Retailer's optimal ordering policy for deteriorating items with maximum lifetime under supplier's trade credit financing

Sheng-Chih Chen a,b, Jinn-Tsair Teng c,*

a Master's Program of Digital Content and Technologies, College of Communication, National ChengChi University, Taipei 11605, Taiwan, ROC
b Center for Creativity and Innovation Studies, National ChengChi University, Taipei 11605, Taiwan, ROC
c Department of Marketing and Management Sciences, Cotsakos College of Business, The William Paterson University of New Jersey, Wayne, NJ 07470, USA

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ABSTRACT
Many products such as fruits, vegetables, pharmaceuticals, volatile liquids, and others not only deteriorate continuously due to evaporation, obsolescence, spoilage, etc. but also have their expiration dates (i.e., a deteriorating item has its maximum lifetime). Although numerous researchers have studied economic order quantity (EOQ) models for deteriorating items, few of them have taken the maximum lifetime of a deteriorating item into consideration. In addition, a supplier frequently offers her/his retailers a permissible delay in payments in order to stimulate sales and reduce inventory. There is no interest charge to a retailer if the purchasing amount is paid to a supplier within the credit period, and vice versa. In this paper, we propose an EOQ model for a retailer when: (1) her/his product deteriorates continuously, and has a maximum lifetime, and (2) her/his supplier offers a permissible delay in payments. We then characterize the retailer's optimal replenishment cycle time. Furthermore, we discuss a special case for non-deteriorating items. Finally, we run several numerical examples to illustrate the problem and provide some managerial insights.

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1. Introduction

[12] further generalized the model to add a permissible delay in payments (i.e., trade credit) under consideration. Recently, Chen et al. [13] proposed an economic production quantity (EPQ) model for deteriorating items with two-level trade credit financing.

In practice, a supplier usually provides her/his retailers a permissible delay in payments to stimulate sales and reduce inventory. During the credit period, the retailer can accumulate the revenue and earn interest on the accumulative revenue. However, beyond this credit period the supplier charges her/his retailers interest on the unpaid balance. Goyal [14] obtained the retailer’s optimal order quantity when the supplier offers a permissible delay in payments. Aggarwal and Jaggi [15] extended Goyal’s EOQ model from non-deteriorating items to deteriorating items. Jamal et al. [16] further generalized Aggarwal and Jaggi’s model for deteriorating items to allow for shortages. Teng [17] established an easy analytical closed-form solution to the problem. Huang [18] considered the trade credit problem to the case in which a supplier offers its retailer a credit period, and the retailer in turn provides another credit period to its customers. Liao [19] extended Huang’s model to an EPQ model for deteriorating items. Teng [20] provided the retailer’s optimal ordering policies to deal with bad credit customers as well as good credit customers. Min et al. [21] proposed an EPQ model under stock-dependent demand and two-level trade credit. Urban [22] extended the model by Min et al. [21] to allow for non-zero ending inventory to increase the retailer’s profitability. Several relevant articles related to this subject are Chen et al. [23], Soni and Shah [24], Hu and Liu [25], Huang [26], Teng et al. [27–29], Zhou et al. [30], and others.

Although most deteriorating items have their expiration dates (i.e., a deteriorating item has its maximum lifetime), none of the above-mentioned papers take the maximum lifetime into consideration. In this paper, we propose an EOQ model for a retailer to obtain its optimal ordering policy when (1) her/his product not only deteriorates continuously but also has a maximum lifetime, and (2) her/his supplier offers a permissible delay in payments. We then characterize the retailer’s optimal ordering policy. Finally, we run several numerical examples to illustrate the theoretical results and provide some managerial insights. Notice that Sarkar [31] established a general EOQ model with quadratic demand, and multiple discount prices and trade credits, which is more general than ours with constant demand and single price and trade credit. However, we are able to: (i) prove the optimal solution not only exists but also is unique (e.g., see Theorems 1 and 2 below), (ii) identify the optimal solution among several alternatives based on two discrimination terms (e.g., see Theorem 3 below), and (iii) show our proposed model is a generalized model for non-deteriorating items (e.g., see Theorem 4 below).

2. Notation and assumptions

The following notation and assumptions are used in the entire paper.

2.1. Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>$o$</td>
<td>the retailer’s average ordering cost per order in dollars</td>
</tr>
<tr>
<td>$c$</td>
<td>the retailer’s purchasing cost per unit in dollars</td>
</tr>
<tr>
<td>$p$</td>
<td>the market price per unit in dollars (with $p &gt; c$)</td>
</tr>
<tr>
<td>$h$</td>
<td>the retailer’s average stock holding cost per unit per year in dollars</td>
</tr>
<tr>
<td>$l_c$</td>
<td>the interest charged per dollar per year in stocks by the supplier’</td>
</tr>
<tr>
<td>$l_e$</td>
<td>the interest earned or return on investment per dollar per year</td>
</tr>
<tr>
<td>$t$</td>
<td>the time in years</td>
</tr>
<tr>
<td>$k(t)$</td>
<td>the retailer’s inventory level in units at time $t$</td>
</tr>
<tr>
<td>$\delta(t)$</td>
<td>the time-varying deterioration rate at time $t$, where $0 \leq \delta(t) \leq 1$</td>
</tr>
<tr>
<td>$m$</td>
<td>the maximum lifetime in years of the deteriorating item</td>
</tr>
<tr>
<td>$n$</td>
<td>the supplier’s permissible delay period to the retailer in years</td>
</tr>
<tr>
<td>$D$</td>
<td>the supplier’s annual constant demand rate in units</td>
</tr>
<tr>
<td>$T$</td>
<td>the retailer’s replenishment cycle time in years (decision variable)</td>
</tr>
<tr>
<td>$Q$</td>
<td>the retailer’s economic order quantity in units</td>
</tr>
<tr>
<td>$\text{TRC}(T)$</td>
<td>the retailer’s total relevant cost per year in dollars</td>
</tr>
<tr>
<td>$T^*$</td>
<td>the retailer’s optimal replenishment cycle time in years</td>
</tr>
<tr>
<td>$\text{TRC}^*$</td>
<td>the retailer’s optimal total relevant cost per year in dollars</td>
</tr>
</tbody>
</table>

2.2. Assumptions

Next, the following assumptions are made to establish the mathematical inventory model.

(1) In today’s global competition, many retailers have no pricing power. As a result, the selling price is hardly changed. Hence, we may assume without loss of generality that the selling price is constant in today’s global competition and low inflation environment.
(2) All deteriorating items have their expiration rates. The physical significance of the deterioration rate is the rate to be closed to 1 when time is approaching to the maximum lifetime $m$. To make the problem tractable, we follow the same assumption as in Sarkar [31] that the deterioration rate is

$$
\theta(t) = \frac{1}{1 + \frac{m}{c_0} t}, \quad 0 \leq t \leq T \leq m.
$$

(1)

Note that (1) it is obvious that the replenishment cycle time $T$ is less than or equal to $m$, and (2) it is a general case for non-deteriorating items, in which $m \to \infty$ and $\theta(t) \to 0$.

(3) Replenishment rate is instantaneous.

(4) In today’s time-based competition, we may assume that shortages are not allowed to occur.

Given the above notation and assumptions, it is possible to formulate the buyer’s optimal replenishment cycle time for deteriorating items with maximum lifetime into a mathematical model.

### 3. Mathematical model

The retailer orders and receives $Q$ units at $t = 0$. Hence, the inventory starts with $Q$ units at $t = 0$, and then gradually depletes to zero at $t = T$ due to the combination effect of demand and deterioration. The graphical representation of the inventory level is shown in Figs. 1 and 2 below. The retailer’s total annual relevant cost consists of the following: (a) the purchasing cost including the cost of deteriorating items, (b) the ordering cost, (c) the holding cost, (d) the interest charged by the supplier, and (e) the interest earned during the credit period. The problem here is for the retailer to determine its optimal replenishment cycle time $T^*$ such that its total annual relevant cost is minimized.

During the replenishment cycle $[0, T]$, the inventory level is depleted by demand and deterioration, and hence governed by the following differential equation:

$$
\frac{dI(t)}{dt} = -D - \theta(t)I(t), \quad 0 \leq t \leq T,
$$

(2)

with the boundary condition $I(T) = 0$. Solving the differential equation (2), we have

$$
I(t) = e^{-\theta(t)} \int_t^T e^{\theta(u)} Du, \quad 0 \leq t \leq T,
$$

(3)

---

**Fig. 1.** The graphical representation for the case of $T < n$.

**Fig. 2.** The graphical representation for the case of $n < T$. 
\[ \delta(t) = \int_0^t \theta(u) \, du. \]

Substituting the time-varying deterioration rate \( \theta(t) = \frac{1}{1 + m - t} \) into (4), we get
\[ \delta(t) = \int_0^t \theta(u) \, du = \int_0^t \frac{du}{1 + m - u} = -\ln(1 + m - u)
\]
\[ = \ln(1 + m) - \ln(1 + m - t) = \ln \left( \frac{1 + m}{1 + m - t} \right). \]

Substituting (5) into (3), we have the inventory level at time \( t \) as
\[ I(t) = D \frac{1 + m - t}{1 + m} \int_t^T \frac{1 + m - u}{1 + m - t} \, du = D (1 + m - t) \ln \left( \frac{1 + m - t}{1 + m - T} \right), \quad 0 \leq t \leq T. \]

Consequently, the retailer’s order quantity is
\[ Q = I(0) = D (1 + m) \ln \left( \frac{1 + m}{1 + m - T} \right). \]

Therefore, the retailer’s holding cost per cycle is
\[ h \int_0^T I(t) \, dt = hD \int_0^T (1 + m - t) \ln \left( \frac{1 + m - t}{1 + m - T} \right) \, dt = hD \left[ \frac{(1 + m)^2}{2} \ln \left( \frac{1 + m}{1 + m - T} \right) + \frac{T^2}{4} - \frac{(1 + m)T}{2} \right]. \]

The retailer has two alternatives to select its replenishment cycle time: either \( T \leq n \) or \( T > n \). If \( T \leq n \), then the retailer sells all \( DT \) units before the supplier’s credit period \( n \). Consequently, there is no interest charged. Otherwise, the retailer cannot sell all \( DT \) units before the supplier’s credit period \( n \), and must finance those unsold items. Let’s discuss them accordingly.

3.1. Case 1. \( T \leq n \)

In this case, the retailer pays off the purchasing cost at \( t = n \) as shown in Fig. 1. Hence, there is no interest charged. However, the retailer accumulates revenue and earns interest on the accumulated revenue during the credit period \([0,n]\). As shown in Fig. 1, the interest earned per cycle is the return rate \( I_e \) multiplied by the area of the trapezoid on the interval \([0,n]\). Therefore, the annual interest earned per cycle is
\[ pl_e \left[ \int_0^n D \, dt + DT(n - T) \right] = pl_e DT \frac{(n - T) + n}{2} = pl_e DT \left( n - \frac{T}{2} \right). \]

Since, the purchasing cost per cycle is \( cI(0) \) dollars and ordering cost per cycle is \( o \) dollars, the retailer’s total annual relevant cost can be expressed as
\[ TRC_1(T) = \text{annual purchasing cost} + \text{annual ordering cost} + \text{annual holding cost} - \text{annual interest earned} \]
\[ = \frac{1}{T} \left( cD(1 + m) \ln \left( \frac{1 + m}{1 + m - T} \right) + o + hD \left[ \frac{(1 + m)^2}{2} \ln \left( \frac{1 + m}{1 + m - T} \right) + \frac{T^2}{4} - \frac{(1 + m)T}{2} \right] \right) - pl_e D \left( n - \frac{T}{2} \right). \]

Now, we finish the case of \( T \leq n \). Next, we discuss the other case of \( T > n \).

3.2. Case 2. \( T > n \)

In this case, the retailer must refinance items in stocks after \( t = n \), as shown in Fig. 2. Hence, the retailer’s interest charged per cycle is \( cl_r \) multiplied by the area under the inventory level \( I(t) \) during \([n,T]\) as follow:
\[ cl_r \int_n^T I(t) \, dt = cl_r D \int_n^T (1 + m - t) \ln \left( \frac{1 + m - t}{1 + m - T} \right) \, dt = cl_r D \left[ \frac{(1 + m - n)^2}{2} \ln \left( \frac{1 + m - n}{1 + m - T} \right) + \frac{T^2 - n^2}{4} - \frac{(1 + m)(T - n)}{2} \right]. \]

On the other hand, the interest earned per cycle is the return rate \( I_e \) multiplied by the area of the triangle \( OA_n \). Consequently, the retailer’s interest earned during the credit period \([0,n]\) is
\[ pl_e \int_0^n D \, dt = pl_e D \frac{n^2}{2}. \]

Similar to (10), we know that the retailer’s total annual relevant cost can be expressed as
\[ TRC_2(T) = \frac{1}{T} \left\{ o + cD(1 + m) \ln \left( \frac{1 + m}{1 + m - T} \right) + hD \left[ \frac{(1+m)^2}{2} \ln \left( \frac{1 + m}{1 + m - T} \right) + \frac{T^2}{4} - \frac{(1+m)T}{2} \right] + cl_n D \left[ \frac{(1+m-n)^2}{2} \ln \left( \frac{1 + m-n}{1 + m - T} \right) + \frac{T^2 - n^2}{4} - \frac{(1+m)(T-n)}{2} - pl_n D n^2 \right] \right\}. \] (13)

Note that from (10) and (13) we have
\[ TRC_1(n) = TRC_2(n). \] (14)

Therefore, the retailer’s objective is to determine the optimal cycle time \( T^* \) such that the total relevant cost per year \( TRC \) is minimized in both (10) and (13). In the next section, we characterize the retailer’s optimal cycle time in each case (i.e., either \( T \leq n \) or \( T \geq n \)), and then obtain the conditions in which the optimal \( T^* \) is in either \( T \leq n \) or \( T \geq n \).

4. Theoretical results and optimal solution

In order to find the optimal solution \( T^* \), taking the first- and second-order derivatives of \( TRC_1(T) \) in (10) with respect to \( T \), and re-arranging terms we get:
\[ \frac{dTRC_1(T)}{dT} = \frac{1}{T^2} \left\{ D(1 + m) \left[ c + \frac{h(1+m)}{2} \right] \left[ \frac{T}{(1+m-T)} - \ln \left( \frac{1+m}{1+m-T} \right) \right] + \frac{DT^2(h + 2pl_m)}{4} - o \right\}. \] (15)

and
\[ \frac{d^2TRC_1(T)}{dT^2} = \frac{1}{T} \left\{ D(1 + m) \left[ c + \frac{h(1+m)}{2} \right] \left[ \frac{3T^2 - 2(1+m)T}{(1+m-T)^2} + 2 \ln \left( \frac{1+m}{1+m-T} \right) \right] + 2o \right\}. \] (16)

Hence, the necessary condition for an optimal solution \( T^* \) in \( TRC_1(T) \) is as follow:
\[ D(1 + m) \left[ c + \frac{h(1+m)}{2} \right] \left[ \frac{T}{(1+m-T)} - \ln \left( \frac{1+m}{1+m-T} \right) \right] + \frac{DT^2(h + 2pl_m)}{4} - o = 0. \] (17)

From Eqs. (15)–(17), we can obtain the following theoretical results. For simplicity, let’s define a discrimination term:
\[ \Delta_1 = \frac{1}{n^2} \left\{ D(1 + m) \left[ c + \frac{h(1+m)}{2} \right] \left[ \frac{n}{(1+m-n)} - \ln \left( \frac{1+m}{1+m-n} \right) - o \right] + \frac{D(h + 2pl_m)}{4} \right\}. \] (18)

**Theorem 1.** If \( T \leq 4 \) (i.e., the retailer’s replenishment cycle time is less than or equal to 4 years), then we have the following results:

1. If \( \Delta_1 \leq 0 \) then \( TRC_1(T) \) is minimized at \( T^* = n \).
2. If \( \Delta_1 > 0 \) then there exists a unique \( T^* \in (0, n) \) satisfies (17) such that \( TRC_1(T^*) \) is minimized.

**Proof.** See Appendix A.

Note that the replenishment cycle time by and large is less than a few months. Hence, the replenishment cycle time less than 4 years (i.e., \( T \leq 4 \)) is true for most of products, especially for deteriorating items. We then analyze the influence of parameters on the optimal solution \( T^* \). Taking the implicit derivative of (17) with respect to each parameter, we have the following results:

**Corollary 1.** If \( T \leq 4 \), and \( \Delta_1 > 0 \), then:

1. The higher the ordering cost \( o \), the longer the replenishment cycle time \( T^* \) (as well as the larger the order quantity \( Q \)).
2. The higher the unit cost \( c \), (as well as the unit price \( p \), the interest earned \( I_e \), and the holding cost \( h \)), the shorter the replenishment cycle time \( T^* \) (as well as the smaller the order quantity \( Q \)).

**Proof.** See Appendix B.

A simple economic interpretation of Corollary 1 is as follows. If the ordering cost is higher, the retailer must reduce the number of orders (i.e., increase the cycle time) to lower the total ordering cost. If the unit cost (as well as the holding cost) is higher, then the retailer must order smaller quantity (i.e., decrease the cycle time) in order to lower the total deterioration and holding cost. Finally, if the unit price (as well as the interest earned) is higher, then the retailer must decrease the cycle time to take the benefit of interest earned more frequently.
Similarly, by taking the first- and second-order derivatives of $TRC_2(T)$ in (13) with respect to $T$, and re-arranging terms we get:

\[
\frac{dTRC_2(T)}{dT} = \frac{1}{T^2} \left\{ D(1 + m) \left[ c + \frac{h(1 + m)}{2} \right] \left[ \frac{T}{1 + m - T} - \ln \left( \frac{1 + m}{1 + m - T} \right) \right] - o + \frac{(h + cIcD)T^2}{4} \right. \\
+ \frac{cIcD(1 + m - n)^2}{2} \left[ \frac{T}{1 + m - T} - \ln \left( \frac{1 + m - n}{1 + m - T} \right) \right] + cIcD \left[ \frac{n^2}{4} - \frac{(1 + m)n}{2} \right] + pIeDn^2 \right\},
\]

and

\[
\frac{d^2TRC_2(T)}{dT^2} = \frac{1}{T^2} \left\{ D(1 + m) \left[ c + \frac{h(1 + m)}{2} \right] \left[ \frac{T}{1 + m - T} - \ln \left( \frac{1 + m}{1 + m - T} \right) \right] - o + \frac{(h + cIcD)T^2}{4} \right. \\
+ 2o + \frac{cIcD(1 + m - n)^2}{2} \left[ \frac{T}{1 + m - T} - \ln \left( \frac{1 + m - n}{1 + m - T} \right) \right] + cIcD \left[ \frac{n^2}{4} - \frac{(1 + m)n}{2} \right] + pIeDn^2 = 0.
\]

Therefore, the necessary condition for an optimal solution $T^*$ in $TRC_2(T)$ is as follow:

\[
D(1 + m) \left[ c + \frac{h(1 + m)}{2} \right] \left[ \frac{T}{1 + m - T} - \ln \left( \frac{1 + m}{1 + m - T} \right) \right] - o + \frac{(h + cIcD)T^2}{4} \right. \\
+ \frac{cIcD(1 + m - n)^2}{2} \left[ \frac{T}{1 + m - T} - \ln \left( \frac{1 + m - n}{1 + m - T} \right) \right] + cIcD \left[ \frac{n^2}{4} - \frac{(1 + m)n}{2} \right] + pIeDn^2 = 0.
\]

Again, for simplicity let’s define

\[
\Delta_2 = \frac{1}{m^2} \left\{ D(1 + m) \left[ c + \frac{h(1 + m)}{2} \right] \left[ m - \ln(1 + m) \right] - o + \frac{(h + cIcD)m^2}{4} + \frac{cIcD(1 + m - n)^2}{2} \left[ m - \ln(1 + m - n) \right] \\
+ cIcD \left[ \frac{n^2}{4} - \frac{(1 + m)n}{2} \right] + pIeDn^2 \right\}.
\]

By using Eqs. (19)–(22), we can get the following results:

**Theorem 2.** If $T \leq 4$, then we have the following results:

1. If $\Delta_1 < 0$ and $\Delta_2 < 0$, then $TRC_2(T)$ is minimized at $T^* = m$.
2. If $\Delta_1 < 0$ and $\Delta_2 > 0$, then there exists a unique $T^* \in (n, m)$ satisfies (21) such that $TRC_2(T^*)$ is minimized.
3. If $\Delta_1 > 0$ and $\Delta_2 > 0$, then $TRC_2(T)$ is minimized at $T^* = n$.

**Proof.** See Appendix C.

Similar to Corollary 1, we can obtain the following results.

**Corollary 2.** If $T \leq 4$, $\Delta_1 < 0$ and $\Delta_2 > 0$, then:

1. The higher the ordering cost $o$, the longer the replenishment cycle time $T^*$ (as well as the larger the order quantity $Q$).
2. The higher the unit cost $c$ (as well as the interest charged $IcD$, the interest earned $IeD$, the unit price $p$, and the holding cost $h$), the shorter the replenishment cycle time $T^*$ (as well as the smaller the order quantity $Q$).

**Proof.** See Appendix D.

A simple economic interpretation of Corollary 2 is similar to those in Corollary 1. Combining Theorems 1 and 2, and Eq. (14), we can prove the following theoretical results:

**Theorem 3.** If $T \leq 4$, then we have the following results:

1. If $\Delta_2 \leq 0$, then $TRC(T)$ is minimized at $T^* = m$.
2. If $\Delta_1 < 0$ and $\Delta_2 > 0$, then there exists a unique $T^* \in (n, m)$ such that $TRC(T^*)$ is minimized.
3. If $\Delta_1 = 0$ and $\Delta_2 > 0$, then $TRC(T)$ is minimized at $T^* = n$.
4. If $\Delta_1 > 0$, then there exists a unique $T^* \in (0, n)$ such that $TRC(T^*)$ is minimized.
Proof. See Appendix E.

For a non-deteriorating item, it is maximum lifetime $m$ is approaching infinity, which is a special case of the proposed model here. Let’s discuss the case for non-deteriorating items now.

5. A special case for non-deteriorating items

The maximum lifetime for non-deteriorating items is approaching infinity. Hence, our proposed model is a generalized model for non-deteriorating items, in which $m$ is approaching infinity. Using Calculus, L’Hospital’s Rule, and simplifying terms, we can simplify the problem for non-deteriorating items as shown below. For details, please see Appendix E.

The retailer’s order quantity per cycle in (7) becomes

$$Q = I(0) = D(1 + m) \ln \left( \frac{1 + m}{1 + m - T} \right) = DT \text{ when } m \to \infty. \tag{23}$$

Similarly, the retailer’s holding cost per cycle in (8) is simplified to

$$\lim_{m \to \infty} hD \left[ \frac{(1 + m)^2}{2} \ln \left( \frac{1 + m}{1 + m - T} \right) + \frac{T^2}{4} - \frac{(1 + m)^2T}{2} \right] = \frac{hDT^2}{2}. \tag{24}$$

Hence, for non-deteriorating items $TRC_1(T)$ in (10) is reduced to:

$$TRC_1(T) = \frac{1}{T} \left[ cDT + o + \frac{DT^2(h + pl_e)}{2} \right] - pl_eDn. \tag{25}$$

Likewise, for non-deteriorating items $TRC_2(T)$ in (13) is simplified to:

$$TRC_2(T) = \frac{1}{T} \left[ cDT + o + \frac{hDT^2}{2} + \frac{cl_e(D(T - n)^2}{2} - \frac{pl_eDn^2}{2} \right]. \tag{26}$$

By using an analogous approach, we can obtain

$$\lim_{m \to \infty} \Delta_1 = \frac{D(h + pl_e)}{2} - \frac{o}{m^2}. \tag{27}$$

Note that $\Delta_2$ does not exist in this special case. Consequently, Theorem 3 can be simplified as follow:

**Theorem 4.** For non-deteriorating items, if $T \leq 4$, then we have the following results:

(1) If $2o > D(h + pl_e)n^2$, then there exists a unique $T^* > n$ such that $TRC(T^*)$ is minimized.

(2) If $2o = D(h + pl_e)n^2$, then $TRC(T)$ is minimized at $T^* = n$.

(3) If $2o < D(h + pl_e)n^2$, then there exists a unique $T^* \in (0, n)$ such that $TRC(T^*)$ is minimized.

Proof. It immediately follows from Eq. (27) and Theorem 3.

**Theorem 4** here is the same as Theorem 1 in Teng [17]. Hence, Teng [17] is a special case in which the maximum lifetime of a product is infinity. Likewise, Goyal [14] is also a special case of our proposed model in which the unit price is equal to the unit cost (i.e., $p = c$), and the maximum lifetime is infinity (i.e., $m \to \infty$).

6. Numerical examples

In this section, we provide several numerical examples to illustrate theoretical results as well as to gain some managerial insights.

**Example 1.** Let’s assume $D = 1000$, $p = $3 per unit, $c = $1 per unit, $o = $30 per order, $h = $1/unit/year, $n = 0.25$ years, $m = 1$ year, $I_e = 0.03$, and $I_o = 0.03$. By using software Maple 9.5, we have the optimal solution as follow:

$$T^* = 0.183281 \text{ and } TRC^*(T^*) = 1283.4039.$$  

**Example 2.** Using the same data as those in **Example 1**, we study the sensitivity analysis on the optimal solution with respect to each parameter. The computational results are shown in Table 1.

The sensitivity analysis reveals that: (1) a higher value of $n$, $p$, or $I_e$ causes lower values of $T^*$ and $TRC^*(T^*)$; (2) by contrast a higher value of $o$ causes higher values of $T^*$ and $TRC^*(T^*)$; (3) a higher value of $D$, $c$, or $h$ causes a lower value of $T^*$ while a higher value of $TRC^*(T^*)$; and (4) conversely a higher value of $m$ causes higher of $T^*$ while a lower value of $TRC^*(T^*)$. Simple
economic interpretations of the above results are as follows. (1) The retailer earns more interest from trade credit if the credit period \(n\) (as well as \(p\), or \(I_c\)) is higher. Hence, the retailer orders less quantity to take the benefit more often, and thus pays less the total relevant cost. (2) If the ordering cost \(o\) is higher, then the retailer pays more the total relevant cost and orders more quantity to reduce the number of orders. (3) If the holding cost \(h\) (as well as \(c\), or \(D\)) is higher, then the retailer pays more the total relevant cost but orders less quantity to reduce holding cost. Finally, (4) If the maximum lifetime \(m\) is higher, the retailer pays less deterioration cost as well as the total relevant cost, and orders more quantity than that the case of shorter \(m\).

### 7. Conclusion and future research

How to determine the optimal ordering policy for deteriorating items with maximum lifetime has received a very little attention by the researchers. In this paper, we have built an EOQ model for the retailer to obtain his/her optimal replenishment cycle time by incorporating the following important and relevant facts: (1) deteriorating products not only deteriorate continuously but also have their maximum lifetime, and (2) a supplier often offers a permissible delay in payment to attract more buyers. Then we have derived the necessary and sufficient conditions, and then characterized the retailer’s optimal replenishment cycle time. Furthermore, we have discussed a special case for non-deteriorating items. Finally, we have used software Maple 9.5 to study the sensitivity analysis on the optimal solution with respect to each parameter to illustrate the model and provide some managerial insights.

For further research, there is a new trend of inventory research that models inventory problems using a thermodynamic approach. For details, readers are referred to Jaber et al. [32], Jaber [33], and Jaber et al. [34]. In addition, this paper can be extended in several ways. For instance, we may consider an integrated solution for both the supplier and the retailer, or a non-cooperative Nash solution. Also, we could generalize the model to allow for shortages, quantity discount, backlogging, etc. Finally, we could consider the effect of inflation rates on the optimal credit period and cycle time simultaneously.

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Appendix A. Proof of Theorem 1

Let’s use (15) to define

\[
F(T) = \frac{d\text{TRC}_1(T)}{dT} = \frac{1}{T^2} \left\{ D(1 + m) \left[ c + \frac{h(1 + m)}{2} \right] \left[ \frac{T}{1 + m - T} - \ln \left( \frac{1 + m}{1 + m - T} \right) \right] - o \right\} + \frac{D(h + 2pL_c)}{4}. 
\] (A1)

Then we get

\[
F(n) = \frac{1}{n^2} \left\{ D(1 + m) \left[ c + \frac{h(1 + m)}{2} \right] \left[ \frac{n}{1 + m - n} - \ln \left( \frac{1 + m}{1 + m - n} \right) \right] - o \right\} + \frac{D(h + 2pL_c)}{4} = \Delta_1. 
\] (A2)

Using L’Hospital’s Rule, we obtain:

\[
\lim_{T \to 0} \frac{1}{(1 + m - T)^2} \ln \left( \frac{1 + m}{1 + m - T} \right) = \lim_{T \to 0} \ln \left( \frac{T - (1 + m - T) \ln \left( \frac{1 + m}{1 + m - T} \right)}{(1 + m - T)^2} \right) = \lim_{T \to 0} \ln \left( \frac{1}{2(T + 2mT - 3T^2)} \right)
\]

\[
= \lim_{T \to 0} \frac{1}{(2 + 2m - 6T)(1 + m - T)} = \frac{1}{2(1 + m)^2}. 
\] (A3)

By using (A1) and (A3), we get

\[
\lim F(T) = \lim_{T \to 0} \frac{D}{2(1 + m)} \left[ c + \frac{h(1 + m)}{2} \right] - \frac{D(h + 2pL_c)}{4} = -\infty. 
\] (A4)

To prove \( F(T) \) is an increasing function (i.e., \( \frac{dF(T)}{dT} > 0 \) or \( \text{TRC}_1(T) \) is a strictly convex function in \( T \), let’s define from (16) that

\[
f(T) = \frac{3T^2 - 2(1 + m)T}{(1 + m - T)^2} + 2 \ln \left( \frac{1 + m}{1 + m - T} \right). 
\] (A5)

It is obvious that \( f(0) = 0 \). Taking the first derivative and re-arranging terms, we have:

\[
\frac{df(T)}{dT} = \frac{6T - 2(1 + m)}{(1 + m - T)^2} \left( 6T^2 - 4(1 + m)T \right) - \frac{2}{1 + m - T} = \frac{6T^2 - 2(1 + m)(1 + m - T) - 6T^2 + 4(1 + m)T + 2(1 + m - T)^2}{(1 + m - T)^3} = \frac{T(8(1 + m) - 10T)}{(1 + m - T)^3} \geq 0, \text{ if } T \leq 4. 
\] (A6)

Combining (A5) and (A6), we know that \( f(T) \) is an increasing function in \( T \), and

\[
f(T) = \frac{3T^2 - 2(1 + m)T}{(1 + m - T)^2} + 2 \ln \left( \frac{1 + m}{1 + m - T} \right) > 0, \text{ for all } 0 < T \leq 4. 
\] (A7)

As a result, we prove

\[
\frac{dF(T)}{dT} = \frac{1}{T} \left\{ (1 + m)D \left[ c + \frac{h(1 + m)}{2} \right] f(T) + 2o \right\} > 0, \text{ if } T \leq 4. 
\] (A8)

If \( \Delta_1 = F(n) \leq 0 \), then it is clear from (A4) and (A8) that \( F(T) \leq 0 \) for all \( T \leq n \). Therefore, \( \text{TRC}_1(T) \) is decreasing in \( T \), and hence minimizing at \( T^* = n \). If \( \Delta_1 = F(n) > 0 \), then by applying (A7) and the Mean Value Theorem into \( F(0) = -\infty \) and \( F(n) > 0 \), there exists a unique \( T^* \in (0, n) \) such that \( F(T^*) = 0 \). Hence, \( \text{TRC}_1(T^*) \) is minimizing at the unique point \( T^* \in (0, n) \). This completes the proof of Theorem 1. \( \square \)

Appendix B. Proof of Corollary 1

Let’s first prove that

\[
L(T) = \frac{T}{(1 + m - T)} - \ln \left( \frac{1 + m}{1 + m - T} \right) > 0, \text{ for all } T > 0. 
\] (B1)

By using the facts that \( L(0) = 0 \) and \( \frac{dL(T)}{dT} = \frac{T}{(1 + m - T)^2} > 0 \), we prove (B1). Taking the implicit derivative of \( T \) in (17) with respect to unit cost \( c \), and simplifying terms, we get

\[
\left[ \frac{T}{(1 + m - T)} - \ln \left( \frac{1 + m}{1 + m - T} \right) \right] + \left[ c + \frac{h(1 + m)}{2} \right] \left[ \frac{T}{(1 + m - T)^2} \right] \frac{dT}{dc} = 0. 
\] (B2)

Consequently, by applying (B1) we derive
\[
\frac{dT}{dc} = \frac{T}{c + \frac{h(1+m)}{2}} \left( \frac{1}{1+m} - \ln \left( \frac{1+m}{1+T} \right) \right) < 0. \tag{B3}
\]

Similarly, taking the implicit derivative of \( T \) in (17) with respect to ordering cost \( o \), and re-arranging terms, we yield
\[
D(1+m) \left[ c + \frac{h(1+m)}{2} \right] \frac{T}{(1+m-T)^2} \frac{dT}{do} = 1 = 0. \tag{B4}
\]

Therefore, we get
\[
\frac{dT}{do} = \frac{(1+m-T)^2}{DT(1+m) \left[ c + \frac{h(1+m)}{2} \right]} > 0. \tag{B5}
\]

Again, taking the implicit derivative of \( T \) in (17) with respect to unit price \( p \), and re-arranging terms, we yield
\[
\left\{ \frac{c + \frac{h(1+m)}{2}}{2} \left( \frac{1}{1+m-T} + \frac{h+2pl_c}{2} \right) \right\}\frac{dT}{dp} + \frac{T}{2} = 0. \tag{B6}
\]

Thus, we have
\[
\frac{dT}{dp} = \frac{-T}{2c + h(1+m) \left[ \frac{1}{1+m-T} + (h+2pl_c) \right]} < 0. \tag{B7}
\]

By using an analogous approach, we can obtain:
\[
\frac{dT}{dI_e} = \frac{-T}{2c + h(1+m) \left[ \frac{1}{1+m-T} + (h+2pl_c) \right]} < 0. \tag{B8}
\]

Finally, taking the implicit derivative of \( T \) in (17) with respect to holding cost \( h \), re-arranging terms, and applying (B1), we have
\[
\left\{ D(1+m) \left[ c + \frac{h(1+m)}{2} \right] \frac{T}{(1+m-T)^2} + \frac{DT(h+2pl_c)}{2} \right\} \frac{dT}{dh} = -\left\{ D(1+m)^2 \left[ \frac{T}{1+m-T} - \ln \left( \frac{1+m}{1+m-T} \right) + \frac{DT^2}{4} \right] \right\} < 0. \tag{B9}
\]

Thus, we know \( \frac{dT}{dh} < 0 \). This completes the proof of Corollary 1. \( \square \)

**Appendix C. Proof of Theorem 2**

From (19), we define
\[
G(T) = \frac{dTRC_2(T)}{dT} = \frac{1}{T^2} \left\{ D(1+m) \left[ c + \frac{h(1+m)}{2} \right] \frac{T}{1+m-T} - \ln \left( \frac{1+m}{1+m-T} \right) - o + \frac{(h+cl_c)DT^2}{4} \right\} + \frac{cl_cD(1+m-n)^2}{2} \frac{T}{1+m-T} \left[ \frac{n}{1+m-n} \right] + cl_cD \left[ \frac{n^2}{4} \left( \frac{1+m-n}{1+m-T} \right) \right] + pl_cD n^2 \right\}. \tag{C1}
\]

Substituting \( T = n \) into (C1), and simplifying terms, we get
\[
G(n) = \frac{1}{n^2} \left\{ D(1+m) \left[ c + \frac{h(1+m)}{2} \right] \left[ \frac{n}{1+m-n} - \ln \left( \frac{1+m}{1+m-n} \right) \right] - o \right\} + \frac{D(h+2pl_c)}{4} = \Delta_1. \tag{C2}
\]

To prove \( G(T) \) is an increasing function (i.e., \( \frac{dG(T)}{dT} = \frac{dTRC_2(T)}{dT} > 0 \) or \( TRC_2(T) \) is a strictly convex function in \( T \)), we define from (20) that
\[
g(T) = \frac{cl_c D(1+m-n)^2}{2} \left[ \frac{3T^2 - 2(1+m)T}{(1+m-T)^2} + 2 \ln \left( \frac{1+m-n}{1+m-T} \right) \right] - \frac{cl_c D}{2} \left[ n^2 - 2(1+m)n \right], \text{ for } T \geq n. \tag{C3}
\]

Substituting \( T = n \) into (C3), and simplifying terms, we get
\[
g(n) = cl_c D n^2 > 0. \tag{C4}
\]

Taking the first derivative of \( g(T) \), and re-arranging terms, we yield
\[
\frac{dg(T)}{dT} = \frac{cl_c D(1+m-n)^2}{2} \left[ \frac{8(1+m)}{1+T} - 10T \right] > 0, \text{ for } n \leq T \leq 4. \tag{C5}
\]
Combining (C3)-(C5), we know $g(T)$ is an increasing function in $T$, and
\[ g(T) > g(n) = cT Dn^2 > 0, \text{ for } n < T \leq 4. \] (C6)

By using the fact that $f(T)$ is an increasing function in $T$ from (A7), we get:
\[ \frac{3T^2 - 2(1 + m)n}{(1 + m - T)^2} + 2 \ln \left( \frac{1 + m}{1 + m - T} \right) = f(T) > f(n) = \frac{3n^2 - 2(1 + m)n}{(1 + m - n)^2} + 2 \ln \left( \frac{1 + m}{1 + m - n} \right) > 0, \text{ for all } n < T \leq 4. \] (C7)

Combining (20), (C1), (C6), and (C7), we have
\[ \frac{dG(T)}{dT} = \frac{d^2 \text{TRC}_2(T)}{dT^2} > \frac{1}{T^3} \left( D(1 + m) \left[ c + \frac{h(1 + m)}{2} \right] \left( \frac{3n^2 - 2(1 + m)n}{(1 + m - n)^2} + 2 \ln \left( \frac{1 + m}{1 + m - n} \right) \right) + 2(1 + cL)Dn^2 - plcDn^2 \right). \] (C8)

Although we could not prove that $\frac{dG(T)}{dT} > 0$ in (C8), we may assume without loss of generality that $\frac{dG(T)}{dT} > 0$ because there is only one negative term (i.e., $-plcDn^2$) in (C8). Further, we run numerous examples and found $\frac{dG(T)}{dT} > 0$ in all examples. If $\frac{dG(T)}{dT} > 0$, from (C2) and $n < T \leq m$, we have
\[ \Delta_1 = G(n) < G(m) = \Delta_2. \] (C9)

Consequently, if $\Delta_1 < 0$ and $\Delta_2 \leq 0$, then $\text{TRC}_2(T)$ is decreasing during time interval $[n, m]$, and thus minimizing at $T^* = m$. If $\Delta_1 = G(n) < 0$ and $\Delta_2 = G(m) > 0$, then by using the Mean Value Theorem we know that there exists a unique $T' \in (n, m)$ such that $G(T') = 0$, and thus $\text{TRC}_2(T)$ is minimizing at $T'$. Finally, if $\Delta_1 = G(n) > 0$ and $\Delta_2 = G(m) > 0$, then $\text{TRC}_2(T)$ is increasing during time interval $[n, m]$, and thus minimizing at $T' = n$. This completes the proof of Theorem 2. \( \square \)

**Appendix D. Proof of Corollary 2**

For simplicity, let’s define
\[ K(T) = \frac{I_c}{2} \left( 1 + m - n \right)^2 \left[ \frac{T}{1 + m - T} - \ln \left( \frac{1 + m - n}{1 + m - T} \right) \right] + \frac{T^2 + n^2}{2} - (1 + m)n, \text{ for all } T \geq n. \] (D1)

Then we get:
\[ K(n) = 0, \text{ and } \frac{dK(T)}{dT} = \frac{I_c}{2} \left( 1 + m - n \right)^2 \left[ \frac{T}{1 + m - T} - \ln \left( \frac{1 + m - n}{1 + m - T} \right) \right] > 0. \] (D2)

Therefore we have
\[ K(T) = \frac{I_c}{2} \left( 1 + m - n \right)^2 \left[ \frac{T}{1 + m - T} - \ln \left( \frac{1 + m - n}{1 + m - T} \right) \right] + \frac{T^2 + n^2}{2} - (1 + m)n > 0 \text{ for all } T > n. \] (D3)

Taking the implicit derivative of $T$ in (21) with respect to unit cost $c$, simplifying terms, and applying (B1) and (D3), we get
\[ \left\{ \left[ c + \frac{h(1 + m)}{2} \right] \left[ \frac{(1 + m)T}{1 + m - T} + \frac{h + cL}{2} \frac{T(1 + m - n)^2}{1 + m - T} \right] - \frac{1}{T} \left[ \frac{T}{1 + m - T} - \ln \left( \frac{1 + m}{1 + m - T} \right) + K(T) \right] \right\} < 0. \] (D4)

Therefore, $\frac{d}{dT} < 0$ so that the higher the unit cost the shorter the replenishment cycle time. Similar to the proof of Corollary 1, we can prove the rest of Corollary 2. \( \square \)

**Appendix E. Proof of Theorem 3**

By using (9) and Theorems 1 and 2, if $\Delta_2 \leq 0$ then we have:
\[ \text{TRC}_2(m) \leq \text{TRC}_2(n) = \text{TRC}_1(n) \leq \text{TRC}_1(T), \text{ for all } T \in (0, m). \] (E1)

Hence, $\text{TRC}(T)$ is minimized at $T^* = m$.

If $\Delta_1 < 0$ and $\Delta_2 > 0$, then there exists a unique $T' \in (n, m)$ such that
\[ \text{TRC}_2(T') \leq \text{TRC}_2(n) = \text{TRC}_1(n) \leq \text{TRC}_1(T), \text{ for all } T \in (0, m). \] (E2)

Therefore, there exists a unique $T' \in (n, m)$ such that $\text{TRC}(T)$ is minimized at $T'$. Similarly, if $\Delta_1 = 0$ and $\Delta_2 > 0$, then we get:
\[ \text{TRC}_1(n) \leq \text{TRC}_1(T) \text{ and } \text{TRC}_2(n) \leq \text{TRC}_2(T), \text{ for all } T \in (0, m). \] (E3)
Consequently, TRC(T) is minimized at T* = n.
Finally, if \( \Lambda_1 > 0 \) then \( \Lambda_2 > 0 \) and there exists a unique \( T^* \in (0, n) \) such that
\[
\text{TRC}_1(T^*) \leq \text{TRC}_1(n) = \text{TRC}_2(n) \leq \text{TRC}_2(T), \quad \text{for all } T \in (0, m).
\] (E4)

Thus, there exists a unique \( T^* \in (0, n) \) such that TRC(T) is minimized at \( T^* \). This completes the proof of Theorem 3. \( \square \)

**Appendix F. Formulas for non-deteriorating items**

Using Calculus and simplifying terms, we have:
\[
\frac{d}{dm} \ln \left( \frac{1 + m}{1 + m - T} \right) = \frac{1 + m - T}{1 + m} \left[ \frac{1}{1 + m - T} - \frac{1 + m}{(1 + m - T)^2} \right] = \frac{-T}{(1 + m)(1 + m - T)}.
\]

Using L'Hospital's Rule, we obtain:
\[
\lim_{m \to \infty} (1 + m) \ln \left( \frac{1 + m}{1 + m - T} \right) = \lim_{m \to \infty} \frac{\frac{\partial}{\partial m} \ln \left( \frac{1 + m}{1 + m - T} \right)}{\frac{\partial}{\partial m} (1 + m)} = \lim_{m \to \infty} \frac{-T}{1 + m} = T.
\] (F1)

Consequently, the retailer’s order quantity per cycle in (7) becomes
\[
Q = I(0) = D(1 + m) \ln \left( \frac{1 + m}{1 + m - T} \right) = DT \quad \text{when } m \to \infty.
\] (23)

Similarly, we can get the following results:
\[
\lim_{m \to \infty} \frac{(1 + m)^2}{2} \ln \left( \frac{1 + m}{1 + m - T} \right) - \frac{(1 + m)^2}{2} = \frac{1}{2} \lim_{m \to \infty} \left[ \ln \left( \frac{1 + m}{1 + m - T} \right) - \frac{T}{1 + m} \right] = \frac{1}{2} \lim_{m \to \infty} \left[ \frac{T}{(1 + m)(1 + m - T)} + \frac{T}{1 + m} \right] = \frac{1}{2} \lim_{m \to \infty} \left[ \frac{T^2(1 + m)}{2(1 + m - T)} \right] = \frac{1}{4} \lim_{m \to \infty} T^2 = \frac{T^2}{4}.
\] (F2)

As a result, we know that the retailer’s holding cost per cycle in (8) is simplified to
\[
\lim_{m \to \infty} hD \left[ \frac{(1 + m)^2}{2} \ln \left( \frac{1 + m}{1 + m - T} \right) + \frac{T^2}{4} - \frac{(1 + m)^2}{2} \right] = \frac{hDT^2}{2}.
\] (24)

Hence, for non-deteriorating items TRC1(T) in (10) is reduced to:
\[
\text{TRC}_1(T) = \frac{1}{T} \left[ cDT + o + \frac{DT^2(h + p_l)}{2} - pl_p Dn.\right.
\] (25)

Likewise, we have the following results:
\[
\lim_{m \to \infty} \frac{(1 + m - n)^2}{2} \ln \left( \frac{1 + m - n}{1 + m - T - n} \right) - \frac{(1 + m - n)(T - n)}{2} = \frac{1}{2} \lim_{m \to \infty} \left[ \ln \left( \frac{1 + m - n}{1 + m - T - n} \right) - \frac{(1 + m - n)(T - n)}{1 + m - n} \right] = \frac{1}{2} \lim_{m \to \infty} \left[ \frac{(T - n)(1 + m - n)}{(1 + m - n)(1 + m - T - n)} + \frac{(T - n)(1 + m - n)}{(1 + m - n)^2} \right] = \frac{1}{2} \lim_{m \to \infty} \left[ \frac{(T - n)(1 + m - n)}{2(1 + m - n - T)} + \frac{(T - n)(1 + m - n)}{1 + m - n} \right] = \frac{1}{2} \lim_{m \to \infty} \left[ \frac{(T - n)(1 + m - n)(1 + m - n - T)}{2(1 + m - n)} \right] = \frac{1}{4} \lim_{m \to \infty} (T - n)(T - 3n).
\] (F3)

Therefore, for non-deteriorating items TRC2(T) in (13) is simplified to:
\[
\text{TRC}_2(T) = \frac{1}{T} \left[ cDT + o + \frac{hDT^2}{2} + \frac{cl_p D(T - n)^2}{2} - \frac{pl_p Dn^2}{2} \right].
\] (26)

Likewise, by applying L’Hospital’s Rule, we obtain:
$$\lim_{m \to \infty} \frac{n}{2m} \ln \left( \frac{1 + m}{1 + m - n} \right)$$

and

$$\lim_{m \to \infty} \frac{n}{2m} \ln \left( \frac{1 + m - n}{1 + m} \right)$$

Combining (17), (24), and (25), we get

$$\lim_{m \to \infty} \Delta I = \frac{D(m + p_k)}{2} \frac{\sigma}{n^2}.$$  

This completes the formulas for non-deteriorating items.

References