Sensitivity Analysis for Optimization Problems Solved by Stochastic Methods

R. H. C. Takahashi, J. A. Ramírez, J. A. Vasconcelos, and R. R. Saldanha

Abstract—The problem of obtaining sensitivity analysis information from optimization stochastic methods is addressed in this work. The sensitivity analysis calculation is obtained using the evaluations of the objective function as samples of the behavior of the objective function in the vicinity of the optimum. A linear optimization problem is formulated for the approximation of a contour surface of the objective function in the vicinity of the optimum. The formulation is applied to TEAM benchmark problems 25 and 22. The results, which are in good agreement with the literature, show the applicability of the formulation in practical optimization problems.

Index Terms—Electromagnetics, optimization, sensitivity analysis, stochastic methods.

I. INTRODUCTION

A DIFFICULT problem that arises in optimization problems is the quantification of the solution “sensitivity.” It is well known that optimal solutions that substantially degrade the optimum objective functional value when subjected to small parameter deviations are undesirable. In the context of electromagnetic device design, these solutions would lead to fabrication processes with strong tolerance specifications, which could render them expensive and difficult. In most of cases the optimization procedures, however, do not furnish any information about the solution sensitivity.

The sensitivity of an objective functional around a point of minimum can be assessed through two basic approaches:

• Estimating the functional Hessian approximation on the minimal solutions [1]. This approach can lead to a sensitivity estimation with relatively low cost. The main drawbacks are associated to: i) the numerical difficulties in the Hessian computation and ii) the issue of how does the Hessian describe the functional behavior for finite perturbations in the parameter vector.

• Evaluating the objective functional for finite perturbations in the parameter vector around the solution. This approach will suffer from a high computational cost associated to the needed computations.

The main idea of this paper is based on the fact that stochastic optimization methods usually perform a large number of objective function evaluations in the search of the global solution, covering a large region of the parameter space with a random distribution. This is an intrinsic feature of approaches such as the genetic algorithms and the simulated annealing and is considered as a major drawback [2]. Once the global solution is achieved the remaining evaluations that have been already performed are discarded.

In this paper a methodology is presented in order to take profit from the data that is generated in the stochastic search. The objective functional evaluations are considered as “samples” of the functional in the space, and will be employed in order to “reconstruct” a contour surface around the solution. The contour surfaces are supposed to be approximable by hyper-ellipsoids that will be estimated in the proposed method. This will lead to the estimation of a region (the interior of the ellipsoid) in which the objective functional is supposed to be less than the value it presents in the ellipsoid surface. In this way, a first approximation of a finite region within which the fabrication process is allowed to let the actually assembled devices vary.

The ellipsoid estimation is formulated as a sequence of linear optimization problems with Linear Matrix Inequalities (LMI’s) constraints [3]. These problems can be efficiently solved with interior point methods, leading to a small additional computational effort that is applied to the already existing data.

II. MATHEMATICAL FORMULATION

Suppose a function \( f(\cdot) \): \( \mathbb{R}^n \mapsto \mathbb{R} \) and let \( x^* \in \mathbb{R}^n \) be its minimum:

\[
x^* = \arg \min_x f(x).
\]

If \( f(x) \) is continuously differentiable and the optimum \( x^* \) is strict, the function has a second-order polynomial approximation around the optimum given by:

\[
f(x) = \alpha + \beta^T x + x^T \Omega x + O(3),
\]

This means that in the region in which this approximation is valid, the contour plots of the objective function can be approximated by ellipsoids:

\[
f(x) = \beta \Leftrightarrow (x - x^*)^T P (x - x^*) = \eta(\beta).
\]

The formulation we discuss here is based on the idea of determining the “smallest ellipsoid” that contains all points whose value of the objective function is smaller than a certain tolerance. This smallest ellipsoid should not contain any point whose value of the objective function is greater than a limit defined as “not acceptable” for a set of points of the objective function in the feasible domain.

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Considering the ellipsoid defined in (3), one has that the semi-axis are oriented according to the directions given by the eigenvectors \( \mathbf{v}_i \) of the matrix \( \mathbf{P} \), and that the length of these semi-axis are proportional to the inverse of the square root of the eigenvalues corresponding to each eigenvector:

\[
\mathbf{e}_i = \sqrt{\frac{\eta(\beta)}{\lambda_i}}
\]  

(4)

in which \( \lambda_i \) is the eigenvalue associated to the eigenvector \( \mathbf{v}_i \) of the matrix \( \mathbf{P} \):

\[
\mathbf{P}\mathbf{v}_i = \lambda_i\mathbf{v}_i.
\]  

(5)

It is possible to formulate the problem of determining the minimum ellipsoid as a problem of determining an ellipsoid with the smallest possible semi-axis. An objective functional that approximately corresponds to this problem is given by:

\[
J(\mathbf{P}) = \sum_{i=1}^{n} \frac{\eta(\beta)}{\mathbf{e}_i^2(\mathbf{P})},
\]  

(6)

The maximization of the functional \( J(\mathbf{P}) \) in variable \( \mathbf{P} \) is correlated with the minimization of the values \( \mathbf{e}_i \) of the ellipsoid semi-axis. It is known that:

\[
J(\mathbf{P}) = \max(\mathbf{P}) = \sum_{i=1}^{n} \lambda_i.
\]  

(7)

This is the basis for the formulation of the minimal ellipsoid determination problem, in terms of a linear optimization problem with Linear Matrix Inequalities (LMI’s) constraints.

Suppose that the functional has been evaluated in a set of points in the parameter space. This can be, for instance, the set of evaluations performed by some stochastic optimization method run. The points \( x_j \in \mathbb{R}^n, j = 1, \ldots, N \) around \( x^* \), for which the objective function has been evaluated, are ordered and divided in two sets:

\[
x_j \in \chi_i \Leftrightarrow f(x_j) < \beta; \quad x_j \in \chi_0 \Leftrightarrow f(x_j) > \beta.
\]  

(8)

The problem of estimation of the ellipsoid generator matrix \( \mathbf{P} \) can be stated as:

\[
\mathbf{P} = \arg \min_{\mathbf{P}} -\mathbf{w}(\mathbf{P})
\]  

(9)

\[
\quad \left\{ \begin{array}{l}
(\mathbf{x}_j - \mathbf{x}^*)^\top \mathbf{P}(\mathbf{x}_j - \mathbf{x}^*) < 1 \quad \forall x_j \in \chi_i \\
(\mathbf{x}_j - \mathbf{x}^*)^\top \mathbf{P}(\mathbf{x}_j - \mathbf{x}^*) > 1 \quad \forall x_j \in \chi_0 \\
\mathbf{P} > \frac{1}{r^2} \mathbf{I},
\end{array} \right.
\]  

(10)

This is a linear optimization problem in the matrix variable \( \mathbf{P} \in \mathbb{R}^{n \times n} \), with constraints of the kind “Linear Matrix Inequalities” (LMIs). The resulting \( \mathbf{P} \) is an estimate of the contour surface generating matrix. The last constraint in equation (10) means that the maximum ellipsoid semi-axis length is less than \( r \). The actual sensitivity analysis algorithm becomes a sequence of solutions of problem (9) and (10), in a search for the smaller value of \( r \) that renders the optimization problem still feasible.

### III. Results

#### A. Problem 1

Problem 1 is the TEAM Benchmark Problem 25, which consists of determining the appropriate design parameters of a die press with electromagnet for orientation of magnetic powder. An illustration of the whole view of the device can be found in [5]. The die molds are set to form the radial flux distribution. A zoom at the die molds region is given in Fig. 1. This problem has four design variables: \( R_1, L_2, L_3 \) and \( L_4 \).

The objective function is defined as:

\[
W = \sum_{i=1}^{n} \left\{ (B_{xi} - B_{xi0})^2 + (B_{yi} - B_{yi0})^2 \right\}
\]  

(11)

where \( n \) is the number of specified points (\( \geq 10 \)) and the subscripts \( \mathbf{p} \) and \( \mathbf{o} \) mean the calculated and specified values respectively. The \( B_x \) and \( B_y \) along the line e-f, see Fig. 2, are specified as:

\[
B_x = 0.35 \cos \theta(T)
\]  

(12)

\[
B_y = 0.35 \sin \theta(T)
\]  

(13)
Fig. 3. Variation of $B_x$ along the segment e-f.

where is the angle measured from the $x$-axis. The $B-H$ curve of the steel is given in [5]. The constraints imposed on $R_1$, $L_2$, $L_3$ and $L_4$ are $(5 < R_1 < 9.4)$, $(12.6 < L_2 < 18)$, $(14 < L_3 < 45)$, $(4 < L_4 < 19)$.

The optimization using a genetic algorithm has produced an optimum at:

$$x = [6.5800 \hspace{0.2cm} 16.1683 \hspace{0.2cm} 26.1693 \hspace{0.2cm} 5.0247]^T \quad (14)$$

with optimal objective functional $J = 0.0019$. In the optimization process, 210 points have been evaluated. Defining the cutting value of $J = 0.0030$, the set $\chi_3$ keeps 31 points, and the set $\chi_0$ the remaining 179 ones. The application of the sensitivity analysis with this data leads to the following matrix $P$:

$$P = 1 \times 10^8 \begin{bmatrix} 1.6384 & 3.6146 & 0.9281 & 1.4793 \\ 3.6146 & 7.9745 & 2.0476 & 3.2637 \\ 0.9281 & 2.0476 & 0.5258 & 0.8380 \\ 1.4793 & 3.2637 & 0.8380 & 1.3357 \end{bmatrix} \quad (15)$$

which defines an hyper-ellipsoid with semi-axis along the vectors:

$$v = \begin{bmatrix} 0.2198 & 0.2130 & 0.8738 & 0.3779 \\ 0.2387 & 0.1800 & -0.4644 & 0.8337 \\ -0.9109 & 0.3491 & 0.0514 & 0.2141 \\ -0.2351 & -0.8946 & 0.1347 & 0.3412 \end{bmatrix} \quad (16)$$

These semi-axis have lengths: $R_3 = 0.8239$; $R_2 = 0.8224$; $R_3 = 0.4001$; $R_4 = 0.0000 0295$. This result suggests that the solution is rather sensitive to disturbances in the direction of $v_4$. With this information, one could either search for another solution that is less sensitive, or define a constructive procedure that guarantees this tolerance in the device physical dimensions.

The graphs shown in Figs. 2 and 3 illustrate the specified and calculated values of $B_x$ and $B_y$.

B. Problem 2

Problem 2 is the TEAM Benchmark Problem 22, which consists of determining the appropriate design parameters of a superconducting magnetic energy storage (SMES) [4], see Fig. 4.

![Fig. 4. SMES basic configuration (not to scale).](image)

TABLE I

<table>
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<th>3v</th>
<th>R_1</th>
<th>R_2</th>
<th>H_1</th>
<th>D_1</th>
<th>R_3</th>
<th>H_2</th>
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<th>J_1</th>
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<tr>
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<td>0.27</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

The determination of the design parameters can be interpreted as an optimization problem with respect to three individual objectives, each of them must be minimized. These are to ensure the stray field is minimal ($f_1$), the stored energy is near 180 MJ ($f_2$) and that the quench condition ensuring superconductivity of the SMES arrangement is met ($f_3$). The problem investigated here is the three variables continuous case, see Table I.

The best solution found using a genetic algorithm was:

$$x = [2.9387 \hspace{0.2cm} 0.6547 \hspace{0.2cm} 0.3288]^T \quad (17)$$

This point has objective functional $J = 0.1607$. The GA performed 152 evaluations of the objective function. The objective functional ranges from the optimum value to the worst one in which $J > 500$. A reference value of $J = 1.5$ has been adopted, which has lead 20 points with objective value smaller than this level, and the remaining ones with objective value greater than it. With this data, the following matrix $P$ has been found with the sensitivity analysis algorithm:

$$P = \begin{bmatrix} 33.6553 & 25.5590 & 20.3787 \\ 25.5590 & 48.3514 & 124.1540 \\ 20.3787 & 124.1540 & 547.5442 \end{bmatrix} \quad (18)$$
The semi-axis of the ellipsoid defined by this matrix are parallel to matrix $P$ eigenvectors:

$$
\mathbf{v} = \begin{bmatrix}
0.5859 & -0.8090 & 0.0472 \\
-0.7945 & -0.5620 & 0.2002 \\
0.1507 & 0.1724 & 0.9720
\end{bmatrix}.
$$

(19)

The length of these semi-axis are, respectively, $R_1 = 0.4687; R_2 = 0.1458; R_3 = 0.0416$. The ellipsoid is shown in Fig. 5, with its projections in the three coordinate planes. This ellipsoid can be interpreted as a first approximation of the region in which the objective function is not greater than 1.5.

**IV. CONCLUSION**

This paper has presented a new methodology for sensitivity analysis of optimization problem solutions. This methodology employs a set of evaluations of the objective functional in a region of the space, and determines an ellipsoid that is an approximation of a contour surface. The evaluations may be generated, for instance, from the optimization procedure developed by a stochastic algorithm. In this case, the sensitivity analysis represents a small additional computational effort that extracts more information from the same data already available. The sensitivity information can be of immense value for the electromagnetic device fabrication process implementation.

**REFERENCES**


