Identification and elimination of parasitic shear in a laminated composite beam finite element

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Abstract

A displacement-based finite element for the analysis of laminated composite beams is formulated using strain gradient notation. The definition of the beam’s longitudinal displacement possesses only the independent term (axial displacement) and a term which is linear in the thickness coordinate \( z \). Thus, the finite element is first-order shear deformable. As strain gradient notation is physically interpretable, the contents of the coefficients of the polynomial expansions are identified a priori. Thus, the modeling capabilities as well as modeling deficiencies of the element are identified during the formulation procedure. A single parasitic shear term (spurious) is found to be present in the transverse shear strain expression of the element, which is responsible for locking. This parasitic shear term is also found to be the cause of a qualitative error existing in the representation of transverse shear strain along the length of a typical beam model. As the spurious term has been clearly identified, it can easily be removed to correct the element. The effectiveness of the procedure is shown through numerical analyses performed using the element containing the spurious term and then corrected for it. The beam model is validated by comparing numerical solutions with analytical solutions provided by the minimization of the total potential energy for a given laminated composite beam.

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1. Introduction

A composite is a material that consists of the combination of two or more materials in a macroscopic scale. Composites usually possess the characteristics of the constituent materials and new characteristics which arise due to their association. Among the different types of composites is the class of laminated composites, which is widely used in structural engineering due to high strength-to-weight and stiffness-to-weight ratios. A laminated composite is obtained by stacking two or more laminae of different materials. A lamina may be either made of a homogeneous material or may be a composite itself. An important type of the latter is the fiber reinforced lamina which consists of fibers of a given material embedded into a matrix of a different material in a given direction. Composites generated by the stacking of fiber reinforced laminae are called fiber reinforced laminated composites or simply laminated composites. For an introductory text on composites, the reader is referred to Jones [1]. For a comprehensive text on the mechanics of laminated composites, the reader is referred to Reddy [2].

Laminated composites find structural applications in automotive, aeronautic and aerospace engineering to name just a few. Structural systems in these applications are largely composed of plates and shells, many times requiring beam reinforcement. Analytical and numerical methods can be employed for the analysis of structural systems composed of laminated composite components. Among all those methods, the finite element method [3] has been widely employed, creating room for research in the development of better and more accurate finite elements. The main objective of this paper is to describe the formulation of a first-order shear deformable finite element for the analysis of laminated composite beams and to identify and eliminate parasitic shear terms which may cause locking of the element.

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High-order deformation theories have been proposed by several authors such as Lo et al. [6,7], Singh and Rao [8], and Bose and Reddy [9]. An interesting work by Reddy and Wang [10] presents an overview of the relationships between classical and shear deformation theories. Computational models ranging from simple to refined which allow for numerical evaluation of all those theories have been developed [11,12]. Reddy [13] makes a thorough presentation of Euler–Bernoulli and shear deformation displacement-based beam finite element models. A locking-free element based on the exact solution of the Timoshenko beam theory is developed which is capable of producing exact nodal values for the generalized displacements. Further, a third-order theory locking-free beam finite element model which is superconvergent is developed as an extension of the procedure employed to develop the previous element. Refined models for shear deformable beams have been proposed recently [14,15]. Shimpi and Ainapure [14] formulate a locking-free beam element based on a layerwise trigonometric shear deformation theory. Despite having two nodes and only three degrees-of-freedom per node, the element incorporates through the thickness sinusoidal variation of in-plane displacement such that the shear stress is zero at the top and bottom surfaces, and the shear stress distribution is parabolic through the laminate’s thickness. Sze et al. [16] formulate a layerwise model for laminated composite beams using the deflection and its derivatives as nodal degrees-of-freedom. Their two-noded model yields poor accuracy due to locking. To overcome the problem, they construct a three-noded “heterosis” element. Subramanian [17] develops $C^1$ finite element for laminated composite beams. Using the constitutive law, transverse normal stresses are accurately computed and display through the thickness parabolic distribution.

Despite all these theoretical and numerical advances in the analysis of laminated composites, this paper focuses on a first-order shear deformation theory element and demonstrates an alternative procedure to identify and eliminate the sources of locking. First-order shear deformation theories are suitable for computing global responses such as displacements and natural frequencies [1,2,18].

In this work, the finite element method through strain gradient notation [19,20] is employed to formulate an element for the analysis of laminated composite beams with the ability to represent transverse shear deformation, which is always present in the behavior of laminates. Strain gradient notation is a physically interpretable notation that can be used to develop finite elements and that allows for the identification and removal of spurious terms, which are inherent to the formulation procedure [21].

Errors can be introduced into the finite element model at the individual element level which makes the element overly stiff. This locking is thus a quantitative error, i.e., an error in magnitude. One source of locking is known as parasitic shear which is caused by incorrect coupling between flexural and shear deformations [21]. Parasitic shear also introduces qualitative errors into laminated composite models. A qualitative error is defined as a misrepresentation of the nature of a deformation such as the one described in this paper. Qualitative errors have been previously shown to occur in the analysis of laminated composite plates [22,23].

The source of parasitic shear is the use of incompatible polynomials as approximations for the field variables. Incompatibility of polynomials manifest itself in two different ways. The polynomial may be incomplete such as is the case in the displacement representation of a four-node quadrilateral element for modeling plane problems. Further, as it occurs in beam and plate analysis, the polynomial may be inconsistent. That is, the polynomial is incomplete and its order is inconsistent with the order of the theory being modeled [21,22].

This paper demonstrates that the cause of locking in the present beam element is the flexural strain $\varepsilon_{x,z}$, which appears in the expansion of $\gamma_{x,z}$. Also, it shows that this parasitic shear term can be identified and eliminated a priori, that is, during the formulation stage, which means that once this is done the analyst does not have to deal with the locking problem during the numerical analysis procedure. Further, it shows that through strain gradient notation the identification and removal of spurious terms from finite elements is a straightforward procedure. Finally, this paper shows that the present spurious term is also the cause of qualitative errors in the representation of the behavior of laminated composite beams. More specifically, it shows that a linearly varying shear force along the length of the structure is misrepresented by a parabolic distribution, and that mesh refinement magnifies this erroneous behavior.

2. Beam finite element

In this section, a two-dimensional beam finite element for the analysis of laminated composites is formulated. As shown in Fig. 1 below, the element is two-noded with
three degrees-of-freedom at each node; namely, the longitudinal displacement \( u \), the vertical displacement \( w \), and the rotation \( q \).

The assumptions for the macromechanical behavior of the laminated beam are as follows: (i) plane sections normal to the longitudinal surface remain plane, but not necessarily normal after bending. Therefore, there is transverse shear deformation of the beam; (ii) there is perfect bond between laminae such that slip or separation cannot be represented, and (iii) strains and stresses normal to the middle surface are negligible.

The beam’s displacement field is defined by

\[
\begin{align*}
  u(x, z) &= u_0(x) - zq(x) \\
  w(x) &= w_0(x)
\end{align*}
\]  

where \( u_0(x) \) is the axial displacement, \( w_0(x) \) is the mid-surface’s vertical displacement, which defines the element’s vertical displacement, and \( q(x) \) is the in-plane rotation. These independent fields are represented by the polynomial expansions below:

\[
\begin{align*}
  w_0(x) &= a_0 + a_1x \\
  u_0(x) &= b_0 + b_1x \\
  q(x) &= c_0 + c_1x
\end{align*}
\]  

In strain gradient notation, the expressions (3)–(5) are written below as

\[
\begin{align*}
  w_0(x) &= [w]_0 + \left[ \frac{\gamma_{xz}}{2} - q \right]_0 x \\
  u_0(x) &= [u]_0 + [e_x]_0 x \\
  q(x) &= \left[ -\frac{\gamma_{xz}}{2} - q \right]_0 + [-e_{xx}]_0 x
\end{align*}
\]  

while the longitudinal displacement \( u(x, z) \) results in the following expression:

\[
  u(x, z) = ([u]_0 + [e_x]_0 x) - z \left( \left[ -\frac{\gamma_{xz}}{2} - q \right]_0 + [-e_{xx}]_0 x \right)
\]  

The quantities in brackets are the polynomial coefficients which in regular notation are unknown at the formulation stage. In strain gradient notation, the physical meanings of the polynomial coefficients are identified at the early steps as revealed by Eqs. (6)–(9). In these equations, \([u]_0\), \([w]_0\) and \([q]_0\) are rigid body displacements; \([e_x]_0\) and \([\gamma_{xz}]_0\) are constant normal and transverse shear strains, and \([e_{xx}]_0\) is the flexural strain. In general, they are referred to as strain gradients. The subscript 0 in the notation is employed to indicate that the strain gradients result from the evaluation of the corresponding Taylor series expansions at a given origin.

The nodal degrees-of-freedom are related to the strain gradients through the following:

\[
\{d\} = \{\Phi\} \{e_{sg}\}
\]  

where the nodal degrees-of-freedom and strain gradients are arranged in the following vectors:

\[
\begin{align*}
  \{d\}^T &= \{u_1 \ w_1 \ q_1 \ u_2 \ w_2 \ q_2 \} \\
  \{e_{sg}\}^T &= \{[u]_0 \ [w]_0 \ [q]_0 \ [e_x]_0 \ [\gamma_{xz}]_0 \ [e_{xx}]_0 \}
\end{align*}
\]  

and \([\Phi]\) is explicitly given below:

\[
\begin{bmatrix}
  1 & 0 & 0 & x_1 & 0 & -x_1 \\
  0 & 1 & -x_1 & 0 & x_1/2 & 0 \\
  0 & 0 & -1 & 0 & -1/2 & -x_1 \\
  1 & 0 & 0 & x_2 & 0 & 0 \\
  0 & 1 & -x_2 & 0 & x_2/2 & 0 \\
  0 & 0 & -1 & 0 & -1/2 & -x_2
\end{bmatrix}
\]  

According to Fig. 1, the element’s origin is defined at its midlength. Thus, in matrix \([\Phi]\), \(x_1 = -L/2\) and \(x_2 = L/2\), which renders the columns of the matrix linearly independent vectors. This has to be verified because the strain gradients form a basis of linearly independent deformation quantities for the element, including the rigid body modes.

The strain field is given by the derivatives of the displacements:

\[
\begin{align*}
  e_x &= [e_x]_0 + [e_{xx}]_0 x \\
  \gamma_{xz} &= [\gamma_{xz}]_0 + [e_{xz}]_0 x
\end{align*}
\]  

Inspection of Eq. (14), the definition of normal strain, show that the coefficients are related to this strain. Therefore, they are both legitimate. However, the expansion for the transverse shear strain, Eq. (15), shows the presence of a shear strain term, but also of the flexural strain term \([e_{xz}]_0\). This indication equation indicates that when the beam undergoes a flexural deformation, there is an increase in shear strain. As this is physically impossible since there are no couplings between transverse shear and flexural strains, the latter term is certainly spurious. The erroneous increase in shear strain energy due to longitudinal strains is known as parasitic shear and it is the source locking, which causes slow convergence. The parasitic shear term appears naturally during the formulation procedure, as it has just been demonstrated, and it must be removed as it will cause strong deleterious effects in the performance of the element. It is important to note that the spurious term has been easily and precisely identified a priori due to the use of strain gradient notation.

To free the element from the effects of parasitic shear, the flexural term \([e_{xz}]_0\) must be removed from the transverse shear expression \(\gamma_{xz}\). Then, that expression simply becomes

\[
\gamma_{xz} = [\gamma_{xz}]_0
\]  

The relation between strains and strain gradients, is given below in matrix form:

\[
\{e\} = [T_{sg}] \{e_{sg}\}
\]  

where the strain vector is

\[
\{e\} = \begin{bmatrix} e_x \\ \gamma_{xz} \end{bmatrix}
\]  

and the transformation matrix \([T_{sg}]\) when the element contains the spurious term is

\[
[T_{sg}] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & z \\ 0 & 0 & 0 & 0 & 1 & x \end{bmatrix}
\]
and when the element is corrected for parasitic shear \( T_{sg} \) is
\[
[T_{sg}] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & z \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\]
(20)

The strain energy of a laminated composite is obtained through the sum of the strain energies of the constituent laminae. For a laminate comprised of \( n \) laminae, after introducing Eqs. (10) and (17) into its definition, the strain energy is written as
\[
U = \frac{1}{2} \{d\}^T \Phi^{-1} [U_M] \Phi^{-1} \{d\}
\]
(21)
where
\[
[U_M] = \sum_{k=1}^{n} \int_{V_k} [T_{sg}] \Phi_k [T_{sg}] \, dV_k
\]
(22)
is the strain energy matrix and \( \Phi_k \) is the constitutive matrix of a typical lamina \( k \) in global coordinates. Each column of the strain energy matrix \( [U_M] \) represents the strain energy associated to a different strain gradient of the basis set of the finite element. The main diagonal terms are the strain energy quantities associated to pure deformation modes or strain gradients. In turn, the off-diagonal terms are the strain energy quantities associated to the coupling of deformation modes or strain gradients.

The constitutive matrix of a lamina of the laminated beam is given by
\[
[\Phi_k] = \begin{bmatrix} \bar{\Phi}_{11} & \bar{\Phi}_{15} \\ \bar{\Phi}_{15} & \bar{\Phi}_{55} \end{bmatrix}
\]
(23)
where the off-diagonal terms are zero because there is no coupling between normal and shear strains.

Carrying out the operations in Eq. (22), the strain energy matrix \( [U_M] \) explicitly given by
\[
[U_M] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{11}(bL) & 0 & B_{11}(bL) \\ 0 & 0 & 0 & A_{55}(bL) & 0 \\ 0 & 0 & 0 & B_{11}(bL) & D_{11}(bL) + A_{55} \frac{B_{11}^2}{L^2} \end{bmatrix}
\]
(24)
where \( A_{11} \) is the extension stiffness, \( A_{55} \) is the transverse shear stiffness, \( D_{11} \) is the bending stiffness, and \( B_{11} \) is the bending-extension coupling stiffness, which are defined by
\[
A_{11} = \sum_{k=1}^{n} \bar{\Phi}_{11} \left( z_k - z_{k-1} \right)
\]
(25)
\[
A_{55} = \frac{5}{4} \sum_{k=1}^{n} \bar{\Phi}_{55} \left[ h_k - h_{k-1} - \frac{4}{3} \left( h_k^3 - h_{k-1}^3 \right) \frac{1}{h^2} \right]
\]
(26)
\[
D_{11} = \frac{1}{3} \sum_{k=1}^{n} \bar{\Phi}_{11} \left( z_k^3 - z_{k-1}^3 \right)
\]
(27)
\[
B_{11} = \frac{1}{2} \sum_{k=1}^{n} \left( \bar{\Phi}_{11} \right)_k \left( z_k^2 - z_{k-1}^2 \right)
\]
(28)

and \( b \) is the width of the beam’s cross-section and \( L \) is the beam’s length. Eq. (26) represents the transverse shear stiffness and thus incorporates the function [18]
\[
f(z) = \frac{5}{4} \left[ 1 - \left( \frac{z}{h/2} \right)^2 \right]
\]
(29)
to represent the parabolic distribution of the shear stress through the laminate’s thickness.

The term \( A_{55}(bL^3/12) \) which appears in the last entry of \( [U_M] \) in Eq. (24) is the term responsible for parasitic shear. When the element is corrected for parasitic shear, that term gets removed from \( [U_M] \).

Further, the stiffness matrix in strain gradient notation is given by
\[
[K] = \Phi^{-1} [U_M] \Phi^{-1}
\]
(30)
which for the present element becomes
\[
[K] = \begin{bmatrix} \frac{U_{11}}{L^2} & 0 & -\frac{U_{15}}{L^2} & -\frac{U_{14}}{L^2} & 0 & \frac{U_{16}}{L^2} \\ 0 & \frac{U_{22}}{L^2} & \frac{U_{24}}{L^2} & 0 & -\frac{U_{26}}{L^2} & \frac{U_{27}}{L^2} \\ -\frac{U_{15}}{L^2} & 0 & \frac{U_{16}}{L^2} & \frac{U_{14}}{L^2} & 0 & -\frac{U_{17}}{L^2} \\ 0 & -\frac{U_{25}}{L^2} & -\frac{U_{26}}{L^2} & 0 & \frac{U_{27}}{L^2} & -\frac{U_{28}}{L^2} \\ \frac{U_{14}}{L^2} & \frac{U_{15}}{L^2} & \frac{U_{16}}{L^2} & \frac{U_{17}}{L^2} & 0 & -\frac{U_{18}}{L^2} \\ \frac{U_{24}}{L^2} & \frac{U_{25}}{L^2} & \frac{U_{26}}{L^2} & \frac{U_{27}}{L^2} & \frac{U_{28}}{L^2} & 0 \end{bmatrix}
\]
(31)

In strain gradient notation, the strains are calculated by
\[
\{\varepsilon\} = [T_{sg}] \Phi^{-1} \{d\}
\]
(32)
which explicitly for the current laminated beam are
\[
\varepsilon_x = \frac{1}{L} (u_2 - u_1) - \frac{x}{L} (q_2 - q_1)
\]
(33)
\[
\gamma_{xz} = \frac{1}{L} (w_2 - w_1) - \frac{x}{L} (q_2 - q_1) - \frac{1}{2} (q_2 + q_1)
\]
(34)

In turn, normal and transverse shear stresses in lamina \( k \) are given by
\[
\sigma_x = \left( \bar{\Phi}_{11} \right)_k \left\{ \frac{1}{L} (u_2 - u_1) - \frac{x}{L} (q_2 - q_1) \right\}
\]
(35)
\[
\tau_{xz} = \left( \bar{\Phi}_{55} \right)_k \left\{ \frac{1}{L} (w_2 - w_1) - \frac{x}{L} (q_2 - q_1) - \frac{1}{2} (q_2 + q_1) \right\}
\]
(36)
Finally, the normal, bending and shear stress resultants are defined as
\[
N_x = A_{11} \varepsilon_x + B_{11} \gamma_{xz}
\]
(37)
\[
M_x = B_{11} \varepsilon_x + D_{11} \gamma_{xz}
\]
(38)
\[
Q_x = A_{55} \gamma_{xz}
\]
(39)
and are calculated explicitly by

\[ N_x = A_{11} \left( \frac{1}{L} (u_2 - u_1) + B_{11} \left( -\frac{1}{L} (q_2 - q_1) \right) \right) \]  
(40)

\[ M_x = B_{11} \left( \frac{1}{L} (u_2 - u_1) + D_{11} \left( -\frac{1}{L} (q_2 - q_1) \right) \right) \]  
(41)

\[ Q_x = A_{55} \left( \frac{1}{L} (w_2 - w_1) - \frac{x}{L} (q_2 - q_1) - \frac{1}{2} (q_2 + q_1) \right) \]  
(42)

3. Numerical analyses

This section presents the solution of a laminated composite cantilever beam comprised of two laminae of graphite/epoxy. The mechanical properties of the graphite/epoxy are \( E_{11} = 138 \) GPa, \( E_{22} = E_{33} = 14.5 \) GPa, \( G_{12} = G_{23} = G_{31} = 5.86 \) GPa, \( v_{12} = v_{23} = v_{31} = 0.21 \). The beam is 2.0 m long and it is loaded with 2000 N/m distributed along its length as shown in Fig. 2. The laminae are of the same thickness, but the graphite fibers of the top laminae are directed 30° with respect to the axis of the beam while the fibers of the bottom laminae are directed along that axis. This is a non-symmetric laminate, thus, coupling between bending and extension will occur [1].

This problem will be analyzed using the finite element model developed in the previous section. In order to validate the finite element model, analytical results should be available for comparison. Hence, an analytical solution for this beam problem will be derived through the minimization of the total potential energy.

The total potential energy for this example beam is given by

\[ \Pi = \frac{1}{2} \int_0^L \left[ b A_{11} \left( \frac{\partial u}{\partial x} \right)^2 - 2b B_{11} \frac{\partial u}{\partial x} \frac{\partial q}{\partial x} + b D_{11} \left( \frac{\partial q}{\partial x} \right)^2 + b A_{55} \left( \frac{\partial w}{\partial x} - q \right) + 2Q_0 w \right] dx \]  
(43)

where \( b \) is the beam’s cross-section width and \( Q_0 \) is the distributed loading.

Through the minimization of Eq. (43), the following equilibrium equations written in terms of displacements are obtained:

\[ A_{11} \left( \frac{\partial^2 u}{\partial x^2} \right) - B_{11} \left( \frac{\partial^2 q}{\partial x^2} \right) = 0 \]  
(44)

\[ Q_0 + b A_{55} \left( \frac{\partial q}{\partial x} - \frac{\partial^2 w}{\partial x^2} \right) = 0 \]  
(45)

\[ B_{11} \left( \frac{\partial^2 u}{\partial x^2} \right) - D_{11} \left( \frac{\partial^2 q}{\partial x^2} \right) + A_{55} \left( q - \frac{\partial w}{\partial x} \right) = 0 \]  
(46)

The essential boundary conditions are provided by the fixed end:

\[ u(0) = 0, \quad w(0) = 0, \quad q(0) = 0 \]  
(47)

and the natural boundary conditions, which arise during the minimization process, when evaluated at the free-end, provide the following:

\[ \left. \frac{\partial u}{\partial x} \right|_{x=L} = 0, \quad \left. \frac{\partial q}{\partial x} \right|_{x=L} = 0 \]  
(48)

\[ \left. \frac{\partial^2 u}{\partial x^2} \right|_{x=L} = 0, \quad \left. q - \frac{\partial w}{\partial x} \right|_{x=L} = 0 \]  
(49)

which mean that bending moment, axial force and shear force are zero at that end.

Solution of Eqs. (44)–(46) using conditions given by Eqs. (47)–(49) provide the following expressions for the displacements:

\[ u(x) = \frac{Q_o B_{11}}{b(B_{11} - A_{11} D_{11})} \left( \frac{x^3}{6} - \frac{L x^2}{2} + \frac{L^3 x}{2} \right) \]  
(50)

\[ w(x) = \frac{Q_o A_{11}}{b(B_{11} - A_{11} D_{11})} \left( \frac{x^4}{24} - \frac{L x^3}{6} + \frac{L^3 x^2}{4} \right) \]  
(51)

\[ q(x) = \frac{Q_o A_{11}}{b(B_{11} - A_{11} D_{11})} \left( \frac{x^4}{6} - \frac{L x^3}{2} + \frac{L^3 x}{2} \right) \]  
(52)

Further, the normal and transverse shear strain expressions for the beam are obtained through the proper derivatives of these displacements according to the elasticity theory. Hence,

\[ \varepsilon_n(x, z) = \frac{Q_o B_{11}}{b(B_{11}^2 - A_{11} D_{11})} \left( \frac{x^2}{2} - L x + \frac{L^2}{2} \right) \]  
(53)

\[ \gamma_{n} (x) = \frac{Q_o}{b A_{55}} (x - L) \]  
(54)

Five uniform meshes comprised of 2, 4, 8, 16 and 32 elements, respectively, will be employed to produce numerical results for the beam problem. These results will be compared to the analytical results provided by Eqs. (50)–(54). For each mesh, two analyses will be carried out; namely, using the model containing parasitic shear first and then using the model corrected for parasitic shear. This will allow for studying the influence of refinement and of the spurious term in the solutions. The results are shown in Figs. 3–12.
Figs. 3 and 4 show, respectively, the solutions for the vertical displacement $w$ along the length of the beam for the models with and without parasitic shear. Both solutions converge to the analytical solution provided by Eq. (51). However, the rate of convergence is greater for the model corrected for parasitic shear, as displayed in Fig. 4.

In turn, Figs. 5 and 6 show the variation of rotation $q$ along the length of the beam. Again, both numerical solutions converge to the analytical one given by Eq. (52), and the rate of convergence presented by the corrected finite element model is much greater.

Further, Figs. 7 and 8 display the variation of the axial displacement $u$ along the length of the beam. This displacement
only exists because the laminated is non-symmetric, and, therefore, presents coupling between bending and axial deformation. As in the previous analyses, the corrected finite element model solution (Fig. 8) converges faster than the parasitic shear model solution (Fig. 7) to the analytic solution given by Eq. (50).

The next four plots show the convergence characteristics of the beam model for strains. Figs. 9 and 10 show, respectively, the variation of normal strain $e_n$ along the length of the beam. Convergence to the solution provided by Eq. (53) occurs for both models, although due to locking, the model with parasitic shear presents a slower convergence.
Figs. 11 and 12 show, respectively, the variation of transverse shear strain $\gamma_{xz}$ along the length of the beam. Convergence to the analytic solution provided by Eq. (54) occurs nicely when using the model corrected for parasitic shear, as displayed by Fig. 12. However, a rather erroneous representation of $\gamma_{xz}$ is provided by the model containing parasitic shear, as shown by Fig. 11. According to Eq. (54), the transverse shear strain varies linearly with the length of the beam. The representation provided by the model containing parasitic shear is parabolic, which is then obviously erroneous. Such representation reveals not only a quantitative error as in the previous solutions, but also an error of...
the qualitative kind. In the present case, it is an error in the shape of the solution. Further, this deleterious effect is augmented by mesh refinement. As shown in Fig. 11, the numerical solution gets farther and farther away from the correct solution as the mesh is refined, revealing a situation where refinement of the model does not attenuate the erroneous representation due to parasitic shear. Therefore, the only remedy is the complete obliteration of the effects of parasitic shear, which here is attained by removing the spurious term from the transverse shear strain expression.
Inspection of the formulation presented above explains the parabolic shape of the plots in Fig. 11. The analytic expression for the beam’s normal strain given by Eq. (53) shows that $e_x$ depends on $z$, $x$, and $x^2$. Thus, $e_{x,z}$, which is the spurious term in the expression for $\gamma_{xz}$, depends on $x$ and $x^2$. Hence, when $e_{x,z}$ multiplies $x$ in the transverse shear strain $\gamma_{xz}$ expression of the finite element (Eq. (15)), it generates a term which is cubic in $x$, responsible then for the erroneous parabolic distribution of $\gamma_{xz}$ along the beam’s length. On the other hand, after elimination of the spurious term, $\gamma_{xz}$ becomes only dependent on the variable $x$, conforming to the analytic expression for the transverse shear strain.

Fig. 11. Transverse shear strain $\gamma_{xz}(x)$ computed using the parasitic shear finite element model.

Fig. 12. Transverse shear strain $\gamma_{xz}(x)$ computed using the corrected finite element model.
strain provided by Eq. (54), thus displaying the correct linear variation along the beam’s length.

4. Summary and conclusions

This paper presented the formulation in strain gradient notation of a two-node beam finite element for the analysis of laminated composites. As it has been shown through the development, the polynomial expansions written in strain gradient notation are transparent to the developer letting him or her know which kinematics quantities are associated to the element. That is, it is possible to learn a priori by inspection what are the modeling capabilities of the finite element as well as its modeling deficiencies. Inspection of the beam element’s transverse shear strain expansion revealed the presence of the flexural term \( \gamma_{xz} \). As this quantity is a gradient of the normal strain in the beam’s longitudinal direction, it is not a genuine term of the \( \gamma_{xz} \) Taylor expansion. Therefore, it is a parasitic shear term, which is responsible for locking. The numerical analyses performed here demonstrated that this spurious term is the source of severe erroneous numerical solution representations. In general, for coarser meshes, parasitic shear has caused significant locking in the beam model. The solutions presented have shown that refinement attenuates this locking effect, leading to convergence. An analytic solution for the laminate in study, derived using the minimization of the total potential energy, is used for verification of convergence. Also, it has been demonstrated that parasitic shear causes an error of the qualitative type in the transverse shear strain solution of the beam model. The solution error is qualitative because the form of the response does not conform to the analytic solution. In the example problem provided, as the loading is uniformly distributed along the beam’s length, solution for transverse shear strains and stresses must vary linearly. However, as shown in the plots, parasitic shear rendered such solutions parabolic, an error which was completely removed when the element was corrected for parasitic shear. On the contrary to what was expected, refinement did not attenuate or remove such qualitative error. Instead, refinement strengthened the spurious term, which is a coefficient to a linear term in \( x \) (longitudinal coordinate), increasing the solutions error proportionally.

The conclusions that can be drawn are the following. The beam element formulated is efficient as the numerical solutions converge to the analytic solution. Strain gradient notation provides an efficient means for detecting spurious terms responsible for modeling deficiencies, which are inherent to the formulation process. Elimination of the spurious term present in the transverse shear strain expression corrected the element for parasitic shear, providing solutions which converge rapidly to the analytic one. Although refinement is able to attenuate or even remove the locking effects caused by parasitic shear, i.e., to solve the quantitative error problem, it has not been able to do the same with the qualitative error. Thus, the only definite way to obliterate the effects of parasitic shear in this beam’s model analyses is to remove the spurious term from the solution a priori. Strain gradient notation presents a clear and effective way of doing this by simply removing the spurious term from the shear strain expansion.

References