Some variants of Cauchy’s method with accelerated fourth-order convergence

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Abstract

In this paper, we present some variants of Cauchy’s method for solving non-linear equations. Analysis of convergence shows that the methods have fourth-order convergence. Per iteration the new methods cost almost the same as Cauchy’s method. Numerical results show that the methods can compete with Cauchy’s method.

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1. Introduction

Solving non-linear equations is one of the most important problems in numerical analysis. In this paper, we consider iterative methods to find a simple root of a non-linear equation \( f(x) = 0 \), where \( f : D \subset \mathbb{R} \rightarrow \mathbb{R} \) for an open interval \( D \) is a scalar function.

In this paper, we derive the iterative methods by Taylor expansion of \( f(x) \)

\[
f(x) \approx \sum_{k=0}^{m} \frac{f^{(k)}(x_n)}{k!}(x - x_n)^k = P_m(x),
\]

where \( P_m \) is the Taylor polynomial of degree \( m \) whose \( k \)th derivatives agree with \( f \) at the point \( x_n \), i.e., \( P_m^{(k)}(x_n) = f^{(k)}(x_n), \) \( k = 0, \ldots, m \). Let the next approximation \( x_{n+1} \) be defined as the root of \( P_m(x) = 0 \) closest to \( x_n \). By solving \( P_1(x) = 0 \), Newton’s method is obtained:

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.
\]

This is an important and basic method [13], which converges quadratically.
In order to derive higher-order methods, we solve \( P_2(x) = 0 \), namely, \( f(x_n) + f'(x_n)(x - x_n) + (1/2!) f''(x_n) (x - x_n)^2 = 0 \) and Cauchy’s method \([14,11]\) is obtained:

\[
 x_{n+1} = x_n - \frac{2}{1 + \sqrt{1 - 2L_f(x_n)}} \frac{f(x_n)}{f'(x_n)},
\]

where

\[
 L_f(x_n) = \frac{f''(x_n)f(x_n)}{f'(x_n)^2}.
\]

This method is an iterative process with cubical convergence.

Moreover, by Taylor approximation of \((1 - 2L_f(x_n))^{1/2}\), it is easy to obtain that

\[
 \frac{2}{1 + \sqrt{1 - 2L_f(x_n)}} = 1 - \sqrt{1 - 2L_f(x_n)} = \sum_{k \geq 0} \left( \frac{\frac{1}{2}k}{k + 1} \right) (-1)^k 2^{k+1} L_f(x_n)^k.
\]

Thus, the method obtained in \([10]\) is expressed as

\[
 x_{n+1} = x_n - \left( \sum_{k=0}^{m} \left( \frac{\frac{1}{2}k}{k + 1} \right) (-1)^k 2^{k+1} L_f(x_n)^k \right) \frac{f(x_n)}{f'(x_n)}, \quad m \geq 1.
\]

This method has \((m + 2)\)th order convergence for approximating square roots. In the case for \( m = 1 \), a famous iterative process of third order, the Euler–Chebyshev method is obtained. For \( m = 2 \), the method with at least third-order method is obtained:

\[
 x_{n+1} = x_n - \left( 1 + \frac{1}{2}L_f(x_n) + \frac{1}{2}L_f(x_n)^2 \right) \frac{f(x_n)}{f'(x_n)}.
\]

The scheme (6) is of fourth-order for quadratic equations \([2]\). However, except for quadratic equations, Cauchy’s method and its variants, Eqs. (5) and (6), only attain third-order of convergence for the general non-linear equations.

On the other hand, Grau and Noguera \([7]\) find a root of the quadratic equation \( f(x_n) + f(z_n) + f'(x_n)(x - x_n) + \frac{1}{2} f''(x_n)(x - x_n)^2 = 0 \) and obtain a variant of Cauchy’s method with fifth-order convergence

\[
 x_{n+1} = x_n - \frac{2(f(x_n) + f(z_n))/f'(x_n)}{1 + \sqrt{1 - 2f''(x_n)/f'(x_n)^2}},
\]

where

\[
 z_n = x_n - \frac{2}{1 + \sqrt{1 - 2L_f(x_n)}} \frac{f(x_n)}{f'(x_n)}
\]

is the Cauchy’s iterate. This method improves the order of convergence and computational efficiency of Cauchy’s method with an additional evaluation of the function.

In this paper, by solving \( P_3(x) = 0 \) and the approach to approximate the third derivative with a finite difference between the second derivatives, we obtain some new variants of Cauchy’s method for solving non-linear equations. These methods are proved to have at least fourth-order convergence. Moreover, per iteration the new methods require the same evaluations of the function, its first derivative and second derivative as Cauchy’s method although the order of convergence is improved. Consequently, the new methods can compete with Cauchy’s method, as we show in some examples.

2. The methods

In what follows, we will derive the new methods and firstly define

\[
 y_n = x_n - \frac{f(x_n)}{f'(x_n)}.
\]
\[ z_n = x_n + \theta(y_n - x_n), \quad (9) \]

where \( \theta \in \mathbb{R} \) and \( \theta \neq 0 \).

Now, we consider \( P_3(x) = 0 \), namely
\[ f(x_n) + f'(x_n)(x - x_n) + \frac{1}{2} f''(x_n)(x - x_n)^2 + \frac{1}{3!} f^{(3)}(x_n)(x - x_n)^3 = 0, \]
which can be rewritten as
\[ f(x_n) + f'(x_n)(x - x_n) + \frac{1}{2} \left[ f''(x_n) + \frac{1}{3} f^{(3)}(x_n)(x - x_n) \right] (x - x_n)^2 = 0. \quad (10) \]

The solution of (10) gives us a new implicit scheme
\[ x_{n+1} = x_n - \frac{2}{1 + \sqrt{1 - 2L_f(x_n)^2}} f'(x_n), \quad (11) \]
where
\[ \tilde{L}_f(x_n) = \left[ f''(x_n) + \frac{1}{3} f^{(3)}(x_n)(x_{n+1} - x_n) \right] f(x_n) \]
\[ f'(x_n)^2. \quad (12) \]

This scheme requires the \((n + 1)\)th iterate \( x_{n+1} \) to calculate the \((n + 1)\)th iterate itself. To obtain the explicit form, we replace the \((n + 1)\)th iterate \( x_{n+1} \) on the right-hand side of (12) with the Newton’s iterate \( y_n \), where \( y_n \) is defined by (8), so we can approximate \( \tilde{L}_f(x_n) \) as
\[ \tilde{L}_f(x_n) \simeq \left[ f''(x_n) + \frac{1}{3} f^{(3)}(x_n)(y_n - x_n) \right] f(x_n) \]
\[ f'(x_n)^2. \quad (13) \]

In order to avoid the computation of the third derivative \( f^{(3)}(x_n) \), we use the approach similar to [4,5] and approximate it with a finite difference between the second derivatives
\[ f^{(3)}(x_n) \simeq \frac{f''(z_n) - f''(x_n)}{z_n - x_n} = \frac{f''(z_n) - f''(x_n)}{\theta(y_n - x_n)}, \quad (14) \]
where \( y_n \) and \( z_n \) are defined by (8) and (9), respectively. This means that
\[ f''(x_n) + \frac{1}{3} f^{(3)}(x_n)(y_n - x_n) \simeq \frac{1}{3\theta} f''(z_n) + \left( 1 - \frac{1}{3\theta} \right) f''(x_n). \quad (15) \]

If we take \( \theta = \frac{1}{3} \) in (15), namely
\[ f''(x_n) + \frac{1}{3} f^{(3)}(x_n)(y_n - x_n) \simeq f'' \left( x_n - \frac{1}{3} \frac{f(x_n)}{f'(x_n)} \right), \quad (16) \]
then we obtain a new method
\[ x_{n+1} = x_n - \frac{2}{1 + \sqrt{1 - 2L_f(x_n)^2}} f'(x_n), \quad (17) \]
where
\[ L_f(x_n) = \frac{f''(x_n) - f(x_n)/(3f'(x_n))}{f'(x_n)^2}. \quad (18) \]

We can see that unlike Cauchy’s method, the new method uses the evaluation of the second derivative \( f'' \) at the point \( (x_n - f(x_n)/(3f'(x_n))) \) instead of \( x_n \), so this method can be viewed as a new variant of Cauchy’s method.
Moreover, we can obtain more new methods by Taylor approximation of \((1 - 2L_f(x_n))^{1/2}\), namely
\[
\sqrt{1 - 2L_f(x_n)} = \sum_{k \geq 0} \left( \frac{1}{k} \right) (-2L_f(x_n))^k.
\] (19)

Using (19) in (17), we can obtain the following form:
\[
x_{n+1} = x_n - \frac{2}{1 + \sum_{k=0}^{m} \left( \frac{1}{k} \right) (-2L_f(x_n))^k} \frac{f(x_n)}{f'(x_n)},
\] (20)
for \(m \geq 2\). In particular, for \(m = 2\), we have
\[
x_{n+1} = x_n - \frac{4}{4 - 2L_f(x_n) - L_f(x_n)^2} \frac{f(x_n)}{f'(x_n)}.
\] (21)

On the other hand, it is clear that
\[
2 \frac{2}{1 + \sqrt{1 - 2L_f(x_n)}} = \frac{1 - \sqrt{1 - 2L_f(x_n)}}{L_f(x_n)} = \sum_{k \geq 0} \left( \frac{1}{k + 1} \right) (-1)^k 2^{k+1} L_f(x_n)^k.
\] (22)

Thus, from (22), we obtain the other approximate form of (17)
\[
x_{n+1} = x_n - \left( \sum_{k=0}^{m} \left( \frac{1}{k + 1} \right) (-1)^k 2^{k+1} L_f(x_n)^k \right) \frac{f(x_n)}{f'(x_n)},
\] (23)
for \(m \geq 2\). The (23) can be also viewed as a variant of the method defined by (5). In particular, for \(m = 2\), we have
\[
x_{n+1} = x_n - \left( 1 + \frac{1}{2} L_f(x_n) + \frac{1}{2} L_f(x_n)^2 \right) \frac{f(x_n)}{f'(x_n)},
\] (24)
which is a variant of the method defined by (6).

In summary, we have obtained three variants of Cauchy’s method which are defined by (17), (20) and (23), respectively. All present methods can be written in the general form:
\[
x_{n+1} = x_n - \left( 1 + \frac{1}{2} L_f(x_n) + \frac{1}{2} L_f(x_n)^2 + \gamma L_f(x_n)^3 + O(L_f(x_n)^4) \right) \frac{f(x_n)}{f'(x_n)},
\] (25)
where \(\gamma \in \mathbb{R}\) and \(L_f(x_n)\) is defined by (18).

We can see that per iteration the new methods require one evaluation of the function, one of its first derivative and one of its second derivative, which are the same as their classical predecessor, Cauchy’s method. The improved order of convergence will be shown in Section 3.

3. Analysis of convergence

Theorem 1. Assume that the function \(f: D \subset \mathbb{R} \rightarrow \mathbb{R}\) for an open interval \(D\) has a simple root \(x^* \in D\). Let \(f(x)\) be sufficiently smooth in the neighborhood of the root \(x^*\), then the order of convergence of the methods defined by (25) is four.

Proof. Let \(e_n = x_n - x^*\) and \(d_n = z_n - x^*\), where \(z_n = x_n - f(x_n)/(3f'(x_n))\). Using Taylor expansion and taking into account \(f(x^*) = 0\), we have
\[
f(x_n) = f'(x^*)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + O(e_n^5)],
\] (26)
where \( c_k = (1/k!) f^{(k)}(x^*)/f'(x^*), \ k \geq 2 \). Furthermore, we have
\[
\frac{f'(x_n)}{f'(x_n)} = f'(x^*)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + O(e_n^4)].
\] (27)

Dividing (26) by (27) gives us
\[
\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + O(e_n^5),
\] (28)
\[
d_n = e_n - \frac{1}{3} \frac{f(x_n)}{f'(x_n)} = \frac{2}{3} e_n + \frac{1}{3} c_2 e_n^2 - \frac{2}{3} (c_2^2 - c_3)e_n^3 + O(e_n^4).
\] (29)

Expanding \( f'''(z_n) \) about \( x^* \), we have
\[
f'''(z_n) = f'(x^*)(2c_2 + 6c_3d_n + 12c_4d_n^2 + O(d_n^3)),
\]
and then from (29), we have
\[
f'''(z_n) = f'(x^*) \left[ 2c_2 + 4c_3e_n + \left( \frac{16}{3} c_4 + 2c_2c_3 \right) e_n^2 + O(e_n^3) \right].
\] (30)

Again dividing (30) by (27) gives us
\[
\frac{f'''(z_n)}{f'(x_n)} = 2c_2 - 4(c_2^2 - c_3)e_n + \left( 8c_2^3 - 12c_2c_3 + \frac{16}{3} c_4 \right) e_n^2 + O(e_n^3).
\] (31)

From (28) and (31), we have
\[
T_f(x_n) = 2c_2 e_n - (3c_2^2 - 2c_3)e_n^2 + (8c_2^3 - 10c_2c_3 + \frac{8}{3} c_4)e_n^3 + O(e_n^4).
\] (32)

Then, from (32), we have
\[
\frac{1}{2} T_f(x_n) + \frac{1}{2} T_f(x_n)^2 + \gamma T_f(x_n)^3 = c_2 e_n - (c_2^2 - 2c_3)e_n^2 + [(8\gamma - 4)c_2^3 - 2c_2c_3 + \frac{8}{3} c_4]e_n^3 + O(e_n^4).
\] (33)

Furthermore, from (28) and (33), we have
\[
\left( 1 + \frac{1}{2} T_f(x_n) + \frac{1}{2} T_f(x_n)^2 + \gamma T_f(x_n)^3 \right) \frac{f(x_n)}{f'(x_n)} = e_n + \left[ (8\gamma - 5)c_2^3 + c_2c_3 - \frac{1}{3} c_4 \right] e_n^4 + O(e_n^5).
\] (34)

Thus, from (25) and (34), we have
\[
e_{n+1} = e_n - \frac{f(x_n)}{f'(x_n)} = \left[ (5 - 8\gamma)c_2^3 - 2c_2c_3 + \frac{1}{3} c_4 \right] e_n^4 + O(e_n^5).
\] (35)

This means that the methods defined by (25) are of fourth-order.

In particular, the method defined by (17) and its variants (20) and (23) \((m \geq 3)\) satisfy the error equation
\[
e_{n+1} = (-c_2c_3 + \frac{1}{3} c_4)e_n^4 + O(e_n^5),
\] (36)

the method defined by (21) satisfies
\[
e_{n+1} = (2c_2^3 - c_2c_3 + \frac{1}{3} c_4)e_n^4 + O(e_n^5),
\] (37)

and the method defined by (24) satisfies
\[
e_{n+1} = (5c_2^3 - c_2c_3 + \frac{1}{3} c_4)e_n^4 + O(e_n^5).
\] (38)

This ends the proof. \( \square \)
Now, we consider the definition of efficiency index [6] as \( p^{1/w} \), where \( p \) is the order of the method and \( w \) is the number of function evaluations (NFE) per iteration required by the method. If we assume that all the evaluations have the same cost as function one, we have that the present methods have the efficiency indexes equal to \( \sqrt{4} \approx 1.745 \), Cauchy’s method \( \sqrt{5} \approx 2.236 \) and Newton’s method \( \sqrt{2} \approx 1.414 \). However, the computational efficiency depends on not only the order and NFE but also the arithmetic of the computation. Cauchy’s method and the methods defined by (7) \( \sqrt{4} \approx 1.745 \), which are better than the ones of the methods defined by (17) \( \sqrt{5} \approx 2.236 \), Cauchy’s method \( \sqrt{3} \approx 1.732 \) and Newton’s method \( \sqrt{2} \approx 1.414 \). However, the computational efficiency depends on not only the order and NFE but also the arithmetic of the computation. Cauchy’s method and the methods defined by (7) and (17) calculate a square root, and for the methods defined by (20) and (23), the larger value of \( m \) may lead to more arithmetics. So the methods defined by (21) and (24), which need no square roots and less arithmetic of the computation, have better efficiency than Cauchy’s method, the methods defined by (17) and the methods defined by (20) and (23) with \( m \geq 3 \).

4. Numerical results

Now, we employ some variants of Cauchy’s method defined by (17), (21) and (24) obtained in this paper to solve some non-linear equations and compare them with Newton’s method (NM), Cauchy’s method (CM), Euler–Chebyshev method (ECM), Halley’s method (HM), super-Halley method (SHM), the method of Grau and Noguera (GNM) and the method defined by (6). ECM \( (x = 0) \), HM \( (x = \frac{1}{2}) \) and SHM \( (x = 1) \) are the third-order methods with the second derivatives, which can be put into the family of third-order methods defined by [8]

\[
x_{n+1} = x_n - \left( 1 + \frac{1}{2} \frac{L_f(x_n)}{1 - zL_f(x_n)} \right) \frac{f(x_n)}{f'(x_n)},
\]

where

\[
L_f(x_n) = \frac{f''(x_n) f'(x_n)}{f'(x_n)^2}.
\]

For the details of ECM, HM and SHM, see [14,3,9,1].

Displayed in Table 1 is NFEs required such that \( |f(x_n)| < 1.E - 15 \). All numerical results are in accordance with the theory and the advantage of the present methods that these methods have fourth-order convergence although they cost almost the same as the classical third-order methods, such as CM, ECM, HM and SHM.

The results in Table 1 show that the present method (17) improves the computational efficiency of its classical method CM. Also we can see that the present method (24) seems to be more efficient than its classical method, Eq. (6).

As far as the results we consider, in general, the present method (17) requires the less NFEs as compared to various methods, while the present method (21) works better than (24). Consequently, the present methods can compete with NM and many third-order methods, such as CM, ECM, HM and SHM.

We use the following functions, which are the same as in [15], respectively,

\[
\begin{align*}
    f_1(x) &= x^3 + 4x^2 - 10, \quad x^* = 1.3652300134140969, \\
    f_2(x) &= x^2 - e^x - 3x + 2, \quad x^* = 0.25753028543986084, \\
    f_3(x) &= x \exp(x^2) - \sin^2(x) + 3 \cos(x) + 5, \quad x^* = -1.20764782713091893, \\
    f_4(x) &= x^3 - 10, \quad x^* = 2.15443469003188372, \\
    f_5(x) &= \cos(x) - x, \quad x^* = 0.73908513321516067, \\
    f_6(x) &= \sin^2(x) - x^2 + 1, \quad x^* = 1.4044916482153411, \\
    f_7(x) &= e^{x^2 + 7x - 30} - 1, \quad x^* = 3.
\end{align*}
\]

Finally, we consider the systems of non-linear equations. For the general systems, the evaluation of the second Fréchet derivative will be too expensive. But, there are some equations with non-expensive or constant second derivatives, such as the Hammerstein equation (see [12]), where the methods evaluating the second derivative are a good alternative.

Let us consider the following Hammerstein equation [16]:

\[
x(s) = 1 - \frac{1}{4} \int_0^1 \frac{s}{t + s x(t)} \, dt, \quad s \in [0, 1].
\]
Table 1
Comparison of various iterative methods for the scalar non-linear equations

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<th>( x_0 )</th>
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<th>CM</th>
<th>ECM</th>
<th>HM</th>
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<th>Eq. (17)</th>
<th>Eq. (21)</th>
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D–Divergence.

Table 2
Comparison of various iterative methods for the systems of equations

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<thead>
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<th>Iter.</th>
<th>NM</th>
<th>ECM</th>
<th>HM</th>
<th>SHM</th>
<th>Eq. (6)</th>
<th>Eq. (21)</th>
<th>Eq. (24)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1053</td>
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<td>0.0479</td>
<td>0.0467</td>
<td>0.0468</td>
<td>0.0125</td>
<td>0.0128</td>
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<td>2.82e − 6</td>
<td>2.31e − 6</td>
<td>1.85e − 6</td>
<td>1.87e − 6</td>
<td>9.00e − 11</td>
<td>1.12e − 10</td>
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<tr>
<td>3</td>
<td>1.81e − 8</td>
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<td>0</td>
<td>0</td>
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<tr>
<td>4</td>
<td>0</td>
<td></td>
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</tr>
</tbody>
</table>

Using the trapezoidal rule of integration with step \( h = 1/m \), we obtain the following system of non-linear equations [1]:

\[
x^i - 1 + \frac{1}{4m} \left( \frac{1}{2} \frac{t_j}{t_j + t_0} x^0 + \sum_{k=1}^{m-1} \frac{t_j}{t_j + t_k} x^k + \frac{1}{2} \frac{t_j}{t_j + t_m} x^m \right) = 0, \quad i = 0, 1, \ldots, m,
\]

where \( t_j = j/m \) and \( x^j = x(t_j) \). In this case, the second Fréchet derivative is diagonal by blocks and non-expensive.

We consider \( m = 20 \) in the quadrature trapezoidal formula. We give the initial guess \( x_0 = 1.5 \). All computations are carried out with double arithmetic precision. Displayed in Table 2 is the 2-norm of vector functions at each iterative step.

The results in Table 2 show that the present methods (21) and (24) can compete with NM and the classical third-order methods ECM, HM, SHM and the method defined by (6).

5. Conclusions

We have shown that it is possible to obtain many variants of Cauchy’s method by solving the Taylor polynomial of degree three. In Theorem 1, we have obtained that the order of convergence of these methods is four. Analysis of efficiency shows that these methods are preferable to their classical predecessor, Cauchy’s method. In the numerical examples presented, we show that these methods can compete with Cauchy’s method.

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References