Robust adaptive parameter estimation of sinusoidal signals

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A B S T R A C T

A novel two-step adaptive identification framework is proposed for sinusoidal signals to estimate the unknown offset, amplitude, frequency and phase, where only the output measurements are used. After representing the sinusoidal signal as a linearly parameterized form, several adaptive laws are developed. The proposed adaptive laws are driven by parameter estimation error information that is derived by applying filter operations on the output measurements, so that globally exponential convergence of the parameter estimation is proved. By using the sliding mode technique, we further improve the design of adaptations to achieve finite-time (FT) parameter estimation. The proposed approaches are independent of any observer/predictor design and robust to bounded measurement noises. The developed estimation methods are finally extended to the full parameter estimation of multi-sinusoids with only output measurements. Comparative simulation results are provided to illustrate the efficacy of the proposed methods.

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1. Introduction

The modeling of periodical signals has been found useful in various scientific disciplines and engineering applications (Abdel-Elrady & Schoukens, 2005; Söderström, Wigren, & Abdel-Elrady, 2005). In particular, the estimation of unknown parameters of sinusoidal signals is essential for the tracking and rejection of such signals, e.g. receptive control (Griñó & Costa-Castelló, 2005). For the estimation of unknown frequency and amplitude, Fourier analysis provides a potential solution. However, this method may not be suitable for control applications where real-time processing is required. The least-squares method (Stoica, Li, & Li, 2000) and Kalman filter (Bittanti & Savaresi, 2000) have also been used. Moreover, the estimation of unknown frequency, amplitude and phase for two sinusoids with close frequencies was studied based on the least-squares grid search method (Barbedo & Lopes, 2009). It is noted that most of these results have been developed from the perspective of signal processing, (e.g. notch filters Regalia, 1991 and frequency locked loop Meyr, Moeneclaey, & Fechtel, 1998), and only local convergence of the parameter estimation can be proved.

To achieve global parameter estimation convergence, several adaptive techniques have been developed for the frequency estimation. In Hsu, Ortega, and Damm (1999), adaptive notch filter was proposed for a single sinusoid signal. For multi-sinusoids, only a few methods have been reported, e.g. lower order estimator (Obregon-Pulido, Castillo-Toledo, & Loukianov, 2002), adaptive identifier (Xia, 2002) and adaptive observer (Marino & Tomei, 2002), where only the frequencies can be estimated.

To address the estimation of other parameters, an amplitude and frequency estimator was considered in Aranovskiy, Bobtsov, Kremlev, Nikolaev, and Slita (2010). The offset, amplitude and frequency estimation of a single sinusoid signal was also studied in Hou (2005), where the phase is not considered. In particular, a quadratic term of the output measurement is employed in Hou (2005) so that the estimator may be sensitive to noises. Moreover, a direct extension of the idea of Hou (2005) to the multiple sinusoids seems to be difficult. For multiple sinusoids, the estimation of the amplitude and frequency was investigated in Hou (2007), where the phase is not considered. However, a direct extension of the idea of Hou (2005) to the multiple sinusoids seems to be difficult. For multiple sinusoids, the estimation of the amplitude and frequency was investigated in Hou (2007), where the phase is not considered. Moreover, the estimation of the amplitude depends on the calculation of an inverse matrix that may encounter a singularity problem. The estimation of frequencies, amplitudes and phases of multi-sinusoids was also studied in Petrović (2012) and Stoica, Moses, Friedlander, and Söderström (1989), where the offset is assumed to be zero. Recent work (Hou, 2012) presented a possible solution for estimating full parameters of multi-sinusoids, where a higher order identifier should be...
adopted to avoid using the unknown derivative of the output measurement, while it may impose significant computational costs.

To the best of authors’ knowledge, the full parameter estimation for sinusoids is still an interesting yet challenging problem, in particular for multi-sinusoids with unknown offset. This paper is devoted to fill this gap and to develop a novel full parameter identification framework for sinusoidal signals. After representing a sinusoidal signal as a linearly parameterized form, a two-step identification procedure is developed: in the first step, the offset and frequency are estimated, for which the estimation is independent of the unknown amplitude and phase; in the second step, the amplitude and phase are estimated on the basis of the estimated frequency. Only the output measurements are employed (i.e., their derivatives are not used) such that the proposed methods are robust to noises. Furthermore, the conventionally used high order observer is avoided, and this leads to faster convergence. On the other hand, a novel methodology of designing adaptive laws for parameter estimation is developed, where the proposed adaptive laws are driven by the parameter estimation error information that is derived by introducing filter operations on the output measurements. In this case, globally exponential error convergence is rigorously proved. We further adopt the sliding mode technique (Edwards & Spurgeon, 1998; Fernandez & Hedrick, 1987; Yu, Yu, Shirinzadeh, & Man, 2005) to improve the design of adaptive laws such that finite-time (FT) estimation of unknown parameters is achieved and the robustness of the proposed methods is also analyzed. Extension of the proposed ideas to multi-sinusoids is finally addressed. Comparative simulation results are provided to show the efficacy of the proposed estimations.

The main contributions of this paper are twofold: (i) a novel full parameter estimation scheme for sinusoidal signal (in particular for multi-sinusoids) is developed, where all parameters (frequencies, amplitudes, phases and offset) can be estimated; (ii) several estimation schemes for multi-sinusoids are developed, where all parameters (frequencies, amplitudes, phases and offset) can be estimated.

Such that finite-time (FT) estimation of unknown parameters is achieved and the robustness of the proposed method is also analyzed. Extension of the proposed ideas to multi-sinusoids is finally addressed. Comparative simulation results are provided to show the efficacy of the proposed estimations.

The paper is organized as follows: Section 2 provides the problem formulation; Section 3 presents the adaptive parameter estimation of a single sinusoidal signal; Section 4 studies a finite-time estimation scheme; The robustness against bounded noises is addressed in Section 5; Section 6 introduces the parameter estimation of multi-sinusoids; Simulations are provided in Section 7 and conclusions are given in Section 8.

2. Problem formulation

This paper will study the parameter estimation of the following sinusoidal signal

\[ y = A_0 + A \sin(\omega t + \phi) \]  

(1)

where the offset \( A_0 \), amplitude \( A \), frequency \( \omega \) and phase \( \phi \) are the unknown parameters to be estimated.

The problem to be addressed is to estimate the unknown offset \( A_0 \), amplitude \( A \), frequency \( \omega \) and phase \( \phi \) by using the available measurement \( y \). In this paper, the output derivative \( \dot{y} \) is not needed. The case with measurement noises will be addressed and the parameter estimation of multi-sinusoids will also be studied in this paper.

Definition 1 (Sastry & Bodson, 1989). A vector or matrix function \( \Phi \) is persistently excited (PE) if there exist constants \( \tau > 0, \epsilon > 0 \) such that \( \int_{t}^{t+\tau} \Phi^T(r)\Phi(r)dr \geq \epsilon I, \forall t \geq 0 \).

3. Adaptive parameter estimation

This section addresses the parameter estimation of sinusoidal signal (1). A two-step estimation procedure will be introduced, e.g. Section 3.1 presents the first step to estimate the offset \( A_0 \) and frequency \( \omega \), while Section 3.2 studies the estimation of amplitude \( A \) and phase \( \phi \).

3.1. Estimation of offset \( A_0 \) and frequency \( \omega \)

To estimate the offset \( A_0 \) and frequency \( \omega \) of (1) by using the output \( y \), we take the derivatives of (1) as

\[ \begin{align*}
\dot{y} &= \omega A \cos(\omega t + \phi) \\
\ddot{y} &= -\omega^2 A \sin(\omega t + \phi) = -\dot{\omega}^2 y + \omega^2 A_0.
\end{align*} \]  

(2)

Consider the second equation of (2) in the s-domain by taking the Laplace transform (ignoring exponentially vanishing terms of initial condition) and filtering both sides with a Hurwitz polynomial \( s^2 + \lambda_1 s + \lambda_2 \), we can represent (2) as

\[ y = \frac{\lambda_1 s}{\Lambda(s)} y + \frac{\lambda_2 - \omega^2}{\Lambda(s)} y + \frac{\omega^2 A_0}{\Lambda(s)} = \Phi_2 \Theta_1 \]  

(3)

where \( \Theta_1 = [\lambda_1, \lambda_2 - \omega^2, \omega^2 A_0]^T \),

\[ \Phi_2 = \left[ \begin{array}{c} y \\ \frac{1}{\Lambda(s)} y \\ \frac{1}{\Lambda(s)} \dot{y} \end{array} \right]. \]  

(4)

Note that Eq. (3) holds only when the system is properly initialized: in general, the right-hand side of (3) differs from \( y \) by exponentially vanishing terms due to initial conditions ignored in the derivation (Xia, 2002). Moreover, the filter \( \Lambda(s) \) is introduced so that the derivative \( \dot{y} \) is not needed, which can diminish the effect of measurement noises compared to Hou (2005).

From (4), it is shown that the offset \( A_0 \) and frequency \( \omega \) can be determined provided that the parameter \( \Theta_1 \) is properly estimated. It is clear that \( \Theta_1 \) is independent of the amplitude \( A \) and phase \( \phi \), which do not appear in (3) and (4). In the following, we will present a novel adaptive estimation of \( \Theta_1 \) without using any observer or predictor (Hou, 2005, 2007; Marino & Tomei, 2002). For this purpose, we define an auxiliary variable \( x \) as

\[ \dot{x} = -\ell x + y, \quad x(0) = 0 \]  

(5)

where \( \ell > 0 \) is a positive constant. It is noted that \( x \) is the filtered version of \( y \) in terms of a filter \( 1/(s + \ell) \).

To facilitate the parameter estimation, we further define auxiliary variables \( x_f, y_f, \Phi_{yf} \) as

\[ \begin{align*}
\dot{x}_f &= x_f + y_f, \quad x_f(0) = 0 \\
\dot{y}_f &= y_f, \quad y_f(0) = 0 \\
\dot{\Phi}_{yf} &= \Phi_{yf} + \Phi_2 \Theta_1, \quad \Phi_{yf}(0) = 0
\end{align*} \]  

(6)

where \( \kappa > 0 \) is a filter parameter. As shown in (6), \( x_f, y_f, \) and \( \Phi_{yf} \) are indeed the filtered version of the variables \( x, y, \) and \( \Phi_2 \) defined in (3)–(5), and thus \( x_f, y_f, \) and \( \Phi_{yf} \) can be obtained by applying a stable filter operation \( (\cdot)_f = (\cdot)/(\kappa s + 1) \) on the corresponding variables \( x, y, \) and \( \Phi_{yf} \), respectively.

Based on (3) and (5), one can derive the following fact

\[ \begin{align*}
y_f &= \frac{1}{\kappa s + 1} \left[ \Phi_1 \Theta_1 \right] = \frac{1}{\kappa s + 1} \left[ \Phi_1 \right] \Theta_1 = \Phi_{yf} \Theta_1 \\
\dot{x}_f &= \frac{1}{\kappa s + 1} \left[ -\ell x + y \right] = -\ell x_f + y_f
\end{align*} \]  

(7)

(8)

where \( \Phi_{yf}, x_f \) and \( y_f \) are calculated from (6).
On the other hand, we can obtain from the first equation of (6) that
\[ x_f = \frac{X - x_j}{\kappa}. \tag{9} \]
According to (7)–(9), one can have
\[ \frac{x - x_f}{\kappa} + \varepsilon x_f = \Phi_f \Theta_1. \tag{10} \]
Eq. (10) reveals partial information on the unknown parameter \( \Theta_1 \) based on the available variables \( x, x_f \) and \( \Phi_f \), which can be explicitly obtained from (2) and (6).

Then we define auxiliary regressor matrix \( P_1 \in \mathbb{R}^{3 \times 3} \) and vector \( Q_1 \in \mathbb{R}^3 \) as
\[
\dot{P}_1 = -\ell P_1 + \Phi_f^T \Phi_f, \quad P_1(0) = 0
\]
\[
\dot{Q}_1 = -\ell Q_1 + \Phi_f^T \left[ \frac{x - x_f}{\kappa} + \varepsilon x_f \right], \quad Q_1(0) = 0
\tag{11}
\]
where \( \ell > 0 \) is the parameter defined in (5).

One can obtain the solution of (11) as
\[
P_1 = \int_0^t e^{-\ell(t-r)} \Phi_f^T(r) \Phi_f(r) dr
\]
\[
Q_1 = \int_0^t e^{-\ell(t-r)} \Phi_f^T(r) \left[ \frac{x(r) - x_f(r)}{\kappa} + \varepsilon x_f(r) \right] dr.
\tag{12}
\]
Finally, denote a vector \( W_1 \in \mathbb{R}^3 \) as
\[
W_1 = P_1 \hat{\Theta}_1 - Q_1
\tag{13}
\]
where \( \hat{\Theta}_1 \) is the estimation of \( \Theta_1 \), which will be given in (16).

By substituting (10) into (11) or (12), we have
\[
Q_1 = P_1 \hat{\Theta}_1
\tag{14}
\]
In this case, Eq. (13) can be rewritten as
\[
W_1 = P_1 \hat{\Theta}_1 - P_1 \Theta_1 = -P_1 \tilde{\Theta}_1
\tag{15}
\]
where \( \tilde{\Theta}_1 = \Theta_1 - \hat{\Theta}_1 \) is the parameter estimation error.

The vector \( W_1 \) in (15) can be derived from the variables \( x, x_f \) and \( \Phi_f \) in (5)–(6) and \( P_1, Q_1 \) in (11)–(13), while it defines explicitly information on the unknown parameter estimation error \( \tilde{\Theta}_1 \). Thus, \( W_1 \) will be used to develop a new adaptive law.

The adaptive law for updating \( \tilde{\Theta}_1 \) is provided by
\[
\dot{\tilde{\Theta}}_1 = -\Gamma_1 W_1, \tag{16}
\]
with \( \Gamma_1 > 0 \) being a constant diagonal matrix.

The positive definiteness property of the matrix \( P_1 \) is important to prove the estimation convergence of (16). To address this issue, the following facts hold:

**Lemma 1.** If \( \Phi_f \) defined in (4) is persistently excited (PE), then the matrix \( P_1 \) defined in (11) is positive definite, i.e., the minimum eigenvalue \( \lambda_{\min}(P_1) > \sigma_1 > 0 \) for positive constant \( \sigma_1 > 0 \).

**Proof.** Because \( \Phi_f \) is the filtered value of \( \phi_f \) in terms of a stable, minimum phase and strictly proper transfer function \( 1/(s^3 + 1) \) in (6), then based on the claims in Sastry and Bodson (1989) and Xia (2002), we know that \( \Phi_f \) is PE as long as \( \Phi_f \) is PE. We further prove that \( P_1 \) is positive definite if \( \Phi_f \) is PE. According to Definition 1, if \( \Phi_f \) is PE, there exists \( \epsilon > 0 \) such that the inequality \( \int_0^t \Phi_f^T(r) \Phi_f(r) dr \geq \epsilon I \) holds, then it can be shown as Na, Herrmann, Ren, Mahyuddin, and Barber (2011), Na, Mahyuddin, Herrmann, and Ren (2013) and Na, Ren, and Xia (2014) that
\[
P_1 = \int_0^t e^{-\epsilon(t-r)} \Phi_f^T(r) \Phi_f(r) dr
\geq e^{-\epsilon t} \int_0^t \Phi_f^T(r) \Phi_f(r) dr \geq e^{-\epsilon t} \epsilon I
\] is true for \( t > \tau \). Consequently, if \( \Phi_1 \) is PE, one may have \( \lambda_{\min}(P_1) > \sigma_1 > 0 \) with \( \sigma_1 = e^{-\epsilon \tau} \epsilon \).

**Lemma 2.** For sinusoid signal \( \omega \), the regressor vector \( \Phi_f \) is PE, so that \( \lambda_{\min}(P_1) > \sigma_1 > 0 \) is true.

**Proof.** We first prove that the alternative vector \( \Phi_f = [\dot{x}, y, 1] = [A \omega \cos(\omega t + \phi), A \omega + \sin(\omega t + \phi), 1] \) is PE. According to the property of sinusoidal functions with a period \( T \), the following facts are true:
\[
\int_{t}^{t+T} \sin(\omega r) \sin(\omega r) dr = T/2
\tag{18}
\]
\[
\int_{t}^{t+T} \cos(\omega r) \cos(\omega r) dr = T/2
\tag{19}
\]
\[
\int_{t}^{t+T} \sin(\omega r) \cos(\omega r) dr = 0
\tag{20}
\]
Then we can obtain the fact (21) holds (as shown in Box I). It can be verified from (21) that \( \int_{t}^{t+T} \Phi_f^T(r) \Phi_f(r) dr > 0 \). In this sense, \( \Phi_f \) is PE based on Definition 1. Moreover, since \( A(s) \) is a Hurwitz polynomial and it is shown that \( \Phi_f = [\dot{x}, y, 1] \), it follows that \( \Phi_f \) is PE as long as \( \Phi_f = [\dot{x}, y, 1] \) is PE. Consequently, we know that \( \Phi_f \) is PE, and thus \( \lambda_{\min}(P_1) > \sigma_1 > 0 \) is true according to Lemma 1.

Now we have the following result:

**Theorem 1.** For the adaptive law (16), the parameter estimation error \( \tilde{\Theta}_1 \) exponentially converges to zero with the rate \( \mu_1 = 2\sigma_1/\lambda_{\max}(\Gamma_1^{-1}) \).

**Proof.** Consider the Lyapunov function as \( V_1 = \frac{1}{2} \tilde{\Theta}_1^T \Gamma_1^{-1} \tilde{\Theta}_1 \). It follows from (15)–(16) that
\[
\dot{V}_1 = \tilde{\Theta}_1^T \Gamma_1^{-1} \dot{\tilde{\Theta}}_1 = \dot{\tilde{\Theta}}_1^T W_1 = -\mu_1 V_1
\tag{22}
\]
where \( \mu_1 = 2\sigma_1/\lambda_{\max}(\Gamma_1^{-1}) \) is a positive constant for all \( t > 0 \). Then exponential convergence of the error \( \tilde{\Theta}_1 \) to zero with the rate \( \mu_1 \) is guaranteed based on (22).

According to Theorem 1, \( \Theta_1 = [\lambda_1, \lambda_2 - \omega^2, \omega^2 A_b]^T \) can be estimated by using adaptation (16) with variables defined in (5), (6), (11) and (13). Then the estimation of the offset \( A_b \) and frequency \( \omega \) of sinusoid signal (1) can be determined accordingly with the estimated parameter vector \( \tilde{\Theta}_1 \). The estimated frequency \( \omega \) will be used to estimate the unknown amplitude and phase in the second step.

### 3.2. Estimation of amplitude \( A \) and phase \( \phi \)

In this section, we are seeking to estimate the amplitude \( A \) and phase \( \phi \). For this purpose, we rewrite (1) as
\[
y = A_0 + a \sin(\omega t) + b \cos(\omega t) = \Phi_2 \Theta_2
\tag{23}
\]
where the amplitude and phase can be obtained based on \( A = \sqrt{a^2 + b^2} \) and \( \phi = \tan^{-1}(b/a) \), and
\[
\Phi_2 = [1, \sin(\omega t), \cos(\omega t)], \quad \Theta_2 = [A_0, a, b]^T
\tag{24}
\]
are the regressor vector and the unknown parameters to be estimated.

Similarly, we define the auxiliary variables \( x, x_f, y_f \) as in (5)–(6) and \( \Phi_{2f} \) as
\[
k \Phi_{2f} + \Phi_{2f} = \Phi_2, \quad \Phi_{2f}(0) = 0
\tag{25}
\]
where \( k > 0 \) is the filter parameter defined in (6).
Following the arguments in Section 3.1, it follows from (23)–(25) that

\[ x - \dot{x}_f = -\frac{\kappa}{\kappa} \Phi_2 \Theta_2. \]  

(26)

Define the matrix \( P_2 \in \mathbb{R}^{3 \times 3} \) and vector \( Q_2 \in \mathbb{R}^3 \) as

\[
\begin{align*}
P_2 &= -\varepsilon P_2 + \Phi_2^T \Phi_2, \quad P_2(0) = 0 \\
Q_2 &= -\varepsilon Q_2 + \Phi_2^T \begin{bmatrix} x - \dot{x}_f \\ \kappa \end{bmatrix}, \quad Q_2(0) = 0
\end{align*}
\]

(27)

and an auxiliary vector \( W_2 \in \mathbb{R}^3 \) as

\[ W_2 = P_2 \dot{\Theta}_2 - Q_2. \]

(28)

Then the adaptive law for updating \( \hat{\Theta}_2 \) is given by

\[ \dot{\hat{\Theta}}_2 = -\Gamma_2 W_2 \]

(29)

with \( \Gamma_2 > 0 \) being a constant diagonal matrix.

To prove the parameter estimation convergence, we need to show that \( \Phi_2 \) is PE and thus \( P_2 \) is positive definite:

**Lemma 3.** The regressor vector \( \Phi_2 \) in (24) is PE, so that \( P_2 \) is positive definite, i.e. \( \lambda_{\min}(P_2) > \sigma_2 > 0 \) for positive constant \( \sigma_2 > 0 \).

**Proof.** Consider \( \Phi_2^T = [1, \sin(\omega t), \cos(\omega t)] \), then according to (18)–(20), we have

\[
\int_{t}^{t+T} \Phi_2^T(r) \Phi_2(r) dr = \int_{t}^{t+T} \begin{bmatrix} 1 \\ \sin(\omega r) \\ \cos(\omega r) \\ \sin^2(\omega r) \\ \cos(\omega r) \sin(\omega r) \\ \cos^2(\omega r) \end{bmatrix} dr = \frac{T}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

(30)

From (30), we know that \( \int_{t}^{t+T} \Phi^T(r) \Phi(r) dr > 0, \quad \forall t \geq 0 \) is true, i.e. \( \Phi_2 \) is PE, which further implies \( \lambda_{\min}(P_2) > \sigma_2 > 0 \) based on similar claims of Lemma 1.

We have the following result:

**Theorem 2.** For the adaptive law (29), the parameter estimation error \( \hat{\Theta}_2 = \Theta_2 - \hat{\Theta}_2 \) exponentially converges to zero with the rate \( \mu_2 = 2\sigma_2/\lambda_{\max}(\Gamma_2^{-1}) \).

**Proof.** From (27) and (28), one can verify that \( W_2 = -P_2 \dot{\Theta}_2 \) holds. Then consider the Lyapunov function as \( V_2 = \frac{1}{2} \dot{\Theta}_2^T \Gamma_2^{-1} \dot{\Theta}_2 \), such that its derivative along (29) is

\[ \dot{V}_2 = \dot{\Theta}_2^T \Gamma_2^{-1} \dot{\Theta}_2 = \dot{\Theta}_2^T W_2 = -\frac{1}{2} P_2 \dot{\Theta}_2 \leq -\mu_2 V_2 \]

(31)

where \( \mu_2 = 2\sigma_2/\lambda_{\max}(\Gamma_2^{-1}) \) is a positive constant for all \( t > 0 \). Consequently, exponential convergence of \( \hat{\Theta}_2 \) to zero with the rate \( \mu_2 \) is guaranteed.

With the estimated parameter \( \hat{\Theta}_2 = [\hat{A}_0, \hat{a}, \hat{b}]^T \), the estimation of the amplitude and phase of sinusoidal signal (1) can be determined by \( \hat{A} = \sqrt{\hat{a}^2 + \hat{b}^2} \) and \( \hat{\phi} = \tan^{-1}(\hat{b}/\hat{a}) \). Note that the estimation of offset \( A_0 \) can be obtained either based on the estimation of \( \Theta_1 \) or \( \Theta_2 \).

**Remark 1.** It should be pointed out that the proposed adaptive laws (16) and (29) are driven by the parameter estimation error \( W_i, \quad i = 1, 2 \) obtained by introducing the filter operations (5), (6) and (25) as Na et al. (2011) and Na et al. (2013). Consequently, the proposed estimation methods do not depend on any observer/predictor design in comparison to Hou (2005, 2007) and Marino and Tomei (2002).

**4. Finite-time parameter estimation**

This section further improves the design of adaptive laws in terms of the sliding-mode technique (Fernandez & Hedrick, 1987; Yu et al., 2005) to achieve finite-time error convergence.

The auxiliary variables \( P_1, Q_1, W_1 \) are defined as (11) and (13), then we design the adaptive law for \( \hat{\Theta}_1 \) as

\[ \dot{\hat{\Theta}}_1 = -\Gamma_1 \frac{P_1^T W_1}{||W_1||}. \]

(32)

**Theorem 3.** For the adaptive law (32), the parameter estimation error \( \hat{\Theta}_1 \) converges to zero in finite time \( t_1 \leq \frac{||\Theta_1(0)|| \lambda_{\max}(\Gamma_1^{-1})}{\mu_1} \).

**Proof.** Consider the Lyapunov function \( V_1 = \frac{1}{2} \dot{\Theta}_1^T \Gamma_1^{-1} \dot{\Theta}_1 \), then it follows from (15) and (32) that

\[ \dot{V}_1 = \dot{\Theta}_1^T \Gamma_1^{-1} \dot{\Theta}_1 = \frac{\dot{\Theta}_1^T P_1^T W_1}{||W_1||} = -||P_1 \dot{\Theta}_1|| \leq -\mu_1 \sqrt{V_1} \]

(33)

where \( \mu_1 = \sigma_1 \sqrt{2/\lambda_{\max}(\Gamma_1^{-1})} \) is a positive constant. Then according to the results in Bhat and Bernstein (1998), \( \lim_{t \to \infty} V_1 = 0 \) is true in finite time \( t_1 \leq 2 \sqrt{V(0)/\mu_1} \), and thus FT convergence of the parameter estimation error \( \dot{\Theta}_1 \) to zero can be guaranteed.

Similar to Theorem 3, with the variables \( P_2, Q_2, W_2 \) defined in (27)–(28), we design the adaptive law of \( \hat{\Theta}_2 \) as

\[ \dot{\hat{\Theta}}_2 = -\Gamma_2 \frac{P_2^T W_2}{||W_2||}. \]

(34)

**Theorem 4.** For the adaptive law (34), the parameter estimation error \( \hat{\Theta}_2 \) converges to zero in finite time \( t_2 \leq \frac{||\Theta_2(0)|| \lambda_{\max}(\Gamma_2^{-1})}{\mu_2} \).

**Proof.** The proof of Theorem 4 is similar to that of Theorem 3, and thus is omitted here.
The idea of the proposed adaptive laws (32) and (34) can be further explained. Consider the fact that \( \hat{\theta}_i = -\hat{\theta}_i, \ i = 1, 2 \), then one may obtain the error dynamics of the proposed adaptations as \( \hat{\theta}_i = -\Gamma P_i^T \hat{P}_i \hat{\theta}_i / ||P_i \hat{\theta}_i|| \), which can be represented as a terminal sliding mode surface (Yu et al., 2005) as \( S(\hat{\theta}_i) = \hat{\theta}_i + \Gamma P_i^T \hat{P}_i \hat{\theta}_i / ||P_i \hat{\theta}_i||. \) Clearly, the origin \( \hat{\theta}_i = 0 \) is the global terminal attractor of the proposed sliding mode surface \( S(\hat{\theta}_i) = 0 \) for invertible matrix \( P_i (i.e. \lambda_{\text{min}} (P_i) > \sigma_i > 0) \). This further implies that the equilibrium point \( \hat{\theta}_i = 0 \) can be reached in finite time along the sliding mode surface \( S(\hat{\theta}_i) = 0 \). Consequently, the precise estimation of \( \hat{\theta}_i \) can be obtained in finite time.

5. Parameter estimation with bounded noises

In the practical systems, additive noises are usually unavoidable. In this section, we will prove that the proposed estimation methods are robust against the measurement noises. Assume there exists a bounded noise \( \nu \) perturbing the output measurement so that (3) is rewritten as

\[
y = \Phi_1 \theta_1 + \nu. \tag{35}
\]

Note that for practical applications, the noises \( \nu \) are usually stochastic but with bounded amplitude (e.g., Gaussian noise \( \nu \sim N(0, \sigma^2) \) with zero mean and \( \sigma^2 \) variance).

In this case, Eq. (7) can be represented as \( y_1 = \Phi_2 \theta_2 + v_1 \), where \( v_1 \) is the filtered version of the noise \( \nu \) in terms of a low-pass filter \( \xi v_1 + v_1 = \nu \), which is similar to that used in (6). Consequently, \( v_1 \) is a bounded stochastic signal as well as \( v \). In this case, one may modify (10) as

\[
\frac{\dot{x} - x}{\kappa} + \xi \sigma = \Phi_2 \theta_2 + v_1 \tag{36}
\]

so that the variable \( W_1 \) defined in (13) fulfills

\[
W_1 = P_1 \hat{\theta}_1 - Q_1 = -P_1 \hat{\theta}_1 + \theta_1 \tag{37}
\]

where \( \zeta = -\int_0^t e^{-\xi(t-s)} \Phi_2^T(r) \nu(r)dr \) is also bounded, i.e. \( \| \zeta \| \leq \sigma \) holds for constant \( \sigma > 0 \).

For sinusoid signal (1), with a bounded noise, we have:

**Corollary 1.** For adaptive law (16) with a bounded noise \( \nu \) in (35), then the estimation error \( \hat{\theta}_i \) is uniformly ultimately bounded (UUB).

**Proof.** Consider the Lyapunov function \( V = \frac{1}{2} \hat{\theta}_1 \Gamma_1^{-1} \hat{\theta}_1 \), then from (37), one may have

\[
\dot{V} \leq -\| \hat{\theta}_1 \|^2 (\sigma_1 / \| \hat{\theta}_1 \| - \sigma). \tag{38}
\]

Then according to the extended Lyapunov’s Theorem (Khalil, 2001), the estimation error \( \hat{\theta}_1 \) ultimately converges to a compact set \( \Omega \) defined as \( \Omega := \{ \hat{\theta}_1 \| \hat{\theta}_1 \| \leq \sigma / \sigma_1 \} \).

**Corollary 2.** For adaptive law (29) with a bounded noise \( \nu \) in (35), then the estimation error \( \hat{\theta}_i \) is UUB.

**Proof.** The proof of Corollary 2 can be conducted following the proof of Corollary 1.

**Corollary 3.** For adaptive law (32) with a bounded noise \( \nu \) in (35), then \( \hat{\theta}_1 \) converges to a compact set in finite time satisfying \( \lim_{t \to \infty} \hat{\theta}_1 = P_1^{-1} \zeta \). 

**Proof.** The derivative of \( P_1^{-1} W_1 \) with respect to time is first investigated. Consider the fact \( P_1^{-1} W_1 = -\hat{\theta}_1 + P_1^{-1} \zeta \) from (37), it follows that

\[
\frac{dP_1^{-1} W_1}{dt} = -\dot{\hat{\theta}}_1 + P_1^{-1} \zeta + P_1^{-1} \dot{\zeta} = \hat{\theta}_1 - P_1^{-1} \dot{P}_1 P_1^{-1} \zeta + P_1^{-1} \dot{\zeta} = \hat{\theta}_1 + \xi \tag{39}
\]

where \( \xi \) is defined as \( \xi = -P_1^{-1} \dot{P}_1 P_1^{-1} \zeta + P_1^{-1} \dot{\zeta} \).

We first prove that \( P_1^{-1} W_1 \) and \( \hat{\theta}_1 \) are bounded. Select a Lyapunov function \( V = \frac{1}{2} W_1^T P_1^{-1} \Gamma_1^{-1} P_1^{-1} W_1 \), and consider the facts that the adaptive gain \( \Gamma_1 \) is a diagonal matrix and the matrix \( P_1 \) is symmetric, then the derivative of \( V \) can be obtained as

\[
\dot{V} = W_1^T P_1^{-1} \Gamma_1^{-1} \frac{dP_1^{-1} W_1}{dt} \leq -W_1^T P_1^{-1} \frac{W_1}{\| W_1 \|} + W_1^T P_1^{-1} \Gamma_1^{-1} \xi \leq (1 - \lambda_{\text{max}}(\Gamma_1^{-1})) \| P_1^{-1} \xi \| \| W_1 \| \leq -\alpha \| W_1 \| \tag{40}
\]

where \( \alpha = 1 - \lambda_{\text{max}}(\Gamma_1^{-1}) \| P_1^{-1} \xi \| \) is a scalar. Consider the fact \( \zeta = -\int_0^t e^{-\xi(t-s)} \Phi_2^T(r) \nu(r)dr \) is also bounded because \( \nu \) and \( \Phi_1 \) are bounded. Moreover, the PE condition \( \lambda_{\text{min}}(P_1) > \sigma_1 > 0 \) holds and thus the matrices \( P_1 \) and \( \dot{P}_1 \) are bounded, and \( P_1^{-1} \) exists and is bounded. Thus, the terms \( \xi \) and \( P_1^{-1} \xi \) are all bounded. In this case, the adaptive gain \( \Gamma_1 \) can be chosen such that \( \lambda_{\text{max}}(\Gamma_1^{-1}) < 1/ \| P_1^{-1} \xi \| \) holds, then we know that \( \alpha = 1 - \lambda_{\text{max}}(\Gamma_1^{-1}) \| P_1^{-1} \xi \| \) can be set to be positive, i.e. \( \alpha > 0 \).

To this end, it follows that \( \dot{V} \leq -\alpha \| W_1 \| \leq 0 \). According to Lyapunov Theorem, \( V(t) \) is finite and bounded, which further implies that \( P_1^{-1} W_1 \) is bounded. Then the estimation error \( \hat{\theta}_1 \) is also bounded according to (37).

Finally, we will derive the upper bound of the parameter estimation error \( \hat{\theta}_1 \). For this purpose, we consider the definition of Lyapunov function \( V \) and have \( \sqrt{2V/\lambda_{\text{max}}(P_1^{-1})} \leq \| W_1 \| \). Then (40) can be rewritten as

\[
\dot{V} \leq -\mu \sqrt{V} \tag{41}
\]

where \( \mu = \alpha \sqrt{2/\lambda_{\text{max}}(P_1^{-1})} \) is a positive constant. Then similar to the proof of Theorem 3, it can be concluded based on Bhat and Bernstein (1998) that \( V(t) \) converges to zero in finite time \( t \leq 2\sqrt{V/\mu}/\mu \), and thus \( \lim_{t \to \infty} P_1^{-1} W_1 = 0 \) is true in finite time. This determines the error bound as \( \lim_{t \to \infty} \hat{\theta}_1 = P_1^{-1} \zeta \) according to (37).

**Corollary 4.** For adaptive law (34) with a bounded noise \( \nu \) in (35), then \( \hat{\theta}_2 \) converges to a compact set in finite time satisfying \( \lim_{t \to \infty} \hat{\theta}_2 = P_1^{-1} \zeta \). 

**Proof.** The proof of Corollary 4 is similar to that of Corollary 3.

From Corollaries 1–4, we know that the proposed estimation methods are robust to bounded noises. In particular for FT estimation schemes (32) and (34), we can calculate the error bounds as \( \lim_{t \to \infty} \hat{\theta}_1 = P_1^{-1} \zeta \).

**Remark 2.** For the proposed estimation methods, the parameters \( \lambda_1, \lambda_2, \ell, \kappa \) and \( \Gamma_i, i = 1, 2 \) are selected by the designers. The parameters \( \lambda_1, \lambda_2 \) should be chosen such that \( A(s) \) is Hurwitz. The parameter \( \ell \) in (5), (11) and (27) introduces a d.c. gain of \( 1/\ell \) for the filter \( 1/(s + \ell) \), and thus \( \ell \) cannot be too large to retain the convergence speed. Another parameter \( \kappa \) in (6) and (25) defines the ‘bandwidth’ of the filter \( 1/(\kappa s + 1) \), and thus \( \kappa \) should be small regarding the robustness. Moreover, large gains \( \Gamma_i, i = 1, 2 \) can achieve faster convergence speed (as shown in Theorems 1 and 2).
but may lead to a more oscillatory transient performance and reduced robustness in the presence of noises. Thus the adaptive gains should be chosen to make a tradeoff between the error convergence performance and the robustness.

6. Parameter estimation of multi-sinusoids

In this section, we will extend the proposed ideas to the parameter estimation of the following multi-sinusoids

\[
y = \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} [A_{0i} + A_i \sin(\omega_i t + \phi_i)]
\]  

(42)

where \(y_i = A_{0i} + A_i \sin(\omega_i t + \phi_i)\) with \(\omega_i \neq \omega_j\) for \(i \neq j\) are the harmonics. Note that the harmonic frequencies \(\omega_i\) are not required to be an integer multiple of the base frequency \(\omega_1\).

To the best of our knowledge, full parameter estimation for (42) including unknown offset \(A_0 = \sum_{i=1}^{n} A_{0i}\) has rarely been addressed. In the following, a constructive 2-step estimation procedure will be presented.

6.1. Estimation of frequencies \(\omega_i\)

For sinusoids in (42), one can obtain that

\[
\begin{pmatrix}
y_1 \\
y_1^{(4)} \\
\vdots \\
y_1^{(2n-2)}
\end{pmatrix} = \begin{bmatrix} A_1 \sin(\omega_1 t + \phi_1) \\ A_2 \sin(\omega_2 t + \phi_2) \\ \vdots \\ A_n \sin(\omega_n t + \phi_n) \end{bmatrix} K + \begin{bmatrix} A_{01} \\ 0 \\ \vdots \\ 0 
\end{bmatrix}
\]

(43)

where

\[
K = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
k_1 & k_2 & \cdots & k_n \\
\vdots & \vdots & \ddots & \vdots \\
k_1^{n-1} & k_2^{n-1} & \cdots & k_n^{n-1}
\end{bmatrix}
\]

is a Vandermonde matrix (Xia, 2002), which is nonsingular for \(\omega_i \neq \omega_j\), \(i \neq j\). Consequently, one can derive that

\[
\begin{pmatrix} A_1 \sin(\omega_1 t + \phi_1) \\ A_2 \sin(\omega_2 t + \phi_2) \\ \vdots \\ A_n \sin(\omega_n t + \phi_n) \end{pmatrix} K^{-1} = \begin{pmatrix} y - A_0 \\ \ddots \\ y^{(2n-2)} \end{pmatrix}.
\]

(46)

Substituting (46) into the last equation of (43) yields

\[
y^{(2n)} = [k_1^{n}, k_2^{n}, \ldots, k_n^{n}] K^{-1} \begin{pmatrix} y - A_0 \\ \ddots \\ y^{(2n-2)} \end{pmatrix}^T
\]

\[
= -\Theta_3 y^{(2n-2)} - \cdots - \theta_n y^{2n-2} - \theta_1 y - A_0.
\]

(47)

It can be verified (Xia, 2002) that

\[
\prod_{i=1}^{n} (s^2 + \omega_i^2) = s^{2n} + \theta_1 s^{2n-2} + \cdots + \theta_{n-1} s^2 + \theta_n.
\]

(48)

Based on (48), the estimation of frequencies \(\omega_i\) can be achieved provided that the unknown vector \([\theta_1, \ldots, \theta_n]^T\) is precisely estimated based on (47). In case \(n\) is not too large, explicit formulations can be easily obtained to determine \(\omega_i\), while for a large \(n\) the frequencies \(\omega_i\) can be numerically computed by finding the roots of the polynomial \(s^{2n} + \theta_1 s^{2n-2} + \cdots + \theta_{n-1} s^2 + \theta_n\) as Hou (2012) and Xia (2002). Thus the problem to be addressed in the following is to estimate \(\theta = [\theta_1, \ldots, \theta_n]^T\) of (47) using output \(y\).

Remark 3. Note that the above derivation is partially inspired by Hou (2012) and Xia (2002). However, the estimation of the offset \(A_0\) has not been studied in the literature (i.e. \(A_0\) is assumed to be known or null). Moreover, the following developments are different to Hou (2012) and Xia (2002), and the direct calculation of the inverse Vandermonde matrix \(K^{-1}\) is also avoided in this paper.

To estimate \(\theta = [\theta_1, \ldots, \theta_n]\), we choose a Hurwitz polynomial \(\lambda(s) = s^{2n} + \lambda_{2n} s^{2n-1} + \cdots + \lambda_{2} s + \lambda_{1}\) and filter both sides of (47) in the s-domain (ignoring exponentially vanishing terms with initial condition) such that

\[
y = \begin{bmatrix} \lambda(2n-1) & \lambda(2n-3) & \cdots & \lambda(2) & \lambda(0) \\ \lambda(2n-2) & \cdots & \lambda(2) & \lambda(0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda(2) & \cdots & \lambda(2) & \lambda(0) \\ \lambda(0) 
\end{bmatrix} y + \Theta_3 A_0
\]

(49)

holds with

\[
\Theta_3 = \begin{bmatrix} \lambda_{2n} - \theta_1 & \lambda_{2n} - \theta_2 & \cdots & \lambda_{2n} - \theta_n \\ \lambda_{2n-1} - \theta_1 & \lambda_{2n-1} - \theta_2 & \cdots & \lambda_{2n-1} - \theta_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_2 - \theta_1 & \lambda_2 - \theta_2 & \cdots & \lambda_2 - \theta_n \\ \lambda_1 - \theta_1 & \lambda_1 - \theta_2 & \cdots & \lambda_1 - \theta_n \
\end{bmatrix},
\]

\[
\Phi_3 = \begin{bmatrix} s^{2n-1} \lambda(s) & s^{2n-3} \lambda(s) & \cdots & s \lambda(s) & 1 \\ \lambda(s) & \lambda(s) & \cdots & \lambda(s) & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda(s) & \lambda(s) & \cdots & \lambda(s) & 1 \\ \lambda(s) & \lambda(s) & \cdots & \lambda(s) & 1 
\end{bmatrix}.
\]

(50)

Now, it is shown that multi-sinusoidal signal (42) is represented in a linearly parameterized form (49) with unknown parameter \(\Theta_3\), so that the estimation of frequencies \(\omega_i\) in (42) can be obtained by estimating \(\Theta_3\) and then solving Eq. (48). This can be taken as further extensions of the methods proposed in Section 3.

To estimate \(\Theta_3\), the filtered variables \(x, y, y, \Phi_3y\) and the auxiliary filtered regressor matrix \(P_0\) and vectors \(Q_0, W_0\) are defined accordingly following their counterparts in (5)–(15), and thus the adaptive law for \(\hat{\Theta}_3\) is given as

\[
\dot{\hat{\Theta}}_3 = -\Gamma_3^T W_3
\]

(51)

with \(\Gamma_3 > 0\) being a constant diagonal matrix.

It should be emphasized here that the proposed regressor vector \(\Phi_3\) in (49) is also PE.

Lemma 4. The regressor vector \(\Phi_3\) in (49) is PE, so that \(P_3\) is positive definite, i.e. \(\lambda_{\min}(P_3) > \sigma_3 > 0\) for constant \(\sigma_3 > 0\).

Proof. This fact can be proved following the merit of the proof of Lemma 2.

We now summarize the main result of this subsection as

Corollary 5. For parameter adaptation (51), the estimation error \(\hat{\Theta}_3 = \Theta_3 - \hat{\Theta}_3\) exponentially converges to zero with the rate \(\mu_3 = 2\sigma_3/\lambda_{\max}(\Gamma_3^{-1})\).

Proof. The proof of Corollary 5 is similar to that of Theorem 1.

To achieve FT estimation of \(\hat{\Theta}_3\), we design the adaptive law as

\[
\dot{\hat{\Theta}}_3 = -\Gamma_3^T P_3^T W_3/\|W_3\|.
\]

(52)

Corollary 6. For parameter adaptation (52), then \(\hat{\Theta}_3\) converges to zero in finite time \(t_3 \leq \|\hat{\Theta}_3(0)\|\lambda_{\max}(\Gamma_3^{-1})/\sigma_3\).

Proof. The proof of Corollary 6 is similar to that of Theorem 3.
6.2. Estimation of amplitude $A_i$, phase $\phi_i$ and offset $A_0$

Finally, we will estimate the unknown amplitude $A_i$, phase $\phi_i$ and offset $A_0$ for multi-sinusoids (42) based on the estimated frequencies $\omega_i$. It is known that

$$y_i = A_{0i} + A_i \sin(\omega_i t + \phi_i) = A_{0i} + a_i \sin(\omega_0 t) + b_i \cos(\omega_0 t)$$ (53)

where $A_i = \sqrt{a_i^2 + b_i^2}$ and $\phi_i = \tan^{-1}(b_i/a_i)$ are the unknown amplitudes and phases to be estimated. Then we rewrite (53) as

$$y = \Phi_i \Theta_i$$ (54)

where

$$\Phi_i = [1, \sin(\omega_1 t), \cos(\omega_1 t), \ldots, \sin(\omega_n t), \cos(\omega_n t)]$$

(55)

are the regressor vector and the parameters to be estimated.

To estimate $\Theta_i$, we define the filtered variables $x, x_0, y_i$ and $\Phi_i$ and the auxiliary matrix $P_4$ and vectors $Q_4, W_4$ similar to their counterparts in (25)–(28), and then propose the adaptive law for $\Theta_i$ as

$$\dot{\Theta}_i = -P_4 W_4$$ (56)

with $P_4 > 0$ being a constant diagonal matrix.

Similar to the proof of Lemma 3, we can prove that the regressor vector $\Phi_i$ in (54) is PE.

**Lemma 5.** The regressor vector $\Phi_i$ in (55) is PE, so that $P_4$ is positive definite, i.e. $\lambda_{\min}(P_4) > \sigma_4 > 0$ for constant $\sigma_4 > 0$.

Then the result of this subsection can be given as

**Corollary 7.** For parameter adaptation (56), the estimation error $\hat{\Theta}_i - \tilde{\Theta}_i$ exponentially converges to zero with the rate $\mu_4 = 2\sigma_4/\lambda_{\max}(P_4^{-1})$.

**Proof.** The proof of Corollary 7 is similar to that of Theorem 2.

Finally, FT estimation of $\hat{\Theta}_i$ can be obtained by proposing the following adaptive law

$$\dot{\Theta}_i = -P_4 \frac{P_4 J_4 W_4}{\|W_4\|}$$ (57)

**Corollary 8.** For parameter adaptation (57), then $\hat{\Theta}_i$ converges to zero in finite time $t_4 \leq \|\hat{\Theta}_i(0)\|/\lambda_{\max}(P_4^{-1})/\sigma_4$.

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Fig. 1. Parameter estimation of $y = 2 + 3 \sin(4t + \pi/4)$.

Fig. 2. Parameter estimation of $y = 2 + 3 \sin(4t + \pi/4)$ with noise $v \sim N(0, 0.5)$.

**Proof.** The proof of Corollary 8 is similar to that of Theorem 3.

With the estimated parameter $\hat{\Theta}_4$, the estimated amplitude $\hat{A}_4$, phase $\hat{\phi}_i$ and offset $\hat{A}_0$ can be calculated in terms of $\hat{A}_i = \sqrt{a_i^2 + b_i^2}$ and $\hat{\phi}_i = \tan^{-1}(b_i/a_i)$.

**Remark 4.** For multiple sinusoidal signal (42), the robustness of the proposed estimation methods (51), (52), (56) and (57) against bounded measurement noises in the output can be studied following the analysis in Section 5, which will not be presented again due to the limited space.

7. Simulations

**Example 1.** We first verify the parameter estimation for a sinusoidal signal $y = 2 + 3 \sin(4t + \pi/4)$ with unknown parameters $A_0 = 2, A = 3, \omega = 4, \phi = \pi/4$. The adaptive law (16) (or (32)) with initial condition $\theta_i(0) = [0, 0, 0]^T$ and learning gain $\Gamma_1 = \text{diag}(50, 500, 500)$ is used to estimate the offset $A_0$ and frequency $\omega$. Then the adaptive law (29) (or (34)) with initial condition $\theta_4(0) = [0, 0, 0]^T$ and learning gain $\Gamma_2 = \text{diag}(10, 10, 10)$ is implemented to estimate the amplitude $A$ and phase $\phi$. Other simulation parameters are set as $\lambda_1 = 2.5, \lambda_2 = 5, \ell = 1, \kappa = 0.001$.

Fig. 1 gives the profiles of the estimated parameters, where the offset $A_0$ and frequency $\omega$ are illustrated in Fig. 1(a), and the amplitude...
A and phase $\phi$ are illustrated in Fig. 1(b). For comparison, the gradient based estimation method (Sastry & Bodson, 1989) for linearly parameterized system (3) and (23) are also simulated, where the gains of gradient scheme can be set to make a tradeoff between the convergence speed and transient performance by means of a trial and error method. It is shown that the proposed adaptive laws (16) and (29) can achieve better transient performance than gradient methods for all estimated parameters, i.e. faster convergence speed and less oscillation. In particular, the proposed FT adaptive laws (32) and (34) converge faster than the other two schemes.

To validate the robustness of the proposed methods, an additive Gaussian noise $v \sim N(0, 0.5)$ with zero mean and 0.5 variance is introduced in the sinusoid signal. Fig. 2 shows the responses of the estimated parameters with the proposed methods and gradient scheme. It is shown that the estimates of frequency, offset, amplitude and phase all converge to a bounded set around their true values even with the noise $v \sim N(0, 0.5)$. However, it is shown in Hou (2012) that the proposed parameter identifier cannot converge to their true values when the noise variance is at the level of 0.01$^2$ or above. This means the proposed methods in this paper are more robust against noises. This may be due to the fact that the output derivative $\dot{y}$ is not used in this paper, and on the other hand, the introduction of filter operations may suppress the effects of noises.

**Example 2.** The parameter estimation of multi-sinusoid signal $y = 2 + 3 \sin(4t + \pi/4) + \sin(t + \pi/6)$ is further tested, where the unknown parameters to be estimated are $A_0 = 2$, $A_1 = 3$, $\omega_1 = 4$, $\phi_1 = \pi/4$ and $A_2 = 1$, $\omega_2 = 1$, $\phi_2 = \pi/6$. The adaptive law (51) (or (52)) is used to estimate the frequencies $\omega_i$. In this case, the filter polynomial $\lambda(s) = s^4 + 8s^3 + 24s^2 + 32s + 6$ is adopted and the simulation parameters and initial condition are selected as $\Gamma_3 = 100 \text{diag}([500, 500, 500, 500, 500], \ell = 1, \kappa = 0.001$ and $\hat{\Theta}_3(0) = [0, 0, 0, 0]$. Then in the second step, adaptive law (56) (or (57)) is used to estimate the amplitude $A_i$, phase $\phi_i$ and offset $A_0$ with the initial condition and the gain as $\hat{\Theta}_4(0) = [0, 0, 0, 0]$, $\Gamma_4 = \text{diag}([10, 10, 10, 10])$. Simulation results are shown in Fig. 3. It can be observed that the proposed estimation schemes work well, and all parameters of multi-sinusoids can be precisely estimated. It is also worthy pointing out that the FT estimations (52) and (57) allow for faster convergence speed.

**8. Conclusions**

This paper addresses adaptive parameter estimation for sinusoidal signals with unknown frequency, amplitude, phase and off-
set. A constructive two-step estimation framework is introduced to estimate all unknown parameters, where only the output measurement is used. By representing the sinusoid signal as a linearly parameterized form, novel adaptive laws are developed, where appropriate parameter estimation error information is explicitly derived and used, so that the proposed methods are independent of observer/predictor synthesis. The idea is then extended to estimate the parameters of multi-sinusoids. Finite-time estimation is also studied by incorporating the sliding-mode methodology into the design of adaptive laws. Globally exponential and finite-time error convergence are achieved and the robustness to bounded noises is also proved. Comparative simulation results exemplify the proposed methods.

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References


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