On the completeness of the system \( \{t^{\lambda_n}\} \) in \( C_0(E) \)✩

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A B S T R A C T
Let \( E = \bigcup_{n=1}^{\infty} I_n \) be the union of infinitely many disjoint closed intervals where \( I_n = [a_n, b_n], 0 < a_1 < b_1 < a_2 < b_2 < \cdots \), and \( \text{dist}(0, I_n) \to \infty \). Let \( \alpha(t) \) be a nonnegative function and \( \{\lambda_n\}_{n=1}^{\infty} \) be a sequence of distinct complex numbers. Motivated by the work of Guantie Deng, in this paper, a theorem on the completeness of the system \( \{t^{\lambda_n}\} \) in \( C_0(E) \) is obtained where \( C_0(E) \) is the weighted Banach space which consists of complex functions continuous on \( E \) with \( f(t)e^{-\alpha(t)} \) vanishing at infinity.

1. Introduction

Fix a weight \( \alpha(t) \), that is a nonnegative continuous function defined on \( \mathbb{R} \) such that

\[
\lim_{|t| \to \infty} \frac{\alpha(t)}{\log |t|} = \infty. \tag{1}
\]

The weighted Banach space \( C_{\alpha} \) consists of complex continuous functions \( f \) defined on the real axis \( \mathbb{R} \) with \( f(t)e^{-\alpha(t)} \) vanishing at infinity, and normed by

\[
\|f\|_{\alpha} = \sup \{|f(t)e^{\alpha(t)}|: t \in \mathbb{R}\}.
\]

Denote by \( M(A) \) the set of functions which are finite linear combinations of exponential system \( \{t^{\lambda}: \lambda \in \Lambda\} \) where \( \Lambda = \{\lambda_n: n = 1, 2, \ldots\} \) is a sequence of complex numbers. Condition (1) guarantees that \( M(A) \) is a subspace of \( C_{\alpha} \). When \( \Lambda = \{\lambda_n: n = 1, 2, \ldots\} \) are just all of the positive integers, the problem of density of \( M(A) \) in \( C_{\alpha} \) in the norm \( \| \cdot \|_{\alpha} \) is the classical Bernstein problem on polynomial approximation in [5] and [6]. A well-known result which was obtained by S. Izumi and T. Kawata in 1937 in [9] is described as follows.

**Theorem A.** (See [9].) Suppose \( \alpha(t) \) is an even function satisfying (1) and \( \alpha(e^t) \) is a convex function on \( \mathbb{R} \). Then a necessary and sufficient condition for polynomials to be dense in the space \( C_{\alpha} \) is

\[
\int_{-\infty}^{+\infty} \frac{\alpha(t)}{1 + t^2} dt = \infty.
\]

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Motivated by the Bernstein problem and the Müntz theorem in [5], combining Malliavin’s uniqueness theorem in [11], in his paper [7], Guantie Deng obtained a necessary and sufficient condition for \( \mathbf{M}(\Lambda) \) to be dense in \( C_\alpha \). The result is described below.

**Theorem B.** (See [7].) Suppose \( \alpha(t) \) is an even function satisfying (1) and \( \alpha(e^t) \) is a convex function on \( \mathbb{R} \). Let \( \Lambda = \{\lambda_n; \; n = 1, 2, \ldots\} \) be a sequence of strictly increasing positive integers and let

\[
\Lambda(r) = 2 \sum_{\lambda_n \leq r} \frac{1}{\lambda_n}, \quad \text{if } r \geq \lambda_1; \quad \Lambda(r) = 0, \quad \text{otherwise},
\]

\[
k(r) = \Lambda(r) - \log^+ r, \quad \log^+ r = \max(\log r, 0), \quad \tilde{k}(r) = \inf\{k(r'): \; r' \geq r\}. \text{ If}
\]

\[
\int_0^{+\infty} \frac{\alpha(\exp(\tilde{k}(t) - a))}{1 + t^2} \, dt = \infty,
\]

(2)

for each \( a \in \mathbb{R} \), then \( \mathbf{M}(\Lambda) \) is dense in \( C_\alpha \).

Conversely, if the sequence \( \Lambda \) contains all of the odd integers, then for \( \mathbf{M}(\Lambda) \) to be dense in \( C_\alpha \), it is necessary that (2) holds for each \( a \in \mathbb{R} \).

Recently, there arose an interest in the Riesz basis property in \( L^2(E) \) (see [14]), where \( E \) is the union of finitely many disjoint intervals:

\[
E = \bigcup_{n=1}^{l} I_n, \quad I_n = (a_n, b_n), \quad 0 < a_1 < b_1 < a_2 < b_2 < \cdots < b_l, \; l \geq 2.
\]

There also arose an interest in approximation in weighted Banach spaces consisting of functions continuous on a set \( E \) which is an infinite union of closed intervals, that is \( E = \bigcup_{n=1}^{\infty} I_n \) and \( I_n \) are disjoint closed intervals on \( \mathbb{R} \), \( \text{dist}(0, I_n) \to \infty \) as \( n \to \infty \).

Let \( C_0(E) \) denote the weighted Banach space consisting of complex functions \( f \) continuous on the union of infinitely many disjoint closed intervals \( E \) with \( \frac{f(t)}{W(t)} \) vanishing at infinity, and normed by

\[
\|f\|_E = \sup \left\{ \frac{|f(t)|}{W(t)}; \; t \in E \right\},
\]

where \( W(t) = \exp(\alpha(t)) \), \( t \in E \); and \( W(t) = \infty \), \( t \in \mathbb{R} \setminus E \). Let \( |I_n| \) be the length of the interval \( I_n \). In Theorem E of [4], the following result is obtained.

**Theorem C.** (See [4], Theorem E.) Suppose \( \alpha(t) \) is an even function satisfying (1) and \( \alpha(e^t) \) is a convex function on \( \mathbb{R} \). Furthermore, suppose

\[
|I_n| \geq c \left( \text{dist}(0, I_n) \right)^{-M}
\]

(3)

for some \( c > 0 \), \( M < \infty \). The polynomials are dense in \( C_0(E) \) if and only if

\[
\int_E \alpha(t)\omega(i, dt, C \setminus E) = +\infty,
\]

(4)

where \( \omega(i, dt, C \setminus E) \) is the harmonic measure for the domain \( C \setminus E \) as seen from \( i \).

We say that the system \( \{t^k\} \) is complete in \( C_\alpha \) if the closure of its linear hull \( \mathbf{M}(\Lambda) \) coincides with \( C_\alpha \) (see [10,12–14,19]). In view of the Müntz theorem (see [5] and [6]), it is natural to ask under what conditions the system \( \{t^{n}\}_{n=1}^{\infty} \) \( \{n = 1, 2, \ldots\} \) is complete in \( C_0(E) \). In this paper, a theorem about this is obtained.

In contrast to the method in [7] which is a combination of Malliavin’s uniqueness theorem in [11] and inverse Fourier transformation that cannot be applied in our situation, we will employ the method in [2] from which with a combination of Theorem C our completeness theorem follows.

Let \( E \) be a union of infinitely many disjoint closed intervals

\[
E = \bigcup_{n=1}^{\infty} I_n, \quad I_n = [a_n, b_n], \quad 0 < a_1 < b_1 < a_2 < b_2 < \cdots < b_n,
\]

(5)

where \( I_n \) satisfies \( \text{dist}(0, I_n) \to \infty \).
Let $\alpha(t)$ be a nonnegative function satisfying
\[ \alpha(t) = \alpha(a_1) + \int_{a_1}^{t} \frac{\phi(\zeta)}{\zeta} \, d\zeta \] with $\phi(t) \geq 0$ and $\phi(t) \uparrow \infty$ as $t \to \infty$.

It is easy to see that $\alpha(t)$ is convex with respect to $\log t$ since
\[ \frac{d(\alpha(t))}{d(\log t)} = \phi(t) \geq 0 \text{ and } \uparrow \infty. \]

So, $\alpha(e^t)$ is a convex function of $t$.

Let $\Lambda = \{\lambda_n: n = 1, 2, \ldots\}$ be a sequence of complex numbers satisfying the following conditions
the $\lambda_n$ are all distinct and $\lim_{n \to \infty} |\lambda_n| = \infty$,
\[ \lim_{n \to \infty} \frac{n}{|\lambda_n|} = D \quad (0 < D < \infty), \]
\[ |\arg(\lambda_n)| < \beta < \frac{\pi}{2}. \]

The main result of this paper is as follows.

**Theorem 1.** Suppose $\alpha(t)$ is defined by (6), and that $E$ is defined by (5) and satisfies (3). Moreover, suppose $\Lambda = \{\lambda_n: n = 1, 2, \ldots\}$ is a sequence of complex numbers satisfying (7)–(9). Let
\[ h = \frac{1}{\eta} + \varepsilon_0, \]
where is $\varepsilon_0$ some positive number and
\[ \eta = \max_{0<\delta<D\cos\beta} \frac{2\delta}{\sqrt{D^2 \sin^2 \beta + \delta^2}} (D \cos \beta - \delta). \]

If (4) holds and if
\[ \int_{\tau_0}^{\tau_1} \frac{\alpha(t)}{t^{1+h}} \, dt = +\infty, \]
then the system $\{t^{\lambda_n}\} (n = 1, 2, \ldots)$ is complete in $C_0(E)$.

It seems that conditions (4) and (12) are very similar. Actually, the following example shows that one of the conditions is not implied by the other.

**Example.** Fix some $\alpha(t)$ which satisfies the hypothesis of Theorem 1 and for some positive constant $h_0 > 1$,
\[ \int_{1+h_0}^{\infty} \frac{\alpha(t)}{t^{1+h}} \, dt < +\infty. \]

It is easy to see the existence of some positive constant $p$ which satisfies
\[ \int_{1+1/p}^{\infty} \frac{\alpha(t)}{t^{1+1/p}} \, dt = +\infty. \]

Now we seek for some set $E$ which satisfies the conditions of Theorem 1. Let $E = \bigcup_{n \in \mathbb{N}} [n^p - \chi, n^p + \chi]$ where $p > 1$ is defined in (14) and $\chi$ is some positive constant satisfying $\chi < 1/2$. By Lemma 2.7 in [4], we have
\[ \omega(i, dt, C \setminus E) \geq \frac{A dt}{1 + t^{1+1/p}}, \]
for $t \in [n^p - \chi, n^p + \chi]$, where $A$ is some positive constant that does not depend on $t$ and $n$. Combination of (14) and (15) yields
\[ \int_{E} \alpha(t) \omega(i, dt, C \setminus E) = +\infty. \]
It remains to show the existence of some complex sequence satisfying (13). By (11), we have
\[ \eta \geq \max_{0 < \delta < D \cos \beta} \frac{2\delta}{\sqrt{D^2 \sin^2 \beta + D^2 \cos^2 \beta}} (D \cos \beta - \delta), \]
thus
\[ \eta \geq \frac{1}{2} D \cos^2 \beta. \]  
(17)
We may select a sequence of complex numbers \( \{\lambda_n\} \) which is defined in (7)-(9), satisfying
\[ h_0 < \frac{1}{D \cos^2 \beta}. \]  
(18)
Thus, by (13), (16) and (18), we know that the conditions (4) and (12) are different.

2. Some lemmas

In order to prove the main result of this paper, we present the lemmas which we will need later in this section.

Define the functions
\[ G(z) = \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{\lambda_n^2} \right) \]  
(19)
and
\[ K(s) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-isy} \frac{dy}{G(iy)}, \quad s = u + iv. \]  
(20)

For sufficiently small \( \delta > 0 \), denote
\[ B_\delta = \{ s = u + iv : |v| \leq \pi D \cos \beta - \delta \pi \}. \]  
(21)

Let \( A \) denote positive constants, which may be different at each occurrence. When conditions (7)-(9) hold, from [15–17], we have

Lemma 1. (See [15–17].) Given \( \varepsilon > 0 \),
\[ \left| \frac{1}{G(iy)} \right| \leq A(\varepsilon) e^{-\pi D \cos \beta + \varepsilon |v|}, \]  
(22)
where \( A(\varepsilon) \) is a constant which depends only on \( \varepsilon \).

Lemma 2. (See [3,17].) The integral in (20) is convergent uniformly and absolutely in \( B_\delta \), hence the function \( K(s) \) is analytic and bounded in \( B_\delta \) for any sufficiently small positive number \( \delta \).

Lemma 3. (See [2,17].) There exists a sequence \( \{t_k\} \) with \( k \geq t_k \geq (1 - a)k \) (\( a \) is some sufficiently small positive number) such that for \( s = u + iv \in B_\delta, \) \( \Re s = u \geq 0 \),
\[ \left| K(s) - \sum_{|\lambda_n| < t_k} \frac{e^{-\lambda_n s}}{K'(\lambda_n)} \right| \leq A t_k e^{-ut_k \sin(\gamma \pi)}, \]  
(23)
and for \( s = u + iv \in B_\delta, \) \( \Re s = u \leq 0 \),
\[ \left| K(s) - \sum_{|\lambda_n| < t_k} \frac{e^{-\lambda_n s}}{K'(\lambda_n)} \right| \leq A t_k e^{-ut_k}, \]  
(24)
where \( A \) is a constant independent of \( s \) and \( t_k \), while \( \gamma \) is a small positive number satisfying
\[ \tan(\gamma \pi) < \frac{\delta}{D \sin \beta}. \]  
(25)

Remark 1. The essential form of (23) and (24) can be found in [15–17]. Here we use the modification form from [2] for convenience.
We also need the following results:

**Lemma 4.** (See Carleman’s Theorem on p. 103 in [10].) Let $\log^{-} r = \max\{-\log r, 0\}$. If $g(w)$ is analytic and bounded in the half-plane $\text{Im}(w) \geq 0$ and
\[\int_{-\infty}^{+\infty} \frac{\log^{-} |g(t)|}{1 + t^2} \, dt = \infty,\]
then $g(w) \equiv 0$.

**Lemma 5.** (See M.M. Dzhrbasian, p. 182 (p. 200 of the English translation) in [16].) Suppose $\alpha(t)$ be given as in (6), let
\[M_n = \sup_{t \geq 0} e^{-\alpha(t)} t^n\]
and
\[\Phi(t) = \sup_{n \geq 1} t^n M_n.\]
Then there exists some constant $A > 0$ such that for $t$ sufficiently large
\[\log \Phi(t) \geq A \alpha(t).\]

**Proof.** We refer to Lemma 1 on p. 95 in [18]. Let $\mu$ denote a measure supported on $E$ satisfying
\[\int_{E'} e^{\alpha(e^t)} \, d|\mu|(e^t) < \infty,\]
where $E'$ denotes the image of $E$ under the transformation $\xi = \ln t$.

We define a function for $s \in B_\delta$ by
\[F(s) = \int_{E'} K(s - \xi) \, d|\mu|(e^t).\]

**Remark 2.** By Lemma 2, when $\xi \in E'$ is fixed $K(s - \xi)$ is analytic for $s \in B_\delta$; when $s \in B_\delta$ is fixed, $K(s - \xi)$ is both measurable and bounded for $\xi \in E'$. Thus, it is not hard to prove that $F(s)$ is analytic and bounded in $B_\delta$ (see Exercise 16, Chap. 10 in [13]; Sect. 3 in [1] and p. 8 in [2]).

The following lemma will be crucial in our proof of the theorem:

**Lemma 6.** Let $\mu$ denote a measure supported on $E$ satisfying
\[\int_{E} e^{\alpha(t)} \, d|\mu|(t) < \infty,\]
where $\alpha(t)$ is a nonnegative continuous function satisfying (4) and (6). $E$ is defined in (5) and satisfying (3). If for $s \in B_\delta$, $F(s) \equiv 0$ where $F(s)$ is defined by (26), then
\[\int_{E} t^n \, d|\mu|(t) = 0, \quad n = 0, 1, 2, \ldots.\]

**Proof.** It is obvious that $s - \xi \in B_\delta$ for $\xi \in E'$ and $s \in B_\delta$. From Lemma 2, we know that the integral
\[K(s - \xi) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i(s-\xi)y}}{G(iy)} \, dy\]
converges uniformly and absolutely with respect to $\xi \in E'$. Interchanging the order of the integrations in (26), we have
\[
F(s) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-isy} G(iy) \left[ \int_{\mathbb{E}'} e^{iy\xi} d\mu(e^\xi) \right] dy = 0. \tag{28}
\]

Define
\[
k(y) = \frac{1}{G(iy)} \int_{\mathbb{E}'} e^{iy\xi} d\mu(e^\xi).
\]

By properly choosing the constant \( \varepsilon > 0 \) such that \( \varepsilon_1 = \pi D \cos \beta - \varepsilon > 0 \), it follows from Lemma 1 and the definition of the measure \( \mu \) that for some positive constant \( A \), we have
\[
||k(y)|| \leq \frac{1}{|G(iy)|} \int_{\mathbb{E}'} |d\mu|(e^\xi) \sup_{\xi \in \mathbb{E}'} |e^{iy\xi}| \leq Ae^{(-\pi D \cos \beta + \varepsilon)|y|} < Ae^{-\varepsilon_1|y|}.
\]

By the Plancherel Theorem (see [13], Theorem 9.13) and (28), we have
\[
\int_{-\infty}^{+\infty} ||k(y)||^2 dy = 0
\]
and for \( y \in \mathbb{R} \), \( k(y) = 0 \), i.e.,
\[
\int_{\mathbb{E}'} e^{iy\xi} d\mu(e^\xi) = 0 \tag{29}
\]
follows from the continuity of \( k(y) \) in \( \mathbb{R} \). Define the function
\[
L(z) = \int_{\mathbb{E}'} e^{z\xi} d\mu(e^\xi),
\]
and take the transformation \( t = e^\xi \), from the theory of transformation (see [8, p. 163]), we have
\[
L(z) = \int_{\mathbb{E}} t^z d\mu(t).
\]
We claim that \( L(z) \) is analytic in the closed right half-plane \( \text{Re } z \geq 0 \). Actually, by the definition of the measure \( \mu \), the analyticity of \( L(z) \) follows from Fubini’s theorem and Morera’s theorem. By (29), \( L(iy) = 0 \) for any \( y \in \mathbb{R} \). Thus, \( L(z) = 0 \) for \( \text{Re } z \geq 0 \). In particular, \( L(n) = 0 \), \( n = 0, 1, 2, \ldots \), i.e.,
\[
\int_{\mathbb{E}} t^n d\mu(t) = 0, \quad n = 0, 1, 2, \ldots \quad \square
\]

3. Proof of the theorem

Our proof follows closely the proof of Theorem 1 in [2].

If the system \( \{t^{in} \} (n = 1, 2, \ldots) \) is not complete in \( C_0(E) \), there exists a non-trivial bounded linear functional \( L \) such that \( L(t^{in}) = 0 \) for all \( n = 1, 2, \ldots \) (For a discussion of the bounded linear functionals in \( C_0(E) \), we refer to [16] for more details.) By the Riesz’s representation theorem (see [13, p. 40]), there exists a complex measure \( \mu \) satisfying
\[
||\mu|| = \int_{\mathbb{E}} e^{z(t)} d|\mu|(t) = ||L||
\]
and
\[
L(h) = \int_{\mathbb{E}} h(t) d\mu(t), \quad h \in C_0(E).
\]
Define
\[
L(z) = \int_{\mathbb{E}} t^z d\mu(t),
\]
Indeed by Lemma 6, it would then follow that we have by (20),

\[ L(\lambda_n) = \int_{E} t^{\lambda_n} d\mu(t) = \int_{E'} e^{\lambda_n t} d\mu(e^{t}) = 0, \tag{30} \]

where \( E' \) is the image of \( E \).

Recall the definition of \( F(s) \) in (26). To prove Theorem 1, it suffices to prove that if (30) holds, then \( F(s) = 0 \) for \( s \in B_{\delta} \). Indeed by Lemma 6, it would then follow that \( \int_{E} t^{\delta} d\mu(t) = 0 \), and then from Theorem C that \( L = 0 \), proving that \( |t^{\lambda_n}| (n = 1, 2, \ldots) \) is complete.

For \( s \in B_{\delta} \), let \( \{t_k\} \) be the sequence defined in Lemma 3, with \( k \geq t_k \geq (1 - a)k \) (\( a \) is a sufficiently small positive number), by (20),

\[ F(s) = \int_{E'} K(s - \xi) d\mu(e^{t}) = \int_{E'} \left[ K(s - \xi) - \sum_{|\lambda| < t_k} \frac{e^{-\lambda(s-\xi)}}{K'(\lambda_n)} \right] d\mu(e^{t}) + \int_{E'} \sum_{|\lambda| < t_k} \frac{e^{-\lambda(s-\xi)}}{K'(\lambda_n)} d\mu(e^{t}) =: F_{1,k}(s) + F_{2,k}(s). \]

By (30), we have

\[ F_{2,k}(s) = \sum_{|\lambda| < t_k} \frac{e^{-\lambda s}}{K'(\lambda_n)} \int_{E'} e^{\lambda t} d\mu(e^{t}) = 0. \]

Hence, for \( s = u + iv \in B_{\delta} \), \( F(s) = F_{1,k}(s) \). By (23) and (24) in Lemma 3, we have

\[ |F(s)| = |F_{1,k}(s)| \leq A_k \left( e^{-u t_k \sin(\gamma \pi)} \int_{E' \cap \{Re(s-\xi) \geq 0\}} |e^{\xi}|^{s \sin (\gamma \pi)} d\mu(e^{t}) + e^{-u t_k} \int_{E' \cap \{Re(s-\xi) \leq 0\}} |e^{\xi}|^{s} d\mu(e^{t}) \right), \]

where \( A \) is a constant independent of \( k \) and \( s \). Hence for \( Re(s) = u \geq 0 \),

\[ |F(s)| \leq A_k \left( \int_{E} |t|^{t_k \sin(\gamma \pi)} d\mu(t) + \int_{E} |t|^{t_k} d\mu(t) \right) \frac{|e^{\xi}|^{s \sin (\gamma \pi)} + e^{-u t_k} \int_{E} |e^{\xi}|^{s} d\mu(e^{t})}{|e^{\xi}|^{t_k \sin (\gamma \pi)}}. \]

Thus, by \( k \geq t_k \geq (1 - a)k \),

\[ |F(s)| \leq \inf_{k \geq 1} A_k \frac{\int_{E} |t|^{t_k} d\mu(t)}{|e^{\xi}|^{t_k \sin (\gamma \pi)}} \leq \inf_{k \geq 1} A_k \|\mu\|_{1, (1 - a)k \sin (\gamma \pi)} \sup_{t \geq 0} |t|^{k} e^{-\alpha(t)}. \]

Let

\[ M_{n} = \sup_{t \geq 0} e^{-\alpha(t)} t^{n} \]

and

\[ \Phi(t) = \sup_{n \geq 1} \frac{t^{n}}{M_{n}}. \]

By Lemma 5, for \( Re(s) \geq 0 \), there exists some constant \( A_2 > 0 \) such that

\[ |F(s)| \leq e^{-A_2 \alpha(t)}, \tag{31} \]

where \( \alpha = A_1 |e^{\xi}|^{(1 - a) \sin (\gamma \pi)} \). In order to use Lemma 4, we transform the domain \( B_{\delta} \) into the upper half-plane \( Im(z) \geq 0 \).

Let

\[ m = D \cos \beta - \delta, \tag{32} \]

then \( B_{\delta} \) is transformed into an angle \( |arg(z_1)| \leq m \pi \) by \( z_1 = e^{\xi} \), and the angle is transformed into the right half-plane \( Re(z_2) \geq 0 \) by \( z_2 = z_1^{1/2m} \), finally, let \( z = iz_2 \), the domain \( B_{\delta} \) is transformed into the upper half-plane \( Im(z) \geq 0 \). More accurately, we have

\[ |e^{\xi}| = |z_1| = |z_2^{2m}| = |(iz)^{2m}| = |z^{2m}| \]

by Fubini’s theorem and Morera’s theorem, we know that \( L(z) \) is analytic in the closed right half-plane \( Re(z) \geq 0 \).
Define \( g(z) = F(\log(-iz)^{2m}) \), it’s obvious that \( g(z) \) is analytic and bounded in the upper half-plane \( \text{Im} z \geq 0 \). By (31), for \( \text{Im} z \geq 0 \) and \( |z| \) sufficiently large, we have

\[
|g(z)| \leq e^{-A_2\alpha(A_3|z|^{2m(1-a)\sin(\gamma\pi)})} = e^{-A_2\alpha(A_3|m'|),
\]

where \( A_3 \) is some positive constant independent of \( z \), \( m \) is given by (32), and

\[
m' = 2m(1-a)\sin(\gamma\pi) = 2(D\cos\beta - \delta)(1-a)\sin(\gamma\pi).
\]

Let \( \tan(\gamma\pi) \to \delta \) in (25), then

\[
sin(\gamma\pi) \to \frac{\delta}{\sqrt{D^2\sin^2\beta + \delta^2}}.
\]

Denote

\[
m'' = \frac{2\delta}{\sqrt{D^2\sin^2\beta + \delta^2}}(D\cos\beta - \delta)(1-a).
\]

By (33), for \( \text{Im} z \geq 0 \) and \( |z| \) sufficiently large, we have

\[
|g(z)| \leq e^{-A_2\alpha(A_3|z|m'')},
\]

It is obvious that \( \delta \) can be chosen such that \( 0 < \delta < D\cos\beta \).

Let

\[
\eta' = \max_{0<\delta<D\cos\beta} m'',
\]

it is easy to see that in (36), if \( m'' \) is replaced by \( \eta' \), the inequality still holds, i.e., for \( \text{Im} z \geq 0 \) and \( |z| \) sufficiently large, we have

\[
|g(z)| \leq e^{-A_2\alpha(A_3|z|^{\eta'})}.
\]

We note that \( \eta' = \eta(1-a) \), choosing \( a \) sufficiently large such that

\[
\frac{1}{\eta'} < \frac{1}{\eta} + \varepsilon_0
\]

where \( \varepsilon_0 \) is from (10). By (10)-(12), we have

\[
\int_{-\infty}^{\infty} \frac{\log|g(t)|}{t^2} \, dt = -\infty.
\]

Hence

\[
\int_{-\infty}^{\infty} \frac{\log|g(t)|}{1 + t^2} \, dt = -\infty.
\]

Let \( \int_{-\infty}^{\infty} \) mean that the upper limit of the integral is a negative number with sufficiently large magnitude. Similarly, we have
\[
\int_{-\infty}^{\infty} \frac{\log |g(t)|}{t^2} \, dt \leq \int_{-\infty}^{\infty} \frac{-A_2 \alpha(A_3 |t|^\eta)}{t^2} \, dt = \int_{-\infty}^{\infty} \frac{-A_2 \alpha(A_3 t^\eta)}{t^2} \, dt = -\infty.
\]

Hence
\[
\int_{-\infty}^{\infty} \frac{\log |g(t)|}{1 + t^2} \, dt = -\infty.
\]

By Remark 2, we know that
\[
\int_{-\infty}^{\infty} \frac{\log |g(t)|}{1 + t^2} \, dt
\]
is bounded near zero, thus
\[
\int_{-\infty}^{\infty} \frac{\log |g(t)|}{1 + t^2} \, dt = -\infty,
\]
and by Lemma 4, \( g(z) \equiv 0 \) and hence \( F(s) \equiv 0 \).

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**References**


