Adaptive Backstepping Fuzzy Control for Nonlinearly Parameterized Systems With Periodic Disturbances

Weisheng Chen, Member, IEEE, Licheng Jiao, Senior Member, IEEE, Ruihong Li, and Jing Li

Abstract—A novel-function approximator is constructed by combining a fuzzy-logic system with a Fourier series expansion in order to model unknown periodically disturbed system functions. Then, an adaptive backstepping tracking-control scheme is developed, where the dynamic-surface-control approach is used to solve the problem of "explosion of complexity" in the backstepping design procedure, and the time-varying parameter-dependent integral Lyapunov function is used to analyze the stability of the closed-loop system. The semiglobal uniform ultimate boundedness of all closed-loop signals is guaranteed, and the tracking error is proved to converge to a small residual set around the origin. Two simulation examples are provided to illustrate the effectiveness of the control scheme designed in this paper.

Index Terms—Dynamic surface control, Fourier series expansion (FSE), fuzzy-logic system (FLS), integral Lyapunov function (ILF), nonlinearly parameterized system, periodic disturbance.

I. INTRODUCTION

OVER THE PAST two decades, much progress has been made in the field of fuzzy control. A survey paper on analysis and design of fuzzy-control systems has been found in [1]. In particular, recently, an adaptive backstepping technique is combined with a fuzzy-logic system (FLS) to develop the so-called adaptive backstepping fuzzy-control (ABFC) approaches, which are especially useful to solve the control problem of lower triangular-structured systems with unknown and mismatched system functions. In fact, the basic idea of ABFC is identical to adaptive backstepping neural-network (NN) control (ABNNC) (see, e.g., [2]–[6], just to name a few), i.e., using approximators to estimate the mismatched uncertainties appearing in systems or controllers online. However, compared with NNs, the main advantage of FLS is that it can combine some experience and knowledge from designers or experts [7]. These experience and knowledge can initiate the estimated parameters in order to make them close to their optimal values, which is very important to improve the control transient performance, especially in the initial phase of the control action. Therefore, this paper focuses on ABFC. ABFC was first proposed in [8] to deal with the tracking problem for a class of strict-feedback systems with $H_{\infty}$ tracking performance and then applied to more general strict-feedback systems [9] and output-feedback ones [10]. Chen and Liu [11] addressed the fuzzy-approximate disturbance-decoupling problem of multi-input–multi-output (MIMO) nonlinear systems by backstepping. Yang et al. [12]–[14] and Ho et al. [15] presented several indirect ABFC schemes by combing the backstepping technique with the small-gain approach, where the controllers contain less-adaptive parameters. Recently, a direct ABFC scheme has been proposed by combining the modified integral Lyapunov function (ILF) with the backstepping technique [16], and the ABFC scheme was further extended to the time-delay systems [17], [18] and the MIMO systems [19]. Very recently, Chen and Zhang [36] proposed a globally stable ABFC scheme for a class of output-feedback nonlinear systems with an unknown high-frequency-gain sign. However, in these ABFC schemes, the FLS is used to approximate the unknown system functions depending only on system states or outputs. When the unmeasured disturbances appear nonlinearly in unknown system functions, all existing ABFC schemes are invalid due to the fact that the disturbances can destroy the universal approximation property of FLS.

Based on the above discussion, this paper will investigate the tracking-control problem of a class of strict-feedback systems in which the unmeasured time-dependent periodic disturbances appear nonlinearly in unknown system functions. The system dynamics is described by the following controllable canonical form:

$$\begin{align*}
\dot{x}_i &= g_i(\bar{x}, \omega_i(t))x_{i+1} + f_i(\bar{x}, \omega_i(t)) \\
&= i = 1, \ldots, n - 1 \\
\dot{x}_n &= g_n(\bar{x}_n, \omega_n(t))u + f_n(\bar{x}_n, \omega_n(t)) \\
y &= x_1
\end{align*}$$

where $\bar{x} = [x_1, \ldots, x_n]^T \in \mathbb{R}^n (1 \leq i \leq n)$; $x = \bar{x}_n \in \mathbb{R}^m$, $y \in \mathbb{R}$, and $u \in \mathbb{R}$ are the system-state vector, output, and control input; $\omega_i(t) : [0, +\infty) \to \mathbb{R}^{m_i}$ ($1 \leq i \leq n$) are unknown, time-varying disturbances with known periods $T_i$ and dimensions $m_i$, i.e., $\omega_i(t + T_i) = \omega_i(t)$; and $f_i : \mathbb{R}^{n+m_i} \to \mathbb{R}$, and $g_i : \mathbb{R}^{n+m_i} \to \mathbb{R} (1 \leq i \leq n)$ are unknown smooth functions.
Compared with the existing work in the ABFC field [8]–[19], the main property of system (1) is that the unmeasured periodic disturbances $\omega(t)$ appear in the unknown system functions in a nonlinear fashion. This is also the main difficulty that has been solved in this paper. However, why do we consider only periodically time-varying disturbances instead of more general ones? Main reasons lie in the following:

1) As pointed out in [20], periodic disturbances often exist in many mechanical control systems such as industrial robots and numerical control machines or disturbances depending on the frequency of the power supply. Very recently, Tomizuka [21] presented some fundamental issues and new challenges to deal with periodic disturbances along with applications to mechanical systems. In addition, there are, indeed, some physical systems that can be described by the model (1) [14], [22].

2) Theoretically, it would be extremely difficult to find a suitable method to solve the tracking problem of system (1) with more general time-varying nonlinearly parameterized disturbances. As stated in [23], a more realistic way is to first classify the disturbances into subsets, e.g., periodic disturbances versus nonperiodic ones, and then look for an appropriate feasible method for each subclass.

3) In fact, much work on the rejection or estimate of periodic disturbances (or parameters) has been widely reported, e.g., see [20]–[28], where, however, the periodic disturbances were allowed to enter the controlled systems only linearly instead of nonlinearly.

It is emphasized that periodic disturbances are generally classified into two subsets: state-dependent periodic disturbances and time-dependent periodic ones. The former often occur in mechanical systems due to internal mechanical oscillations, such as rotary dynamic systems [29], and these kinds of disturbances can be denoted by $\omega(x)$ satisfying $\omega(x + T) = \omega(x)$, where $T > 0$ is the period, and $x$ is the system state. The latter usually exist in some physical systems due to some external disturbances (or parameters) has been widely reported, and $\omega$ is state-dependent, i.e., $\omega_1(t) = \omega_1(x_t)$, then the unknown functions $f_1(\bar{x}_t, \omega(t))$ and $g_1(\bar{x}_t, \omega(t))$ will become the functions only of the system state $\bar{x}_t$, i.e., $f_1(\bar{x}_t, \omega(t)) = f_1(\bar{x}_t)$, and $g_1(\bar{x}_t, \omega(t)) = g_1(\bar{x}_t)$.

In this case, some existing ABFC approaches [8]–[19] can be directly applied to solve the control problem of system (1). Therefore, in this paper, we will focus on the time-dependent periodic disturbances.

From the aforementioned discussion, it can be seen that the tracking problem of system (1) has theoretical and practical importance. The main obstacle is how to deal with the unknown system functions affected by periodic disturbances in a nonlinear fashion. To encounter this obstacle, in our previous work, we proposed two new approximators by combining Fourier series expansion (FSE) and NNs [30]; then, both were used for ABNNC in [31] and [32], respectively. However, it is well known that NNs cannot utilize some of the experience and knowledge from the designer and expertise, but FLS can. Therefore, in this paper, we combine FSE with FLS to establish a new

FSE–FLS-based approximator to model each suitable disturbed uncertainty, where FSE is used to estimate the time-varying disturbances, and then, the estimated values are further used as the FLS inputs to approximate the unknown disturbed system functions, which is different from all existing fuzzy approximators that are employed to only model the disturbance-independent functions [8]–[19]. The main advantage of FSE–FLS-based approximator is its good capability to compensate for the nonlinearly parameterized periodic disturbances due to the introduction of FSE. Furthermore, based on the proposed FSE–FLS-based approximator, we develop a semiglobally stable ABFC scheme using the dynamic-surface-control (DSC) approach and the ILF technique, where the DSC approach is used to solve the problem of “explosion of complexity” in the backstepping design procedure, and a new time-varying parameter-dependent ILF is used to analyze the stability of the closed-loop system.

The remainder of this paper is organized as follows. Section II gives preliminaries, problem formulations, and the FSE–FLS-based approximator. In Section III, we present the design procedure of ABFC algorithm. Section IV gives the stability analysis of closed-loop systems and the main result of this paper. In Section V, two simulation examples are provided to illustrate the effectiveness of the proposed control scheme. In Section VI, we conclude the work of this paper.

II. PRELIMINARIES, PROBLEM FORMULATION, AND FOURIER-SERIES-EXPANSION–FUZZY-LOGIC-SYSTEM-BASED APPROXIMATOR

A. Preliminaries

The following notations will be used throughout this paper. $I_m$ denotes the $m$-dimensional unit matrix. $\text{Tr}(\cdot)$ represents the trace operator. For a given matrix $A$, $A^T$ denotes its transpose. $\|\cdot\|$ denotes Euclidean norm of a vector or its induced matrix norm, $\|B\|_F$ denotes the Frobenius norm, i.e., for a given matrix $B = [b_{ij}] \in \mathbb{R}^{m \times n}$, $\|B\|_F = \sqrt{\text{Tr}(B^TB)}$, and $[\xi_i] = \sum_{i=1}^n |\xi_i|$ with $\xi = [\xi_1, \xi_2, \ldots, \xi_n]^T \in \mathbb{R}^n$, and $\lambda_{\max}(C)$ and $\lambda_{\min}(C)$ denote the largest and smallest eigenvalues of a square matrix $C$, respectively.

**Definition I** [35]: The time-varying vector $\chi(t)$ is semiglobally uniformly ultimately bounded (SGUUB), if for any $\Omega$, a compact subset of $\mathbb{R}^n$ and all $\chi(t_0) \in \Omega$, there exists an $\varepsilon > 0$ and a number $T(\varepsilon, \chi(t_0))$ such that $\|\chi(t)\| < \varepsilon$ for all $t \geq t_0 + T$.

**Lemma I** [33]: Let us suppose that function $\hat{V}(t) \geq 0$ is a differentiable function defined for $t \geq 0$. If $\hat{V}(t) \leq -\gamma\hat{V}(t) + \kappa$, where $\gamma$ and $\kappa$ are positive constants; then

$$\hat{V}(t) \leq \left(\hat{V}(0) - \frac{\kappa}{\gamma}\right) e^{-\gamma t} + \frac{\kappa}{\gamma}.$$ 

Now, we introduce an FLS comprising a static-system mapping from $U \subset \mathbb{R}^m$ to $\mathbb{R}^n$. The fuzzy IF–THEN rule is written as

$$R^{(i)}: \text{if } x_1 \text{ is } F_1^i \text{ and } \ldots \text{ and } x_n \text{ is } F_n^i, \text{ then } y \text{ is } G^i,$$

where $F_1^i$ and $G^i$ are fuzzy sets with the membership functions $\mu_{F_1^i}(x_1)$ and $\mu_{G^i}(y)$, respectively. The FLS with a
Lemma 2 means that for arbitrary small $g \in \mathbb{R}^l$, where $m$ is the number of fuzzy rules, $x = [x_1, x_2, \ldots, x_n]^T$, and $\vec{g}$ is the point at which $\mu_{F_i}(\vec{g}) = 1$. In (2), the fuzzy-membership functions $\mu_{F_i}(x_i)$ are commonly chosen as the Gaussian functions with the form

$$\mu_{F_i}(x_i) = \exp \left[ -\left( \frac{x_i - a_i^l}{b_i^l} \right)^2 \right]$$

where $a_i^l$ and $b_i^l$ denote the centers and widths of $\mu_{F_i}(x_i)$, respectively. If we view $\vec{g}^T, a_i^l$, and $b_i^l$ as adjustable parameters, then the fuzzy-membership functions $\mu_{F_i}(x_i)$ can be further written as

$$\mu_{F_i}(x_i) = \exp \left[ -\left( A_i^l Z_i \right)^T \right]$$

where $A_i^l = [1/b_i^l, -a_i^l/b_i^l]$ is a vector of adjustable parameters; $Z_i = [x_i, 1]^T$ is an unknown vector-valued function. Equation (2) can be also rewritten as

$$o(x) = W^T S(A^T Z)$$

where $W = [\vec{g}^T, \vec{g}^T, \ldots, \vec{g}^T]^T$ is a vector of adjustable parameters; $Z = [x^T, 1]^T$ is a vector-valued function; $A^T = [A_1^T, A_2^T, \ldots, A_m^T]^T$ is a matrix of adjustable parameters with

$$(A_i^l)^T = \begin{bmatrix} 1/b_i^l & 0 & \cdots & 0 & -a_i^l/b_i^l \\ 0 & 1/b_i^l & \cdots & 0 & -a_i^l/b_i^l \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1/b_i^l & -a_i^l/b_i^l \end{bmatrix}, \quad l = 1, \ldots, m$$

and

$$S(A^T Z) = [\varphi_1(A^T Z), \varphi_2(A^T Z), \ldots, \varphi_m(A^T Z)]^T$$

is a vector-valued function with $\varphi_i(A^T Z)$ defined as

$$\varphi_i(A^T Z) = \frac{\prod_{l=1}^m \mu_{F_l}(x_i)}{\sum_{i=1}^m \prod_{l=1}^m \mu_{F_l}(x_i)} \quad l = 1, \ldots, m.$$  

The following lemma shows the universal approximation property of FLS (2) or (3).

**Lemma 2 [7]:** For any given real continuous function $g(x)$ on a compact set $U \subset \mathbb{R}^n$ and arbitrary $\varepsilon' > 0$, there exists an FLS $o(x)$ in the form of (3) such that

$$\sup_{x \in U} |o(x) - g(x)| < \varepsilon'.$$

**Remark 1:** Lemma 2 means that for arbitrary small approximation-error bound $\varepsilon' > 0$, there must exist suitable ideal parameters of FLS (3), i.e., the centers $a_i^l$, the widths $b_i^l$, and the number of fuzzy rules $l$, such that $\sup_{x \in U} |o(x) - g(x)| < \varepsilon'$. It must be pointed out that if the number of fuzzy rules $l$ is fixed, then the approximation error cannot be made arbitrarily small only by tuning $a_i^l$ and $b_i^l$. To decrease the approximation error, we must obtain as many fuzzy rules as possible, i.e., increasing $l$ is helpful in decreasing the approximation error, which is very similar to NNs [35].

### B. Problem Formulation

The control objective of this paper is formulated as follows. For a given reference signal $y_r(t)$, find a dynamic adaptive control law with the form

$$u = u(x, y_r, \dot{y}_r, \dot{\chi}, t)$$

$$\dot{\chi} = \nu(x, y_r, \dot{y}_r, \dot{\chi}, t)$$

where $\dot{\chi}$ usually denotes the filter signals and the estimates of unknown parameters, such that, while maintaining all the closed-loop signals SGUUB, the output tracking error $y(t) - y_r(t)$ satisfies

$$\lim_{t \to \infty} |y(t) - y_r(t)| \leq \varpi$$

where $\varpi \geq 0$ is a constant, which can be designed as small as desired.

The following assumptions on system (1) are used to achieve the desired control objective described earlier.

**Assumption 1:** The signs of $g_i(x_t, \omega_t(t))$, $i = 1, \ldots, n$, are known, and there exist constants $g_{00} > 0$ and known smooth functions $g_i(x_t)$ such that $g_{00} \leq g_i(x_t, \omega_t) \leq \tilde{g}_i(x_t) \forall x_t \in \mathbb{R}^n$. 

**Assumption 2:** The reference signal $y_r(t)$, as well as $\dot{y}_r(t)$ and $\ddot{y}_r(t)$, are continuous and bounded.

**Remark 2:** Assumption 1 is widely employed in approximator-based adaptive backstepping control schemes (see, e.g., [3] and [23]). This assumption implies that smooth functions $g_i(x_t, \omega_t(t))$ are strictly either positive or negative. Without losing generality, it is assumed that $0 < g_{00} \leq g_i(x_t, \omega_t(t)) \leq \tilde{g}_i(x_t)$. Assumption 2 is standard in the DSC design [4]. It should be mentioned that $\dot{y}_r(t)$ will be used for stability analysis in DSC design but not for control design.

### C. Fourier-Series-Expansion–Fuzzy-Logic-System-Based Approximator

In this paper, the main design obstacle is that the unknown periodic disturbances $\omega_t(t)$ cannot be used as the FLS inputs. Let us consider the periodic property of $\omega_t(t)$. We first employ FSE to estimate $\omega_t(t)$ and then utilize the measured system signals $\chi_t$ and the estimated values of $\omega_t$ as the FLS inputs to approximate some suitable unknown functions $h_t(\chi_t, \omega_t(t))$.

Without loss of generality, we consider an unknown function $h_t(\chi, \omega(t))$, where $\chi \in \Omega_\chi \subset \mathbb{R}^n$ is a measured signal, with $\Omega_\chi$ being a compact set, and $\omega(t) = [\vartheta_1(t), \ldots, \vartheta_m(t)]^T \in \Omega_\omega \subset \mathbb{R}^m$ is an unknown continuous disturbance of known period $T$, with $\Omega_\omega$ being a compact set. On the one hand, the continuous and periodic disturbance vector $\omega(t)$ can also be expressed by a linearly parameterized FSE as follows [25]:

$$\omega(t) = B^T \phi(t) + \delta_\omega(t), \quad ||\delta_\omega(t)|| \leq \delta_\omega$$  

(4)
where $B = [b_1, \ldots, b_m] \in R^{q \times m}$ is a constant matrix with $v_i \in R^n$ being a vector consisting of the first $q$ coefficients of the FSE of $\bar{\nu}_i(t)$ \ ((where $q$ is an odd integer), $\delta_i(t)$ is the truncation error with the minimum upper bound $\bar{\delta}_i > 0$, which can be arbitrarily decreased by increasing $q$, and

$$\phi(t) = [\phi_1(t), \ldots, \phi_q(t)]^T$$

with

$$\phi_1(t) = 1$$
$$\phi_{2j}(t) = \sqrt{2} \sin(2\pi j t / T)$$
$$\phi_{2j+1}(t) = \sqrt{2} \cos(2\pi j t / T), \quad j = 1, \ldots, (q - 1)/2$$

whose derivatives up to $n$th order are smooth and bounded.

On the other hand, if $\omega(t)$ is measured, the unknown continuous function $h(\chi, \omega(t))$ can be approximated over the compact set $\Omega = \Omega_1 \times \Omega_\omega$ by the FLS (3), i.e.,

$$h(\chi, \omega(t)) = W^T S(\hat{A}^T Z) + \hat{\delta}_h(\chi, \omega(t))$$

where $Z = [\chi^T, \omega^T(t), 1]^T \in R^{t+m+1}$, and $\hat{\delta}_h(\chi, \omega(t))$ is the approximation error with the minimum upper bound $\bar{\delta}_h > 0$, which can be decreased by increasing the number of fuzzy rules according to Remark 1.

However, since $\omega(t)$ is unknown, we must design a new function approximator based on (4) and (5). Note that $\hat{A}^T Z$ can be divided into three parts, i.e., $\hat{A}^T Z = A^T \bar{\nu} + A^T \omega(t) + A^T \delta_\omega(t)$, in which by replacing the unknown periodically time-varying disturbance $\omega(t)$ with (4), we have

$$\hat{A}^T Z = A^T \bar{\nu} + A^T \delta_\omega(t) + A^T \omega(t) + A^T \delta_\omega(t)$$

where $V^T = [A^T, A^T B, A^T]$, and $Z(\chi, \phi) = [\chi^T, \phi^T(t), 1]^T$. Substituting (6) into (5) leads to

$$h(\chi, \omega(t)) = W^T S(V^T Z(\chi, \phi) + A^T \delta_\omega(t)) + \hat{\delta}_h(\chi, \omega).$$

(7)

Based on (7), we construct a novel FSE–FLS-based approximator

$$G(\chi, t) = W^T S(V^T Z(\chi, \phi))$$

(8)

to model the unknown function $h(\chi, \omega(t))$ as follows:

$$h(\chi, \omega(t)) = W^T S(V^T Z(\chi, \phi)) + \delta(\chi, t)$$

(9)

where

$$\delta(\chi, t) = \delta_h + W^T S(V^T Z(\chi, \phi) + A^T \delta_\omega(t))$$

$$- W^T S(V^T Z(\chi, \phi))$$

(10)

Remark 3: It can be seen from (9) that if the FLS input $(\chi, \omega(t))$ always remains over the compact set $\Omega = \Omega_1 \times \Omega_\omega$, then the approximation error $\delta(\chi, t)$ is bounded and can be arbitrarily decreased by increasing the values of $p$ and $q$, which implies that the new approximator (8) has good approximation capability. However, once the FLS input is out of the compact set $\Omega$, then the approximation error bound cannot be guaranteed. This is the reason why the stability of closed-loop system obtained in this paper is only semiglobal instead of global. In fact, as pointed out in [5], how to identify this compact set and guarantee the global stability of closed-loop system is an open problem in the field of fuzzy control or NN control. In this paper, for convenience of stability analysis, we keep this open problem as the work to be investigated in future and still assume the FLS input always remains over some suitable compact set. Thus, the approximation error is always bounded.

Based on Remark 3, we still assume $|\delta(\chi, t)| \leq \bar{\delta}$, where $\bar{\delta}$ denotes the minimum upper bound of $\delta(\chi, t)$. In general, the parameters $W$ and $V$ are unknown and need to be estimated in the controller design. Let $\bar{W}$ and $\bar{V}$ be the estimates of $W$ and $V$, respectively, and the weight-estimation errors are $\bar{W} = W - \bar{W}$ and $\bar{V} = V - \bar{V}$.

**Lemma 3:** For the FSE–FLS-based approximator (8), the estimation error can be expressed as

$$W^T S(V^T Z(\chi, \phi)) - W^T S(V^T Z(\chi, \phi))$$

$$= W^T (S - S'V^T Z(\chi, \phi)) + W^T S'V^T Z(\chi, \phi) + d$$

(11)

where

$$\hat{S} = S(V^T Z(\chi, \phi))$$

$$\tilde{S}' = [\tilde{\varphi}_1', \tilde{\varphi}_2', \ldots, \tilde{\varphi}_p'] \in R^{m \times p}$$

with

$$\tilde{\varphi}_i' = \frac{\partial \varphi_i(\omega)}{\partial \omega} |_{\omega = \omega^T Z(\chi, \phi)}, \quad i = 1, \ldots, p$$

and the residual term $d$ is bounded by

$$|d| \leq \|V\|F \|\bar{Z}(\chi, \phi)\|W'\| + \|W\| \|S'V^T Z(\chi, \phi)\| + |W|_r.$$  

(12)

**Proof:** The proof is similar to that of [34, Lemma 3.1], and is omitted here.

### III. ADAPTIVE DYNAMIC–SURFACE–CONTROL DESIGN BASED ON INTEGRAL LYAPUNOV FUNCTION

In this section, we present the design steps for system (1) by combining the DSC approach [4] with the ILF technique [3], where the DSC approach is used to solve the problem of “explosion of complexity” in the backstepping design procedure, and the ILF technique is used to analyze the stability of closed-loop system; however, the FSE–FLS-based approximator (8) is utilized to approximate some suitable unknown periodically time-varying and nonlinearly parameterized functions $h_1(\chi_i, \omega_i(t))$, i.e.,

$$h_1(\chi_i, \omega_i(t)) = W_i^T S_i(V_i^T \bar{Z}_i(\chi_i, \phi)) + \delta_i(\chi_i, t)|\delta_i(\chi_i, t)| \leq \bar{\delta}_i$$

(13)

where $(\chi_i, \omega_i) \in \Omega'$ with $\Omega'$ being compact sets, and the additional inherent approximation errors, and the estimation errors will be considered.

**Step 1:** Let us define $z_1 = y - y_r$. From the first equation in (1) for $i = 1$, we have

$$\dot{z}_1 = g_1(x_1, \omega_1(t))x_2 + f_1(x_1, \omega_1(t)) - \dot{y}_r.$$  

(14)
Let us define $\beta_1(x, y, t) = \bar{g}_i(x)/g_i(x, y, t)$ and an ILF

$$V_{z1} = \int_0^{z_1} \sigma \beta_1(\sigma + y_r, \omega_1(t))d\sigma.$$  \hfill (15)

By changing the variable $\sigma = \tau z_1$ and using Assumption 1, we may rewrite $V_{z1}$ as

$$V_{z1} = z_1^2 \int_0^{1} \tau \beta_1(\tau z_1 + y_r, \omega_1(t))d\tau.$$  \hfill (16)

Noting that

$$1 \leq \beta_1(\tau z_1 + y_r, \omega_1(t)) \leq \frac{\bar{g}_i(\tau z_1 + y_r)}{g_{10}}$$

we have

$$\frac{z_1^2}{g_{10}} \leq V_{z1} \leq \frac{z_1^2}{g_{10}} \int_0^{1} \tau \bar{g}_i(\tau z_1 + y_r)d\tau$$

which implies that $V_{z1}$ is a positive-definite function with respect to $z_1$. Then, the time derivative of $V_{z1}$ can be expressed as

$$V_{z1} = z_1 \beta_1(x_1, \omega_1(t))x_1 + \bar{g}_i \int_0^{z_1} \frac{\partial \beta_1(\sigma + y_r, \omega_1)}{\partial \sigma} d\sigma$$

$$+ \int_0^{z_1} \frac{\partial \beta_1(\sigma + y_r, \omega_1)}{\partial \omega_1} \omega_1 d\sigma$$

$$= z_1 \beta_1(x_1, \omega_1(t))\left[ g_i(x_1, \omega_1(t))x_2 + f_i(x_1, \omega_1(t)) - \bar{g}_i \right]$$

$$+ \bar{g}_i \left[ \sigma \beta_1(\sigma + y_r, \omega_1) \right]^{z_1}_0 - \int_0^{z_1} \beta_1(\sigma + y_r, \omega_1)d\sigma$$

$$+ \int_0^{z_1} \frac{\partial \beta_1(\sigma + y_r, \omega_1)}{\partial \omega_1} \omega_1 d\sigma$$

$$= z_1 \left[ g_i(x_1)x_2 + \beta_1(x_1, \omega_1)f_i(x_1, \omega_1) \right.$$

$$- \bar{g}_i \int_0^{1} \beta_1(\tau z_1 + y_r, \omega_1)d\tau$$

$$+ \int_0^{1} \frac{\partial \beta_1(\tau z_1 + y_r, \omega_1)}{\partial \omega_1(t)} \omega_1 d\tau \right]$$

$$= z_1 \left[ g_i(x_1)x_2 + h_1(x_1, \omega_1(t)) \right]$$  \hfill (17)

where $\chi_1 = [x_1, y_r, \bar{g}_j, \omega_1]^T$, $\chi_1(t) = [\omega_1^T(t), \omega_1^T(t)]^T$, and $h_1(x_1, \omega_1(t))$

$$= \beta_1(x_1, \omega_1(t))f_i(x_1, \omega_1(t)) - \bar{g}_i \int_0^{1} \beta_1(\tau z_1 + y_r, \omega_1(t))d\tau$$

$$+ \int_0^{1} \frac{\partial \beta_1(\tau z_1 + y_r, \omega_1(t))}{\partial \omega_1(t)} \omega_1(t)d\tau.$$  \hfill (18)

The first virtual control $\alpha_3$ is designed as

$$\alpha_3 = \frac{1}{g_i(x_1)} \left[ - k_1(t)z_1 - \bar{W}_i^T S_1(\bar{W}_i^T \bar{Z}_1(\chi_1, \phi)) \right]$$  \hfill (19)

where the FSE-FLS-based approximator $\bar{W}_i^T S_1(\bar{W}_i^T \bar{Z}_1(\chi_1, \phi))$ is introduced to approximate $h_1(x, \omega_1(t))$, the time-varying gain $k_1(t)$ is designed as

$$k_1(t) = \frac{\bar{g}_i(x_1)}{2} + \frac{1}{\epsilon_1} \left( 1 + \int_0^{1} \tau \bar{g}_i(\tau z_1 + y_r)d\tau \right.$$

$$+ \| \bar{Z}_1(\chi_1, \phi) \bar{W}_i^T S_1^2(\bar{W}_i^T \bar{Z}_1(\chi_1, \phi)) \|^2 \right)$$  \hfill (20)

where constant $\epsilon_1 > 0$, and the unknown parameter vectors are updated by

$$\begin{align*}
\dot{\bar{W}}_i &= \Gamma_{w1}[\bar{S}_1 - \bar{S}_1^T \bar{W}_i^T \bar{Z}_1(\chi_1, \phi)]z_1 - \sigma_{w1}\bar{W}_i \\
\dot{\bar{V}}_i &= \Gamma_{v1}[\bar{Z}_1(\chi_1, \phi) \bar{W}_i^T S_1^2(\bar{W}_i^T \bar{Z}_1(\chi_1, \phi))] + d_1 - 1.
\end{align*}$$  \hfill (21)

Let us introduce a new state variable $\zeta_2$, and let $\alpha_1$ pass through a first-order filter $\zeta_2$ with time constant $\tau_2$ to obtain $\zeta_2$

$$\tau_2 \frac{d\zeta_2}{dt} = -\zeta_2 + \alpha_1, \quad \zeta_2(0) = \alpha_1(0).$$  \hfill (24)

Let us define $z_i = x_i - \zeta_i$. From $i$th equation in (1), we obtain

$$\dot{z}_i = g_i(\bar{x}_i, \omega_i(t))\bar{x}_{i+1} + f_i(\bar{x}_i, \omega_i(t)) - \zeta_i.$$  \hfill (25)

Let us define

$$\beta_i(\bar{x}_i, \omega_i(t)) = \frac{\bar{g}_i(\bar{x}_i)}{g_i(\bar{x}_i, \omega_i(t))}$$

and the following ILF:

$$V_{z1} = \int_0^{z_1} \sigma \beta_1(\bar{x}_i, \sigma + \zeta_i, \omega_i(t))d\sigma.$$  \hfill (26)

Then, similar to the derivation in (17), the time derivative of $V_{z1}$ can be expressed as

$$\dot{V}_{z1} = z_1 \left[ g_i(\bar{x}_i)\bar{x}_{i+1} + h_i(\bar{x}_i, \omega_i(t)) \right]$$  \hfill (27)

where $\chi_i = [\bar{x}_i, \zeta_i, \bar{z}_i]^T$, $\omega_i(t) = [\omega_i^T(t), \omega_i^T(t), \omega_i^T(t)]^T$, and

$$h_i(\bar{x}_i, \omega_i(t))$$

$$= \beta_i(\bar{x}_i, \omega_i(t))f_i(\bar{x}_i, \omega_i(t))$$

$$- \zeta_i \int_0^{1} \frac{\partial \beta_i(\bar{x}_i, \tau z_1 + \zeta_i, \omega_i(t))}{\partial \omega_i(t)} d\tau.$$
\[ + z_i \int_0^1 \tau \frac{\partial \beta_i(\bar{x}_{i-1}, \tau z_i + \bar{\zeta}_i, \omega_i(t))}{\partial \bar{x}_{i-1}} \bar{\dot{x}}_{i-1} d\tau \]
\[ + z_i \int_0^1 \tau \frac{\partial \beta_i(\bar{x}_{i-1}, \tau z_i + \bar{\zeta}_i, \omega_i(t))}{\partial \omega_i(t)} \dot{\omega}_i(t) d\tau. \tag{28} \]

Let us design the following virtual control:
\[ \alpha_i = \frac{1}{\bar{g}_i(\bar{x}_i)} \left( -\bar{g}_i(\bar{x}_{i-1}) z_{i-1} - k_i(t) z_i - \bar{W}_i^T S_i(\bar{V}_i^T \bar{Z}_i(\chi_i, \phi)) \right) \]
\[ - \bar{W}_i^T S_i(\bar{V}_i^T \bar{Z}_i(\chi_i, \phi)) \]
\[ + ||\bar{Z}_i(\chi_i, \phi) \bar{W}_i^T \bar{S}_i^T||_F^2 + ||\bar{S}_i^T \bar{V}_i \bar{Z}_i(\chi_i, \phi)||^2 \tag{29} \]

with constant \( \varepsilon_i > 0 \), and the unknown parameter vectors are updated by
\[ \begin{align*}
\dot{\bar{W}}_i &= \Gamma_{wi}[\bar{S}_i - \bar{S}_i^T \bar{V}_i \bar{Z}_i(\chi_i, \phi)] z_i - \sigma_{wi} \bar{W}_i \\
\dot{V}_i &= \Gamma_{vi}[\bar{S}_i \bar{Z}_i(\chi_i, \phi) \bar{W}_i^T \bar{S}_i^T - \sigma_{vi} \bar{V}_i]
\end{align*} \tag{31} \]

with \( \Gamma_{wi} = \Gamma_{wi}^T > 0 \), \( \Gamma_{vi} = \Gamma_{vi}^T > 0 \), \( \sigma_{wi} > 0 \), and \( \sigma_{vi} > 0 \). Similar to the derivation of (22), substituting (29) into (27) yields
\[ \bar{V}_{zi} = -\bar{g}_i(\bar{x}_{i-1}) z_{i-1} z_i - k_i(t) z_i^2 - \Psi_i z_i + z_i \bar{g}_i(\bar{x}_i)(x_{i+1} - \alpha_i) \tag{32} \]

where
\[ \Psi_i = \bar{W}_i^T S_i(\chi_i, \phi) \bar{V}_i \bar{Z}_i(\chi_i, \phi) - h_i(\chi_i, \omega_i(t)) \]
\[ = \bar{W}_i^T (\bar{S}_i - \bar{S}_i^T \bar{V}_i \bar{Z}_i(\chi_i, \phi)) + \bar{W}_i^T \bar{S}_i^T \bar{V}_i \bar{Z}_i(\chi_i, \phi) + d_i - \delta_i. \tag{33} \]

Let us introduce a new state variable \( \zeta_{i+1} \), and let \( \alpha_i \) pass through a first-order filter \( \zeta_{i+1} \) with time constant \( \varepsilon_i \) to obtain \( \zeta_{i+1} \) as follows:
\[ \dot{\zeta}_{i+1} = -\zeta_{i+1} + \alpha_i, \quad \zeta_{i+1}(0) = \alpha_i(0). \tag{34} \]

**Step n:** Let us define \( z_n = x_n - \zeta_n \). By using the last equation in (1), we obtain
\[ \dot{z}_n = g_n(\bar{x}_n, \omega_n(t)) u + f_n(\bar{x}_n, \omega_n(t)) - \dot{\zeta}_n. \tag{35} \]

Let us define
\[ \beta_n(\bar{x}_n, \omega_n(t)) = \frac{g_n(\bar{x}_n)}{g_n(\bar{x}_n, \omega_n(t))} \]
and the nth ILF
\[ V_{zn} = \int_0^{z_n} \sigma \beta_n(\bar{x}_{n-1}, \sigma + \zeta_n, \omega_n(t)) d\sigma. \tag{36} \]

Then, using the derivations similar to the previous steps, the time derivative of \( V_{zn} \) can be expressed as
\[ \dot{V}_{zn} = z_n [\bar{g}_n(\bar{x}_n) u + h_n(\chi_n, \omega_n(t))] \tag{37} \]

**TABLE I**

<table>
<thead>
<tr>
<th>Controller parameters</th>
<th>Criteria of choice</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control gains ( \varepsilon_i )</td>
<td>Positive</td>
</tr>
<tr>
<td>Adaptive gain matrices ( \Gamma_{wi}, \Gamma_{vi} )</td>
<td>Positive definite</td>
</tr>
<tr>
<td>( \sigma )-modification coefficients ( \sigma_{wi}, \sigma_{vi} )</td>
<td>Positive, very small</td>
</tr>
<tr>
<td>Filter parameters ( \varepsilon_i )</td>
<td>Positive, vary small</td>
</tr>
</tbody>
</table>

where \( \chi_n = [\bar{x}_n, \zeta_n, \dot{\zeta}_n]^T \), \( \omega_n(t) = [\omega_1^T(t), \ldots, \omega_n^T(t), \omega_{n+1}^T(t)]^T \), and
\[ h_n(\chi_n, \omega_n(t)) = \beta_n(\bar{x}_n, \omega_n(t)) f_n(\bar{x}_n, \omega_n(t)) \]
\[ - \dot{\zeta}_n \int_0^1 \beta_n(\bar{x}_{n-1}, \tau z_n + \zeta_n, \omega_n(t)) d\tau \]
\[ + z_n \int_0^1 \tau \frac{\partial \beta_n(\bar{x}_{n-1}, \tau z_n + \zeta_n, \omega_n(t))}{\partial \bar{x}_{n-1}} \bar{\dot{x}}_{n-1} d\tau \]
\[ + z_n \int_0^1 \tau \frac{\partial \beta_n(\bar{x}_{n-1}, \tau z_n + \zeta_n, \omega_n(t))}{\partial \omega_i(t)} \dot{\omega}_i(t) d\tau. \tag{38} \]

Let us design the control input \( u \) as
\[ u = -\frac{1}{\bar{g}_n(\bar{x}_n)} [-\bar{g}_i(\bar{x}_{i-1}) z_{i-1} z_i - k_i(t) z_i - \Psi_i z_i + z_i \bar{g}_i(\bar{x}_i)(x_{i+1} - \alpha_i)] \]
\[ - \bar{W}_i^T S_i(\bar{V}_i^T \bar{Z}_i(\chi_i, \phi)) \tag{39} \]

where the FSE-FLS-based approximator \( \bar{W}_n^T S_n(\bar{V}_n^T \bar{Z}_n(\chi_n, \phi)) \) is introduced to approximate \( h_n(\chi_n, \omega_n(t)) \). The gain \( k_n(t) \) is specified as
\[ k_n(t) = \frac{1}{\varepsilon_n} \left( 1 + \int_0^1 \tau \bar{g}_n(\bar{x}_{n-1}, \tau z_n + \zeta_n) d\tau \right) \]
\[ + ||\bar{Z}_n(\chi_n, \phi) \bar{W}_n^T \bar{S}_n^T||_F^2 + ||\bar{S}_n^T \bar{V}_n \bar{Z}_n(\chi_n, \phi)||^2 \tag{40} \]

with constant \( \varepsilon_n > 0 \), and the unknown parameter vectors are updated by
\[ \begin{align*}
\dot{\bar{W}}_n &= \Gamma_{wn}[\bar{S}_n - \bar{S}_n^T \bar{V}_n \bar{Z}_n(\chi_n, \phi)] z_n - \sigma_{wn} \bar{W}_n \\
\dot{V}_n &= \Gamma_{vn}[\bar{S}_n(\chi_n, \phi) \bar{W}_n^T \bar{S}_n^T - \sigma_{vn} \bar{V}_n]
\end{align*} \tag{41} \]

with \( \Gamma_{wn} = \Gamma_{wn}^T > 0 \), \( \Gamma_{vn} = \Gamma_{vn}^T > 0 \), \( \sigma_{wn} > 0 \), and \( \sigma_{vn} > 0 \). Substituting (39) into (37) yields
\[ \dot{V}_{zn} = -\bar{g}_i(\bar{x}_{i-1}) z_{i-1} z_i - k_i(t) z_i^2 - \Psi_z z_n \tag{42} \]

where, similar to the previous steps, the function-estimation error \( \Psi_z \) is expressed as
\[ \Psi_z = \bar{W}_n^T S_n(\bar{V}_n^T \bar{Z}_n(\chi_n, \phi)) - h_n(\chi_n, \omega_n(t)) \]
\[ = \bar{W}_n^T (\bar{S}_n - \bar{S}_n^T \bar{V}_n \bar{Z}_n(\chi_n, \phi)) + \bar{W}_n^T \bar{S}_n^T \bar{V}_n \bar{Z}_n(\chi_n, \phi) + d_n - \delta_n. \tag{43} \]

In the above controller-design process, there are many parameters to be chosen. Similar to some existing literature on the DSC method [4], the criteria of choice of controller parameters is shown in Table I.
IV. Stability Analysis

Noting
\[
x_{i+1} = x_i + \alpha_i = x_{i+1} - \zeta_i + \zeta_i - \alpha_i = z_{i+1} + \varrho_{i+1}, \quad i = 1, \ldots, n - 1
\]  
(44)
here, we denote the filter error signal as
\[
\varrho_{i+1} = \zeta_i - \alpha_i, \quad i = 1, \ldots, n - 1.
\]
According to (24) and (34), \(\varrho_{i+1}\) can be decreased by increasing the design parameter \(t_{i+1}\). Note that
\[
\dot{\varrho}_2 = -\frac{\varrho_2}{\varrho_2} + B_2(z_1, z_2, \varrho_2, \dot{\hat{W}}_1, \ddot{V}_1, \omega_1, \phi, y_r, \ddot{y}_r) \tag{45}
\]
\[
\dot{\varrho}_{i+1} = -\frac{\varrho_{i+1}}{t_{i+1}} + B_{i+1}(\tilde{z}_{i+1}, \tilde{\varrho}_i, \dot{\hat{W}}_i, \dot{V}_i, \tilde{\omega}_1, \phi, y_r, \ddot{y}_r, \ddot{y}_r) \tag{46}
\]
where
\[
\tilde{z}_{i+1} = (z_1, \ldots, z_{i+1}), \quad \tilde{\varrho}_i = (\varrho_2, \ldots, \varrho_i), \quad \tilde{\hat{W}}_i = (\dot{\hat{W}}_1, \ldots, \dot{\hat{W}}_i), \quad \tilde{V}_i = (\dot{V}_1, \ldots, \dot{V}_i), \quad \tilde{\omega}_i = (\omega_1, \ldots, \omega_i),
\]
\[
\begin{align*}
B_2(z_1, z_2, \varrho_2, \dot{\hat{W}}_1, \ddot{V}_1, \omega_1, \phi, y_r, \ddot{y}_r) & = \frac{\partial \alpha_1}{\partial \delta_1} \ddot{z}_1 + \frac{\partial \alpha_1}{\partial y_r} \ddot{y}_r + \frac{\partial \alpha_1}{\partial \hat{W}_1} \dot{\hat{W}}_1 + \\
& + \frac{\partial \alpha_1}{\partial \phi} \dot{\phi}(t) \\
B_{i+1}(\tilde{z}_{i+1}, \tilde{\varrho}_i, \dot{\hat{W}}_i, \dot{V}_i, \tilde{\omega}_1, \phi, y_r, \ddot{y}_r, \ddot{y}_r) & = \frac{\partial \alpha_i}{\partial \delta_i} \ddot{z}_i + \frac{\partial \alpha_i}{\partial z_i} \dot{z}_i + \frac{\partial \alpha_i}{\partial y_i} \dot{y}_i + \frac{\partial \alpha_i}{\partial \hat{W}_i} \dot{\hat{W}}_i + \\
& + \frac{\partial \alpha_i}{\partial \phi} \dot{\phi}(t)
\end{align*}
\]  
(47)
which are continuous functions.

The main results are summarized as follows.

Theorem 1: Under Assumptions 1 and 2, let us consider the closed-loop adaptive system consisting of the plant (1), virtual control functions (19) and (29), filters (24) and (34), and the control law (39) with the adaptive laws (21), (31), and (41). Then, for any initial condition satisfying \(\sum_{j=1}^{n} V_{ij} + (1/2) \sum_{j=1}^{n} \hat{W}_{ij} \Gamma_{w}^{-1} \hat{W}_j + (1/2) \sum_{j=1}^{n} \hat{W}_j \Gamma_{w}^{-1} \hat{W}_j \geq M_0\), where \(M_0\) is any positive constant, there exist \(\epsilon_i, \kappa_i, \sigma_w, \sigma_v, \sigma_{v_i}, \Gamma_{w_i}, \Gamma_{v_i}, p_i, \) and \(g_i, i = 1, \ldots, n\), such that all signals of the closed-loop system are SGUB, and the tracking error \(z_i\) may be made arbitrarily small by adjusting the design parameters in the controller, appropriately.

Proof: Let us consider the following Lyapunov function:
\[
\hat{V} = \sum_{i=1}^{n} V_{zi} + \frac{1}{2} \sum_{i=1}^{n} \left[ \hat{W}_i \Gamma_{w}^{-1} \hat{W}_i + \mathbf{Tr} \left\{ \dot{\hat{W}}_i \Gamma_{w}^{-1} \hat{W}_i \right\} \right] + \frac{1}{2} \sum_{i=1}^{n-1} g_i^2.
\]  
(48)
Noting (44), along the trajectories of (21) and (22), (31) and (32), (41) and (42), (45) and (46), the time derivative of \(\hat{V}\) is given as
\[
\dot{\hat{V}} = -k_1(t) z_i^2 - \Psi_1 z_i + z_i \bar{g}_i(x_i)(z_i + \varrho_2) + \\
+ \sum_{i=2}^{n-1} \left[ -\bar{g}_{i-1}(\bar{x}_{i-1}) z_i z_i - k_{i-1}(t) \right] z_i^2 - \Psi_2 z_i + z_i \bar{g}_i(x_i)(z_i + \varrho_1) + \\
- \bar{g}_n(\bar{x}_n) z_n z_n - k_n(t) z_n^2 - \Psi_n z_n + \\
+ \sum_{i=1}^{n} \left[ \hat{W}_i \Gamma_{w}^{-1} \hat{W}_i + \mathbf{Tr} \left\{ \dot{\hat{W}}_i \Gamma_{w}^{-1} \hat{W}_i \right\} \right]
\]  
(49)
Using (23), (33), and (43), we have
\[
\begin{align*}
\Psi_1 z_i & = \hat{W}_i (\hat{S}_i - \hat{S} \hat{V}_i \hat{S}_i z_i, \phi) z_i + \\
& + \hat{W}_i (\hat{S}_i + \hat{V}_i \hat{S}_i z_i, \phi) z_i + (d_i - \delta_i) z_i.
\end{align*}
\]  
(50)
Substituting (21), (31), (41), and (50) into (49), and noting that
\[
\hat{V}_i \hat{S}_i \hat{V}_i \hat{S}_i z_i, \phi = \mathbf{Tr} \left\{ \dot{\hat{V}}_i \hat{S}_i \hat{V}_i \hat{S}_i z_i, \phi \right\}
\]
we have
\[
\dot{\hat{V}} = - \sum_{i=1}^{n} k_i(t) z_i^2 + (d_i + \delta_i) z_i + \sum_{i=1}^{n-1} z_i \bar{g}_i(x_i) \varrho_{i+1} + \\
+ \sum_{i=1}^{n-1} \left( \frac{\varrho_{i+1}}{\varrho_2} + \varrho_{i+1} B_{i+1} \right) + \\
- \sum_{i=1}^{n} \left[ \sigma_w \hat{W}_i \hat{S}_i \hat{W}_i + \sigma_v + \sigma_v \mathbf{Tr} \left\{ \dot{\hat{W}}_i \hat{S}_i \hat{V}_i \hat{S}_i \right\} \right].
\]  
(51)
Since for any \(B_0 > 0\) and \(M_0 > 0\), the sets \(P = \{ (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5) : \omega_1 \omega_2 + \cdots + \omega_n \omega_n + \Phi_1 + \cdots + \Phi_n \Phi_n + y_1^2 + y_2^2 + w_1^2 \leq B_0 \}\), and \(P = \{ \sum_{i=1}^{n} V_{ij} + (1/2) \sum_{j=1}^{n} \hat{W}_{ij} \Gamma_{w}^{-1} \hat{W}_j - (1/2) \sum_{j=1}^{n} \hat{W}_j \Gamma_{w}^{-1} \hat{W}_j \geq M_0 \}, i = 1, \ldots, n\), are compact, respectively, \(P \times \Pi\) is also compact. Therefore, \(B_{i+1}\) has a maximum \(M_{i+1}\) on \(P \times \Pi\), and then, the following inequalities can be easily derived:
\[
(d_i - \delta_i) z_i \leq \left[ \| \hat{V}_i \| \hat{S}_i \hat{V}_i \hat{S}_i \| \right] \hat{Z}_i (x_i, \phi) \hat{W}_i \hat{S}_i \hat{V}_i \hat{S}_i \| \leq \hat{Z}_i (x_i, \phi) \hat{W}_i \hat{S}_i \hat{V}_i \hat{S}_i \| + \hat{Z}_i (x_i, \phi) \hat{W}_i \hat{S}_i \hat{V}_i \hat{S}_i \| \leq \frac{\hat{Z}_i (x_i, \phi) \hat{W}_i \hat{S}_i \hat{V}_i \hat{S}_i \|}{\hat{Z}_i (x_i, \phi) \hat{W}_i \hat{S}_i \hat{V}_i \hat{S}_i \|} + \frac{\hat{Z}_i (x_i, \phi) \hat{W}_i \hat{S}_i \hat{V}_i \hat{S}_i \|}{\hat{Z}_i (x_i, \phi) \hat{W}_i \hat{S}_i \hat{V}_i \hat{S}_i \|}
\]
\[
\begin{align*}
&+ \frac{z_i^2 \varepsilon_i}{2} \frac{\varepsilon_i}{4} \|V_i\|_F^2 + \frac{\varepsilon_i}{4} \|W_i\|_F^2 + \varepsilon_i |\|W_i\|_F^2 + \delta_i^2| \\
&
i = 1, \ldots, n
\end{align*}
\]

Substituting (20), (30), (40), and (50)–(56) into (51) yields
\[
\dot{V} \leq -n \sum_{i=1}^{n} \left[ \frac{z_i^2}{\varepsilon_i} \left( \frac{1}{2} + \int_0^1 \tau \tilde{g}_i(\tau z_i + \zeta) d\tau \right) \\
+ \frac{\sigma_{\nu i}}{2} \|W_i\|_F^2 + \frac{\sigma_{wi}}{2} \|W_i\|_F^2 \right] \\
+ \sum_{i=1}^{n-1} \left[ -\frac{1}{\ell_{i+1}} + \frac{1}{2} \frac{M^2_{i+1}}{2} \varepsilon_i \right] \varepsilon_i + \nu
\]
where
\[
\nu = \sum_{i=1}^{n} \left[ \frac{\varepsilon_i}{4} \|V_i\|_F^2 + \frac{\varepsilon_i}{4} \|W_i\|_F^2 + \left( \|W_i\|_F^2 + \delta_i^2 \right) \\
+ \|W_i\|_F^2 \right] + \frac{n-1}{2} \zeta.
\]
Choosing
\[
\frac{1}{\ell_{i+1}} = \frac{1}{2} + \frac{1}{2} \frac{M^2_{i+1}}{2} + a_0
\]
and noting that
\[
\frac{z^2}{g_{i0}} \int_0^1 \tau \tilde{g}_i(\bar{x}_{i-1}, \tau z_i + \zeta) d\tau \geq V_{z_i}
\]
which can be easily obtained by the similar derivation of (16), we then have
\[
\dot{V} \leq -\ell \dot{V} + \nu
\]
where
\[
\ell = \min_{1 \leq i \leq n} \left\{ \frac{\varepsilon_i}{\varepsilon_i} \frac{\sigma_{wi}}{\lambda_{\max} (\Gamma^{-1}_{wi})} \right\}
\]
Letting \( \ell > \nu/M \), then \( \dot{V} \leq 0 \) on \( \dot{V} = M \). Thus, \( \dot{V} \leq M \) is an invariant set, i.e., if \( \dot{V}(0) \leq M \), then \( \dot{V}(t) \leq M \) for all \( t \geq 0 \). Thus, (59) holds for all \( \dot{V}(0) < M \) and \( t \geq 0 \).

Inequality (59) implies
\[
0 \leq \dot{V} \leq \frac{\nu}{\ell} + \left( \dot{V}(0) - \frac{\nu}{\ell} \right) e^{-\ell t} \quad \forall t \geq 0.
\]
According to Lemma 1, the above equation means that \( \dot{V}(t) \) is eventually bounded by \( \nu/\ell \). Thus, all signals of the closed-loop system, i.e., \( z_i, W_i, \tilde{V}_i \), and \( g_{i+1} \) are uniformly ultimately bounded. Moreover, by increasing the values of \( 1/\varepsilon_i \) and reducing the values of \( \lambda_{\max} (\Gamma^{-1}_{wi}), \lambda_{\max} (\Gamma^{-1}_{wi}) \), and \( \ell \), i.e., increasing the value of \( \ell \), the quantity \( \nu/\ell \) can be made arbitrarily small. Thus, the tracking error \( z_1 \) may be made arbitrarily small. This concludes the proof.

V. SIMULATION STUDY

In this section, two numerical simulation examples are given to demonstrate the effectiveness of the proposed control method. One is a mathematically constructive system, and the other is a physically well-known system—the van der Pol oscillator.

Example 1: Let us consider the following second-order system:
\[
\begin{align*}
\dot{x}_1 &= [0.8 + 0.2 \cos(x_1 \omega_1(t))]x_2 + \frac{x_1 \omega_1(t)}{x_1^2 \omega_1(t) + 0.5} \\
\dot{x}_2 &= [0.7 + 0.3 \sin(x_1 x_2 \omega_2^2(t))]u \\
&\quad + (x_1 x_2 \omega_2^2(t))^{3} e^{-x_1 x_2 \omega_2^2(t)} \\
y &= x_1
\end{align*}
\]
where the unknown time-varying disturbances \( \omega_1(t) = |\sin(0.5t)| \) and \( \omega_2(t) = |\cos t| \) with known periods \( T_1 = 2\pi \) and \( T_2 = \pi \), respectively. The reference signal is chosen as \( y_r(t) = \sin(t) \). It is easily verified that Assumptions 1 and 2 hold with \( \tilde{g}_1(x_1) = g_2(x_2) = 1 \). Based on the control approach developed in Section III, the virtual control \( \alpha_1 \) is given as
\[
\alpha_1 = -k_1(t) z_1 - \tilde{W}_i^T S_1 (\tilde{V}_i^T \tilde{Z}_1(\chi_1, \phi))
\]
Choosing
\[
k_1(t) = \frac{1}{2} + \frac{1}{\varepsilon_1} \left[ \frac{3}{2} + \left( \tilde{Z}_1(\chi_1, \phi) \tilde{S}_1 \right)^2 \right]
\]
the adaptive laws are given by
\[
\begin{align*}
\dot{\chi}_1 &= y - y_r \\
\chi_1 &= [x_1, y_r, y_r]^T
\end{align*}
\]
and the first-order adaptive filter is designed as
\[
\varepsilon_2 \dot{\zeta}_2 + \zeta_2 = \alpha_1, \quad \zeta_2(0) = \alpha_1(0).
\]
Then, the control law \( u \) is designed as
\[
\begin{align*}
u &= -k_2(t) z_2 - z_2 - \tilde{W}_i^T S_2 (\tilde{V}_i^T Z_2(\chi_2, \phi)) \end{align*}
\]
where
\[
\begin{align*}k_2(t) &= \frac{1}{2} + \frac{1}{\varepsilon_2} \left[ \frac{3}{2} + \left( \tilde{Z}_2(\chi_2, \phi) \tilde{W}_2 \tilde{S}_2 \right)^2 \right] \\
&\quad + \left( \tilde{S}_2 \tilde{V}_i^T Z_2(\chi_2, \phi) \right)^2
\end{align*}
\]
Choosing \( \varepsilon_2 = 1 \) and \( \varepsilon_2 = 1 \), the adaptive controller is designed as
\[
\begin{align*}
\dot{\chi}_2 &= x_2 - \zeta_2 \\
\chi_2 &= [x_1, x_2, \zeta_2]^T
\end{align*}
\]
and the adaptive laws are designed as

\[
\begin{align*}
\dot{\hat{W}}_2 &= \Gamma_{w2}[\{S_2 - \bar{S}_2 V_2^T \bar{Z}_2(\chi_2, \phi)\} s_2 - \sigma_{w2} \hat{W}_2] \\
\dot{\hat{V}}_2 &= \Gamma_{v2}[\bar{Z}_2(\chi_2, \phi) \hat{W}_2^T \bar{S}_2 \bar{s}_2 - \sigma_{v2} \hat{V}_2].
\end{align*}
\] (66)

It is assumed that there exist some fuzzy rules of \(h_1(\chi_1, \omega_1)\) and \(h_2(\chi_2, \omega_2)\), which are from the a priori knowledge of \(g_i(\bar{x}_i, \omega_i)\) and \(f_i(\bar{x}_i, \omega_i)\) \((i = 1 \text{ and } 2)\). These rules are given as follows.

Fuzzy rules of \(h_1(\chi_1, \omega_1)\):

\[
\begin{align*}
F_{h_1}^{(1)}: \text{IF } \chi_1 \text{ and } \omega_1 \text{ approach } -0.9, \text{ THEN } h_1 \text{ approaches 1.7}; \\
F_{h_1}^{(2)}: \text{IF } \chi_1 \text{ and } \omega_1 \text{ approach } -0.6, \text{ THEN } h_1 \text{ approaches 1.1}; \\
F_{h_1}^{(3)}: \text{IF } \chi_1 \text{ and } \omega_1 \text{ approach } -0.3, \text{ THEN } h_1 \text{ approaches 0.4}; \\
F_{h_1}^{(4)}: \text{IF } \chi_1 \text{ and } \omega_1 \text{ approach } 0, \text{ THEN } h_1 \text{ approaches 0}; \\
F_{h_1}^{(5)}: \text{IF } \chi_1 \text{ and } \omega_1 \text{ approach 0.3, THEN } h_1 \text{ approaches } -0.1; \\
F_{h_1}^{(6)}: \text{IF } \chi_1 \text{ and } \omega_1 \text{ approach 0.6, THEN } h_1 \text{ approaches } -0.05; \\
F_{h_1}^{(7)}: \text{IF } \chi_1 \text{ and } \omega_1 \text{ approach 0.9, THEN } h_1 \text{ approaches } -0.2.
\end{align*}
\]

Fuzzy rules of \(h_2(\chi_2, \omega_2)\):

\[
\begin{align*}
F_{h_2}^{(1)}: \text{IF } \chi_2 \text{ and } \omega_2 \text{ approach } -0.9, \text{ THEN } h_2 \text{ approaches 1.4}; \\
F_{h_2}^{(2)}: \text{IF } \chi_2 \text{ and } \omega_2 \text{ approach } -0.6, \text{ THEN } h_2 \text{ approaches 0.8}; \\
F_{h_2}^{(3)}: \text{IF } \chi_2 \text{ and } \omega_2 \text{ approach } -0.3, \text{ THEN } h_2 \text{ approaches 0.4}; \\
F_{h_2}^{(4)}: \text{IF } \chi_2 \text{ and } \omega_2 \text{ approach 0, THEN } h_2 \text{ approaches 0}; \\
F_{h_2}^{(5)}: \text{IF } \chi_2 \text{ and } \omega_2 \text{ approach 0.3, THEN } h_2 \text{ approaches } -0.4; \\
F_{h_2}^{(6)}: \text{IF } \chi_2 \text{ and } \omega_2 \text{ approach 0.6, THEN } h_2 \text{ approaches } -0.8; \\
F_{h_2}^{(7)}: \text{IF } \chi_2 \text{ and } \omega_2 \text{ approach 0.9, THEN } h_2 \text{ approaches } -0.6.
\end{align*}
\]

In simulation, the initial states of system are set to be \(x_1(0) = -0.05\) and \(x_2(0) = 0\). We choose the numbers of FSE components as \(p_1 = p_2 = 9\), and the initial values of coefficients of FSE are taken randomly in the interval \([1]\). For the FLS, the membership functions are chosen as Gaussian functions

\[
\mu_{F_i}(x_i) = \exp\left[-\left(\frac{x_i - a_i^j}{b_i^j}\right)^2\right]
\]

which are shown in Section II. Then, the centers \(a_i^j\) and the adjustable vectors \(W_i\) are initiated according to the above fuzzy rules. All initial values of widths \(b_i^j\) are set to be 1. The initial values of \(\hat{W}_i\) and \(\hat{V}_i\) in the new FSE–FLS approximators \(\hat{W}_i^T S_i(\bar{Z}_i(\chi_i, \phi))\) \((i = 1 \text{ and } 2)\) are computed based on the initial values of above parameters.

Moreover, according to the criteria shown in Table I, the controller parameters are chosen as \(\varepsilon_1 = \varepsilon_2 = 1\), \(\sigma_{w1} = \sigma_{w2} = \sigma_{\bar{v}1} = \sigma_{\bar{v}2} = 0.0001\), and \(\tau_2 = 0.02\). The adaptive gains are chosen as \(\Gamma_{w1} = \Gamma_{w2} = 2I\) and \(\Gamma_{v1} = \Gamma_{v2} = 5I\). The simulation results are shown in Figs. 1 and 2.

Fig. 1(a) and (b) shows the system output, the reference signal, and the tracking error. Fig. 1(c) and (d) shows the fuzzy rules \(h_1(\chi_1, \omega_1)\), its approximation \(\hat{W}_1^T S_1(\bar{Z}_1(\chi_1, \phi))\), and the approximation error, from which we can see that since we sufficiently use some fuzzy rules, the tracking error and the approximation error are both small, even in the initial phase. This implies that the closed-loop system has the good transient performance. Moreover, the other closed-loop signal curves, including the estimates of \(\|V\|, \|W\|, i = 1, 2\), the filter \(\zeta_2\), and the control \(u\), are shown in Fig. 2, from which it can be seen that these closed-loop signals are all bounded.

In order to show the difference between our control scheme and the existing control methods, we apply the ABFC idea in \([8]–[19]\) to system (61), where no FSE is combined with the FLS, and only pure FLS is used to approximate the unknown system functions, i.e., we replace the FSE–FLS-based approximator in (62)–(66) by pure FLS. For impartial comparison, in
simulation, we still use the DSC approach and keep all design parameters as before. The simulation results are shown in Fig. 3. It can be seen that because of the existence of periodically time-varying disturbances, the ultimate tracking error and the ultimate approximation error are obviously larger than those shown in Fig. 1, which also confirms the ability of the FSE–FLS-based approximator to compensate for the unknown function disturbed by unknown and periodically time-varying disturbances.

In our previous work [30], we proposed the FSE–NN-based approximators, including the FSE–radial basis function neural network (RBFNN)-based approximator and the FSE–multilayer neural network (MNN)-based approximator. Now, we compare both the approximators. Without loss of generality, we replace the FSE–FLS-based approximators in (62)–(66) with the FSE–RBFNN-based approximators. In simulation, since NNs cannot use any a priori knowledge from system (61), the initial values of unknown parameters of FSE–RBFNN-based approximators are taken randomly in the interval $[1]$. For an impartial comparison, the other controller parameters and initial conditions are kept as before. The simulation results are shown in Fig. 4. It can be seen that although the tracking error and the approximation error still converge to a small neighborhood around the origin, the transient performance of closed-loop system is not ideal in the initial phase. Because no useful information can be used to improve the initial values of estimated parameters, these estimated parameters are far away from their optimal values, which will undoubtedly deteriorate the transient performance in the initial phase.

Finally, to show the application of the proposed control method, we further consider a time-varying disturbed physical system, i.e., the van der Pol oscillator, which is investigated by the learning method in [22], where the disturbance is assumed to be unknown, but the system functions are known.

However, in the following, we assume that the periodic disturbance and the system functions are both unknown.

**Example 2:** Let us consider the following van der Pol oscillator:

\[
\begin{align*}
\dot{x}_1 &= -x_2 + x_1 - \frac{1}{3}x_1^3 + p + F(t) \\
\dot{x}_2 &= u + 0.1(x_1 + a - bx_2) \\
y &= x_1
\end{align*}
\]

where $F(t) = q \cos(wt)$ is a periodic exciting signal. As given in [22], when the system parameters are chosen as $w = 1$, $a = 0.7$, $b = 0.8$, $p = 0$, and $q = 0.74$, the system (67) without control will present the chaotic behavior. Obviously, the period of the time-varying disturbance $F(t)$ is still $2\pi$, but the function $g_1(x_1, \theta(t)) = -1$ is negative. Therefore, we must change the sign of the virtual control $\alpha_1$, i.e., $\alpha_1$ is modified as follows:

\[
\alpha_1 = k_1(t)z_1 + \hat{W}_1^T S_1(V_1^T Z(\chi_1, \phi))
\]

but the control law, the parameter adaptive laws, and the filter are still kept as before. Similarly, we assume that there exist the following fuzzy rules to initiate the estimated parameters.

**Fuzzy rules of $h_1(\chi_1, \omega_1)$:**

\[
F_{h_1}^{(1)}: \text{IF } \chi_1 \text{ and } \varpi_1 \text{ approach } -0.9, \text{ THEN } h_1 \text{ approaches } 0.7; \\
F_{h_1}^{(2)}: \text{IF } \chi_1 \text{ and } \varpi_1 \text{ approach } -0.6, \text{ THEN } h_1 \text{ approaches } 0.5; \\
F_{h_1}^{(3)}: \text{IF } \chi_1 \text{ and } \varpi_1 \text{ approach } -0.3, \text{ THEN } h_1 \text{ approaches } 0.3; \\
F_{h_1}^{(4)}: \text{IF } \chi_1 \text{ and } \varpi_1 \text{ approach } 0, \text{ THEN } h_1 \text{ approaches } 0; \\
F_{h_1}^{(5)}: \text{IF } \chi_1 \text{ and } \varpi_1 \text{ approach } 0.3, \text{ THEN } h_1 \text{ approaches } -0.3; \\
F_{h_1}^{(6)}: \text{IF } \chi_1 \text{ and } \varpi_1 \text{ approach } 0.6, \text{ THEN } h_1 \text{ approaches } -0.5; \\
F_{h_1}^{(7)}: \text{IF } \chi_1 \text{ and } \varpi_1 \text{ approach } 0.9, \text{ THEN } h_1 \text{ approaches } -0.7.
\]

**Fuzzy rules of $h_2(\chi_2, \omega_2)$:**

\[
F_{h_2}^{(1)}: \text{IF } \chi_2 \text{ and } \varpi_2 \text{ approach } -0.9, \text{ THEN } h_2 \text{ approaches } 0.05;
\]
The reference signal is changed to be $y_r(t) = \sin(t)\sin(0.5t)$. The simulation is run under the same control parameters and the initial conditions, and the simulation results are shown in Fig. 5, from which we can see that the control performance is still satisfactory. However, if we assume that no fuzzy rules can be used, then the initial values of the estimated parameters are taken randomly in the interval $[-1, 1]$. In this case, we show the simulation results in Fig. 6. Similar to the case of using the FSE–RBFNN-based approximator, the transient performance is not good in the initial simulation phase, but the ultimate tracking error and the approximation error are still ideal.

The above two simulation examples further verify the theoretical results obtained in this paper, and the simulation comparison shows that the control approach, which is proposed in this paper, is, indeed, superior to some existing control methods.

VI. CONCLUSION

In this paper, we proposed a novel FSE–FLS-based function approximator to approximate the unknown system functions depending nonlinearly on the periodic disturbances. Then, based on this approximator, we developed an adaptive DSC scheme for strict-feedback and periodically time-varying systems with unknown control-gain functions. Moreover, the tracking error is proven to converge to a small neighborhood around the origin, while keeping all closed-loop signals SGUUB. Further work could aim at removing the requirement that the periods of the disturbances are known. The approximation problem of nonperiodically disturbed functions should be also investigated.

ACKNOWLEDGMENT

The authors would like to thank the Associate Editor and the anonymous reviewers for their valuable comments and suggestions that have improved the presentation of this paper.

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