of the operator, we find

\[
\left[ \frac{|A_m|}{|B_m|} \right] R(\vartheta)^m \leq \left[ \frac{|A_m|}{|B_m|} \right],
\] (3.19)

with \( m \in \mathbb{N}_0 \). By virtue of the Lebesgue’s theorem, the Fourier coefficients \( \alpha_m, \beta_m \) go to zero as \( m \to +\infty \), and the order of convergence to zero increases with the smoothness of the boundary values \( F(\vartheta) \). According to the inequalities (3.19), the coefficients \( A_m \) and \( B_m \) turn to be infinitesimal as \( m \to +\infty \) since \( R(\vartheta) \) is bounded. This means that the vectorial operator defined by the linear system (3.6) is compact. As a matter of fact, we can split up such operator in the sum of two contributions such that the former is finite-dimensional and the latter features maximum norm as small as, we wish.

In a similar way, the exterior Dirichlet problem

\[
\begin{align*}
\Delta u(x, y) &= 0, & (x, y) &\in \mathbb{R}^2 \setminus \mathcal{D}, \\
u(x, y) &= f(x, y), & (x, y) &\in \partial \mathcal{D},
\end{align*}
\] (3.20)
subject to the null condition at infinity
\[
\lim_{\rho \to +\infty} v(x, y) = 0,
\] (3.21)
may be addressed. In particular, the following theorem can be easily proved:

**Theorem 3.2.** Under the hypotheses of Theorem 3.1, the exterior Dirichlet problem for the Laplace equations (3.20)-(3.21) admit a classical solution
\[
v(x, y) \in C^2(\mathbb{R}^2 \setminus D),
\] (3.22)
such that the following Fourier-like series expansion holds:
\[
v(rR(\theta) \cos \theta, rR(\theta) \sin \theta) = U(r, \theta)
\] = \sum_{m=1}^{\infty} [rR(\theta)]^{-m} (A_m \cos m\theta + B_m \sin m\theta). \tag{3.23}

The coefficients \( A_m \) and \( B_m \) in (3.23) are the solution of the infinite linear system
\[
\sum_{m=1}^{\infty} \begin{bmatrix} X_{n,m}^+ & Y_{n,m}^+ \\ X_{n,m}^- & Y_{n,m}^- \end{bmatrix} \begin{bmatrix} A_m \\ B_m \end{bmatrix} = \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix},
\] (3.24)
where
\[
X_{n,m}^{[\pm]} = \frac{\epsilon_n}{2\pi} \int_0^{2\pi} R(\theta)^{-m} \cos m\theta \begin{bmatrix} \cos n\theta \\ \sin n\theta \end{bmatrix} d\theta,
\] \tag{3.25}
\[
Y_{n,m}^{[\pm]} = \frac{\epsilon_n}{2\pi} \int_0^{2\pi} R(\theta)^{-m} \sin m\theta \begin{bmatrix} \cos n\theta \\ \sin n\theta \end{bmatrix} d\theta,
\] \tag{3.26}
with \( m, n \in \mathbb{N} \).

**Remark 3.2.** Note that the above formulas still hold under the assumption that the function \( R(\theta) \) is a piecewise continuous function, and the boundary values are described by square integrable, not necessarily continuous functions, so that the relevant Fourier coefficients \( \alpha_m, \beta_m \) in (3.3) are finite quantities.
4. Numerical Example

The developed methods can be readily applied to the solution of differential problems in a two-dimensional domain $D$, whose boundary is described by a general $k$-type Gielis curve

$$\rho = R(\vartheta) = \sum_{j=1}^{k} \tilde{r}_G(\vartheta; \gamma_j(\vartheta), a_j, b_j, m_j, n_{1j}, n_{2j}, n_{3j}), \quad (4.1)$$

where

$$\tilde{r}_G(\vartheta; \gamma(\vartheta), a, b, m, n_1, n_2, n_3) = \gamma(\vartheta) \left( \left| \frac{1}{a} \cos \frac{m\vartheta}{2} \right| + \left| \frac{1}{b} \sin \frac{m\vartheta}{2} \right| \right)^{1/n_1} \quad (4.2)$$

denotes the ordinary Gielis polar equation [4] modulated by a given smooth function $\gamma(\vartheta)$. It is to be pointed out that formulas (4.1)-(4.2) provide a unified description of a wide variety of natural-shaped and abstract curves ranging from Lamé curves (also known as superellipses) to multi-fold symmetric figures. We emphasize that all two-dimensional normal-polar domains may be described, or approximated accurately, by selecting suitable modulating functions $\gamma(\vartheta)$ and parameters $a_j, b_j, m_j, n_{1j}, n_{2j},$ and $n_{3j}$ in (4.1)-(4.2).

As an example, by using Theorem 3.1, the interior Dirichlet problem for the Laplace equation has been solved in the $k$-type Gielis domain of order $k = 3$

$$\rho = R(\vartheta) = \tilde{r}_G\left(\vartheta; \frac{1}{3} \cos \frac{3\vartheta}{2}, 1, 1, 3, 2, 2, 2\right) + \tilde{r}_G\left(\vartheta; \frac{1}{3} \cos \frac{3\vartheta}{2}, 1, 1, 4, 1, 1, 1\right)$$

$$+ \tilde{r}_G\left(\vartheta; \frac{1}{3} \sin \frac{5\vartheta}{2}, 1, 1, 5, 1, 1, 1\right), \quad (4.3)$$

under the assumption that the boundary values are given by $f(x, y) = x + \cos y$. As it can be readily noticed in Figure 1, each of the terms in (4.3) is a multi-lobed rose or Grandi curve inscribed in a conventional Gielis domain, the number of lobes being directly affected by the period parameters $m_j$ [see Equations (4.1), (4.2)].
Figure 1. Sketch of the $k$-type Gielis domain of order $k = 3$ described by Equation (4.3), and resulting from the superposition of modulated Grandi curves.

In order to assess the performance of the proposed approach in terms of numerical accuracy and convergence rate, the relative boundary error function has been defined as follows:

$$e_N = \frac{\| U_N(1, 9) - F(9) \|}{\| F(9) \|},$$  \hspace{1cm} (4.4)$$

$\| \cdot \|$ denoting the usual $L^2(\partial D)$ norm, and $L^2(\partial D)$ the partial sum of order $N$ relevant to the Fourier-like expansion approximating the solution of the specific Dirichlet problem for the Laplace equation, namely;

$$U_N(r, \theta) = \sum_{m=\nu}^{N} [rR(\theta)]^{\chi(m)}(A_m \cos m\theta + B_m \sin m\theta),$$  \hspace{1cm} (4.5)$$

where $\nu = 0, \chi(m) = m$ for the interior problem and $\nu = 1, \chi(m) = -m$ for the exterior one.

Under the specified assumptions, the relative boundary error $e_N$ as function of the order $N$ of the truncated series expansion (4.5) exhibits the behaviour shown in Figure 2. As it appears from Figure 3, the selection of the order $N = 7$ leads to a very accurate Fourier-like representation ($e_N < 2\%$) of the solution, whose approximate spatial distribution is plotted in Figure 4.
Figure 2. Relative boundary error $e_N$ as function of the order $N$ of the partial sum $U_N(r, \theta)$ approximating the solution of the interior Dirichlet problem for the Laplace equation in the $k$-type Gielis domain of order $k = 3$ described by the Equation (4.3).

Figure 3. Angular behaviour of the partial sum $U_N(1, \theta)$ of order $N = 7$ relevant to the interior Dirichlet problem for the Laplace equation in the $k$-type Gielis domain described by Equation (4.3). A good agreement with the exact solution can be noticed.
Figure 4. Spatial distribution of the partial sum \( v_N(r\theta \cos \phi, r\theta \sin \phi) = U_N(r, \phi) \) approximating the solution of the interior Dirichlet problem for the Laplace equation in the \( k \)-type Gielis domain described by Equation (4.3).

**Remark 4.1.** The \( L^2 \) norm of the difference between the exact solution and the relevant truncated Fourier-like expansion series is generally small. Point-wise convergence seems to be verified with the only exception of a set of measure zero consisting of the quasi-cusped points of the boundary. In these points, Gibbs-like oscillations of the approximate solution may occur.

### 5. Conclusion

The use of stretched co-ordinate systems allows the straightforward application of the classical Fourier projection method for solving the interior and exterior Dirichlet problems for the Laplace equation in two-dimensional \( k \)-type Gielis domains. In this way, analytically based expressions of the solution can be derived by using suitable quadrature rules, so overcoming the need for cumbersome numerical techniques such as finite-difference methods. The proposed approach has been successfully validated by means of a dedicated numerical procedure based on the computer-aided algebra tool Mathematica\textsuperscript{®}. A point-wise convergence of the expansion series of the solution seems to be verified.
with the only exception of a set of measure zero consisting of the quasi-
cusped points of the boundary. In these points, Gibbs-like oscillations
may occur. The computed results are found to be in good agreement with
the theoretical findings on Fourier series.

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