Coherent structures in turbulent boundary layers with adverse pressure gradients

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(Received 22 December 2009; final version received 23 May 2010)

The coherent structures in turbulent boundary layers (TBLs) subjected to adverse pressure gradients (APGs) were investigated by analyzing a database of direct numerical simulations. The equilibrium adverse pressure gradient flows were established by using a power law free-stream distribution. The population trends of the spanwise vortices show that the outer regions of the APG TBLs are densely populated with hairpin-like vortices. These vortical structures induce low-momentum regions in the middle of the boundary layers, which result in an outer peak in the Reynolds shear stress. The 3-D features of the hairpin packets were deduced from their spatial characteristics in the spanwise-wall-normal plane. The conditionally averaged velocity fields show that there are counter-rotating $v-w$ swirling motions that represent cross-sectional evidences of the packets. Moreover, two-point correlations and linear stochastic estimations were used to provide statistical information about the hairpin packet motions in the cross-stream planes of the APG TBLs.

Keywords: adverse pressure gradient; turbulent boundary layer; population trends of spanwise vortices; coherent structure

Nomenclature

Roman symbols

$D^{2D}$ 2-D local velocity gradient tensor, $= \partial u_i/\partial x_j$, $i, j = 1, 2$
$I$ indicator function,
$L$ cross-sectional length,
$m$ power of free-stream velocity, $U \sim x^m$
$N$ ensemble averaged number of spanwise vortex,
$R_{AB}$ two-point correlations between A and B,
$Re$ Reynolds number,
$S$ contribution to total mean shear,
$y^*$ viscous length scale.

Greek symbols

$\beta$ nondimensional pressure gradient parameter,
$\delta$ boundary layer thickness.

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ISSN: 1468-5248 online only
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DOI: 10.1080/14685248.2010.496786
http://www.informaworld.com
θ_{in} momentum thickness of incoming boundary layer,
λ_{ci} swirling strength,
Λ_{ci} swirling-strength parameter,
Λ_{ci}\_normalized normalized swirling strength,
Π vortex population density,
Ψ population fraction,
ω_z spanwise vorticity.

Abbreviations
APG adverse pressure gradient,
DNS direct numerical simulation,
LSE linear stochastic estimation,
p.d.f. probability density function,
TBL turbulent boundary layer,
ZPG zero pressure gradient.

Superscript
+ normalized by wall unit,
rms root mean square.

Subscript
p prograde spanwise vortex,
r retrograde spanwise vortex.

1. Introduction
Turbulent boundary layers (TBLs) subjected to an adverse pressure gradient (APG) are frequently encountered in many engineering applications, such as diffusers, turbine blades, and the trailing edges of airfoils. The performances of such flow devices are significantly affected by the presence of APGs. Understanding the coherent structures in APG TBLs is essential to the comprehension of boundary layer turbulence. Since these organized motions play crucial roles in the production and dissipation of wall turbulence, the study of coherent structures contributes to the accuracy of turbulence models [1] and to the advances in flow control strategies [2]. Therefore, elucidating the behavior of the coherent structures in APG TBLs provides insight into the physics of APG flow and improves its modeling.

Most previous studies of coherent structures and, in particular, of the role of hairpin vortices and packets have been carried out for a zero pressure gradient (ZPG) boundary layer. Adrian et al. [3] showed in their particle image velocimetry (PIV) measurements that hairpin vortices form packets in which they convect with nearly the same velocities. Large packets in the outer region propagate faster than small packets in the log region, and the streamwise spacing of individual hairpins in the outer region is expected to be greater than that in the log region. Tomkins and Adrian [4] found that streamwise-aligned successive hairpins induce the formation of elongated regions of low-momentum fluid in the log layer. They conducted linear stochastic estimation (LSE) of the conditionally averaged fields of the low-momentum regions (LMRs) and obtained the spanwise characteristics of the structures in the streamwise–spanwise plane. Wu and Christensen [5] determined the distributions of the spanwise vortices and their contributions to the mean shear by analyzing the population trends of spanwise vortices in both channel and ZPG TBLs: they found that the heads of the
hairpin vortices mostly aggregate in the log layer. Spanwise vortices with induced motions generate significant total mean shear within the log layer. Hutchins et al. [6] performed stereo PIV measurements to examine the organized motions in the inclined cross-stream planes and deduced the 3-D spatial nature of the hairpin packets from the planar data. By using conditionally averaged velocity fields, they characterized the averaged $v - w$ swirling motions around the LMRs and the ramp-like packet arrangements. Carlier and Stanislas [7] examined the vortical structures in the spanwise-wall-normal plane by using multiple plane stereo PIV and determined the spatial relationship between these vortical structures and the sweep–ejection motions. The sweep and ejection motions were found to be coupled with low- and high-speed streaks respectively.

Several experimental studies of the turbulence characteristics of APG flows have been performed. Bradshaw [8] and Skåre and Krogstad [9] conducted experiments to obtain turbulence statistics for equilibrium APG TBLs [10], and found that the turbulence intensities have peak values in the outer region in the presence of strong APGs. Both the turbulence statistics and the turbulence structures of APG TBLs were examined by Krogstad and Skåre [11]. By using two-point velocity correlations, they found that the streamwise correlation length of the streamwise velocity decreases when a strong APG is applied. Perry and Marusic [12] performed theoretical calculations to extend the attached-eddy hypothesis to APG TBLs by making the assumption that each individual eddy is independent of the pressure gradient; the Reynolds stress generation is then correlated with attached eddies by inspection of the invariants of the velocity gradient tensor [13]. Zhou et al. [14] investigated the evolution of a hairpin vortex to explore the mechanism of formation of hairpin packets, and suggested that any primary hairpin vortices above a threshold strength autogenerate new hairpin vortices both upstream and downstream. Recently, Lee and Sung [15] conducted direct numerical simulations (DNSs) of APG TBLs and found that the coherent structures in the outer layer are more activated than those in ZPG TBLs, which might be due to the higher turbulence intensities in APG TBLs, and results in the development of a second peak in the turbulent energy [16]. Low-speed streaky structures that are intensified by the pressure gradient are present in the outer regions of APG flows and can be related to the outer maximum observed in the turbulent kinetic energy [17]. An enhancement of the outer kinetic energy is associated with the presence of large-scale outer streaky structures in APG TBLs.

In the present study, the turbulence statistics and coherent structures in APG TBLs were scrutinized by utilizing the DNSs of Lee and Sung [15] for $\beta = 0.73$ and 1.68. The DNS of a ZPG TBL ($\beta = 0$) was also examined for comparison. To obtain evidence of the presence of outer vortical structures, the instantaneous velocity fields were obtained in 3-D as well as planar views. Further, statistical information about the vortex distribution was obtained from the population trends of the spanwise vortices to examine the relationship in APG flows between the distribution of hairpin vortices and the contribution of the vortices to the total mean shear. Finally, two-point correlations and LSE were used to analyze the turbulence structures of TBLs with APGs in the spanwise-wall-normal plane. Our detailed comparison of the statistical structures demonstrates that spatial information about the hairpin packets in the cross-stream plane provides a comprehensive understanding of the 3-D nature of APG TBLs.

2. Computational details

A schematic diagram of the 3-D computational domain is shown in Figure 1(a). The governing equations are integrated in time by using the fractional step method with the implicit velocity decoupling procedure proposed by Kim et al. [18]. Based on a block
LU decomposition, both velocity–pressure decoupling and the additional decoupling of intermediate velocity components are achieved through approximate factorization. In this approach, the terms are first discretized in time by using the Crank–Nicholson method, and then the coupled velocity components are solved without iteration. All terms are resolved by using a second-order central difference scheme in space with a staggered mesh. The details of the numerical algorithm can be found in Kim et al. [18].

The no-slip boundary condition is imposed at the solid wall, and the boundary conditions at the top surface of the computational domain are provided by specifying that the streamwise variation of the free-stream velocity has the form of a power law relation

\[
\begin{align*}
    u &= U_\infty(x) = \\
    &\begin{cases} 
        U_0, & \text{for } x < 0 \\
        U_0 \left(1 - \frac{x}{x_0}\right)^m, & \text{for } x \geq 0,
    \end{cases} \\
    \frac{\partial v}{\partial y} &= -\frac{\partial u}{\partial y}, \\
    \frac{\partial w}{\partial y} &= 0,
\end{align*}
\]

which produces an equilibrium TBL. Periodic boundary conditions are applied in the spanwise direction. Since the boundary layer develops spatially in the downstream direction, it is necessary to use nonperiodic boundary conditions in the streamwise direction. To overcome the difficulties associated with such boundary conditions, and to avoid the simulation of the laminar and transitional regions arising near the leading edge, an auxiliary simulation based on the method of Lund et al. [19] was carried out to acquire time-dependent turbulent inflow data at the inlet; the Reynolds numbers, defined based on the momentum thickness at the inlet \(Re_\theta = U_\infty \theta_{in}/\nu\), were 300 for the APG flows and 1410 for the ZPG flow. Further, the convective boundary condition at the exit was specified as \((\partial u/\partial t) + c(\partial u/\partial x) = 0\), where \(c\) is the local bulk velocity. A TBL with a ZPG was also simulated to compare its turbulence characteristics with those of the APG flows. In the present study, three cases \((m = 0, -0.15, \text{ and } -0.2)\) were examined, which correspond to a ZPG \((\beta = 0)\) and moderate \((\beta = 0.73)\).
Table 1. Computational details.

<table>
<thead>
<tr>
<th>m</th>
<th>β</th>
<th>Δx^+</th>
<th>Δy^+</th>
<th>Δz^+</th>
<th>Δt^+</th>
<th>L_x/θ_in × L_y/θ_in × L_z/θ_in</th>
<th>Grid</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>15</td>
<td>0.2</td>
<td>40</td>
<td>0.2</td>
<td>240 × 30 × 40</td>
<td>1025 × 161 × 513</td>
</tr>
<tr>
<td>−0.15</td>
<td>0.73</td>
<td>12.5</td>
<td>0.17</td>
<td>24</td>
<td>0.25</td>
<td>1600 × 80 × 80</td>
<td>2049 × 121 × 257</td>
</tr>
<tr>
<td>−0.2</td>
<td>1.68</td>
<td>12.5</td>
<td>0.17</td>
<td>24</td>
<td>0.25</td>
<td>1600 × 120 × 160</td>
<td>2049 × 161 × 513</td>
</tr>
</tbody>
</table>

and strong (β = 1.68) APGs, respectively, for comparable ranges of Reθ in the equilibrium region. The nondimensionalized pressure gradient parameter β is defined as (δ*/τ_w)dP/dx. Here, δ* is the displacement thickness and τ_w is the wall shear stress. The mean velocity (Figure 1(b)) and Reynolds shear stress (Figure 1(c)) obtained from the dataset for ZPG flow are in good agreement with those of DeGraaff and Eaton [21] and Spalart [14], while the Reynolds shear stress of DeGraaff and Eaton [21] shows a slight deviation. Table 1 summarizes the relevant computational parameters; the reader is referred to [15] for further computational details.

3. Results and discussion

3.1. Instantaneous flow fields

The mean velocity profiles show that when an APG is applied, the standard logarithmic law of the wall becomes violated and the wake occupies a larger fraction of the boundary layer [15, 22, 23]. Furthermore, the turbulence intensities are at a maximum in the outer region of the APG flow, whereas they are at a maximum in the buffer layer of the ZPG flow. Recent DNS studies have shown that the main reason for the outer peak is the presence of more intense outer layer structures, i.e., the low-speed streaky structures in the outer region are intensified by the pressure gradient [16, 17].

To examine the outer layer structures in detail, the turbulent swirling strength, λ_ci, can be employed to distinguish the vortical structures from the background turbulence [15]. A top view of the 3-D flow fields in the swirling strength in a strong APG flow (β = 1.68) is shown in Figure 2(b); the outer vortical structures that are present near the top of the

Figure 2. Top view of the outer vortical structures in (a) β = 0 and (b) β = 1.68. The prograde and retrograde vortices are shown as blue and red isosurface plots of the swirling strength (11% of the maximum λ_ci). The black and white areas are the low- and high-speed regions (10% below and above the local mean velocity).
boundary layer are illustrated. A ZPG flow is shown in Figure 2(a) for comparison. The prograde and retrograde vortices, which were first introduced by Wu and Christensen [5], are shown as blue and red isosurface plots respectively. Several elongated low-speed regions (the black isosurface plots) are present in the outer region, and each of them lies beneath a group of streamwise-aligned hairpin-like structures. For example, there are three low-speed regions with a width of \(0.4\delta - 0.5\delta\) in Figure 2(b): these regions are highly elongated and exceed the entire streamwise domain. Even though most of the prograde vortices do not exhibit a complete hairpin vortex form, the low-speed structures are clearly enclosed by a series of hairpin vortices, which is consistent with the structures of Tomkins and Adrian [4]. It is also evident that the retrograde vortices in the outer region are mainly located near the prograde vortex cluster. Since the prograde vortices in a packet convect with similar streamwise velocities, the nested retrograde vortices also flow with similar velocities.

To visualize the wall-normal features in the streamwise-wall-normal plane (\(x-y\) plane), the spanwise swirling motions that contain heads of both symmetric and asymmetric hairpin-like vortices were examined. To detect \(u-v\) swirling motions at a given location, the spanwise swirling strength, \(\lambda_{ci,z}\), was obtained from the velocity gradient tensor in the \(x-y\) plane \((D^{z^2} - D^{x-y})\) rather than from the full velocity gradient tensor [24]. Although \(\lambda_{ci,z}\) is very useful for identifying vortices, it does not indicate the rotational direction. Hence, a swirling-strength parameter \(\Lambda_{ci,z}\) was defined [5],

\[
\Lambda_{ci,z}(x, y) = \lambda_{ci,z}(x, y) \times \frac{\omega_z(x, y)}{|\omega_z(x, y)|},
\]

in order to specify the rotational direction, where \(\omega_z\) is the instantaneous fluctuating spanwise vorticity. The negative and positive \(\Lambda_{ci,z}\) correspond to the prograde and retrograde spanwise vortices respectively. The instantaneous velocity fields with Galilean frames in the \(x-y\) plane are shown in Figure 3. Contours of instantaneous \(\Lambda_{ci,z}\) are shown as 10% of the maximum \(\Lambda_{ci,z}\) (up to 60% of the maximum) to highlight the locations of the spanwise vortices. For both the APG and ZPG flows, more prograde than retrograde spanwise vortices are illustrated.
vortices are observed. As shown in Figure 3(a), spanwise vortices are densely aggregated in the log layer of the ZPG boundary layer. However, there is significant aggregation of vortices in the wake region as well as in the log layer of the strong APG flow in Figure 3(b). The wake region of the APG flow contains more strong swirling motions than are found in the ZPG flow. We found that the contour size of the swirling strength in the wake flow becomes larger as the wall-normal location increases. These findings imply that the APG has a significant influence on vortical structures and on the distribution of structures in the wake flow.

3.2. Vortex identification

Since the swirling strength is obtained from the velocity gradient tensor, $\Lambda_{ci,z}$ can vary dramatically with the wall-normal location, i.e., there can be a large $u$-velocity gradient in the inner layer and a small $u$-velocity gradient in the wake region. Thus, it is necessary to normalize $\Lambda_{ci,z}$ along the wall-normal location. Since $\Lambda_{ci,z}$ is non-zero within the core of the vortex and a non-zero value of $\Lambda_{ci,z}$ can be positive or negative, the root mean square of $\Lambda_{ci,z}$ was chosen to represent the characteristic magnitude at a given $y$-location, i.e., the normalized swirling strength $\tilde{\Lambda}_{ci,z}$ is defined as $\tilde{\Lambda}_{ci,z} = \Lambda_{ci,z}/\Lambda_{ci,z}^{rms}$ [5]. The variations of the root mean square of $\Lambda_{ci,z}$ nondimensionalized by the wall unit with $y$-location and $\beta$ are shown in Figure 4. Each profile has an inner peak in the buffer region regardless of $\beta$ and the value at the peak increases with increasing $\beta$. Although $(\Lambda_{ci,z}^{rms})^+$ decreases rapidly with increasing $y$-location beyond the buffer region in the ZPG flow, the rate of decrease is slow for the APG flows. This result means that the characteristic magnitude of $\Lambda_{ci}$ nondimensionalized by the wall unit is large for the APG boundary layer.

In order to quantify the distribution of the magnitude of $\tilde{\Lambda}_{ci,z}$, the probability density functions (p.d.f.s) of $\tilde{\Lambda}_{ci,z}$ were evaluated at several $y$-locations of the ZPG ($\beta = 0$) and strong APG TBLs ($\beta = 1.68$). Figures 5(a) and 5(b) show the p.d.f.s in the range $30 < y^+ < 0.9\delta^+$ for $\beta = 0$ and $\beta = 1.68$ respectively. Each jagged line represents an ensemble-averaged p.d.f. at each $y$-location; the lines are very similar in the given ranges. The p.d.f. makes it possible to determine a universal threshold level for the identification of a vortex; a similar method was reported by Nagaosa and Handler [25], although in this study a strong APG was applied ($\beta = 1.68$). Thus, a threshold of $|\tilde{\Lambda}_{ci,z}| = 1.5$ was set for both the APG and ZPG boundary layers; this threshold is the same as that used in a previous experimental study [5]. On the other hand, the collapsed profiles of the APG flow are larger than those of the ZPG flow, especially in the large $|\tilde{\Lambda}_{ci,z}|$ region. For example,
the p.d.f. for the centerline of the collapsed profiles for $\beta = 1.68$ is three times as large as that for $\beta = 0$ in the $\tilde{\Lambda}_{ci,z} < -6$ region. These findings imply that strong swirling motions of hairpin heads are more common in the APG flow than in the ZPG flow. The profiles of the retrograde vortices, i.e., those with positive $\tilde{\Lambda}_{ci,z}$, also exhibit larger values for the APG flow; however, these differences are smaller than those for the prograde vortices.

Figures 5(c) and 5(d) show the p.d.f.s at several $y^+$-locations close to the wall ($y^+ = 10, 15, 20, 25, \text{ and } 30$). Although the p.d.f.s of the prograde spanwise vortices have similar profiles over this range, those of the retrograde spanwise vortices are sensitive to the $y^+$-location near the wall because of the large viscous mean shear in the region. The minimum location for applying a universal threshold level to the retrograde spanwise vortices is $y^+ \approx 30$. In addition to monitoring the magnitude of $\tilde{\Lambda}_{ci,z}$, a constraint for determining the contour size is also required to identify the spanwise vortices. A threshold level for the size of an identified vortex can be prescribed in terms of the cross-sectional length $L$ of the vortex sections. In the present study, the size of the smallest vortex section was limited such that the length of its major axis must exceed $22y^*$, which is the same limit as used in the previous study [5]. Since the limit is smaller than the average log layer diameter range, $L = 35y^* - 44y^*$, as reported by Carlier and Stanislas [7], and $L = 45y^* - 52y^*$, as reported by Wu and Christensen [5], it is suitable for vortex identification.

### 3.3. Vortex population densities

The effects of APGs on individual vortical structures were reported in the previous study of Lee and Sung [17]. They found that the swirling motion of each individual hairpin in the outer region is stronger in APG flows than in ZPG flows. The population densities of vortices provide a possible link between the structures and the turbulence statistics of APG flows.
flows. In this section, we investigated this approach by considering the population trends of spanwise vortices [5]. These trends provide statistical information about the distribution of vortex cores and their contribution to the total mean shear. The vortex population density, \( \Pi_{p(r)} \), represents the average number of prograde or retrograde spanwise vortices per unit \( x - y \) area at a given wall-normal location, and is defined as follows [5]:

\[
\Pi_{p(r)}(y/\delta) = \frac{N_{p(r)}(y/\delta)}{(3W_y/\delta)(W_x/\delta)}.
\]  

Here, \( N_{p(r)} \) is the ensemble-averaged number of prograde or retrograde spanwise vortices, \( W_x \) is the streamwise field of view, and \( W_y \) is the wall-normal height of the window used to assess the population trends. The use of an averaging window of height \( 3W_y/\delta \) decreases the scatter in these profiles.

The outer-scaled prograde population densities, \( \Pi_p \), are shown in Figure 6(a) as a function of \( y/\delta \). The largest \( \Pi_p \) is found in the buffer layer; \( \Pi_p \) rapidly decreases with increasing \( y \) up to the top of the log layer (\( y/\delta \approx 0.2 \)) in both the APG and ZPG flows. The densities continuously decrease with \( y \)-location beyond the log layer for the ZPG flow. This result is consistent with those in Figure 3(a), which shows the increases in the space between the prograde vortices with increases in the distance from the wall. In contrast to the ZPG flow, the densities for the APG flows are keeping constant beyond the log layer. This result provides statistical evidence for the presence of outer vortical structures in the APG boundary layer. Furthermore, in previous studies [9, 16, 23], the Reynolds shear stress forms a peak in the outer region when an APG is applied to TBL flows. The outer peak of the turbulence intensities is located at a height similar to that of keeping the prograde vortex densities in the outer region. Thus, it is likely that the strong velocity fluctuations of the spanwise vortices contribute to the intensities in the outer region. Although the population densities of the spanwise vortices do indicate the locations most populated with vortices, they cannot provide evidence for the nesting of structures in the streamwise direction. However, we found that the prograde vortices are mainly advected in a group by the instantaneous flow fields (Figure 2). On the other hand, the outer-scaled retrograde population densities, \( \Pi_r \), are shown in Figure 6(b) as a function of \( y/\delta \). The profiles of \( \Pi_r \) are far less sensitive to the APG and there is a local maximum near the top of the log layer. The profiles also pass through a minimum near zero as the wall is approached; similar results were derived from the p.d.f.s of \( \tilde{\Lambda}_{ci} \) (see Figures 5(c) and 5(d)). These findings imply
that none of the mechanisms of formation of retrograde vortices [4, 26] can be sustained near the wall.

Figures 7(a) and 7(b) show outer-scaled population densities as a function of $L$ in the log layer ($y/\delta = 0.1$) and in the wake region ($y/\delta = 0.5$) respectively. Each bar indicates the density of spanwise vortices that are larger than $L$ accumulatively, for example, the densities for $L = 11y^*$ contain those for $L = 22y^*$. Although the densities for $L = 22y^*$ have similar values in the log layer, the density of smaller vortices decreases and that of larger vortices increases as $\beta$ increases. In the wake region ($y/\delta = 0.5$), where there is a local maximum in $\Pi_{1p}$ for a strong APG flow ($\beta = 1.68$), the densities of the APG flows are larger than those of the ZPG flow for all given $L$s. Furthermore, we found that the population densities of retrograde vortices as a function of $L$ are proportional to those of prograde vortices for both regions. The retrograde vortices nest in the vicinity of clusters of the prograde vortices, as shown for the instantaneous flow fields of a strong APG TBL in Figure 2.

The proportion of each type of spanwise vortex that resides at a given wall-normal location can be monitored by calculating the fractions of prograde and retrograde spanwise vortices. The fraction of retrograde spanwise vortices $\Psi_r$ is obtained from [5],

$$\Psi_r(y) = \frac{\Pi_r(y)}{\Pi_p(y) + \Pi_r(y)},$$ \hspace{1cm} (4)

and is shown in Figure 8(a). The fraction of prograde spanwise vortices $\Psi_p$ is defined as $\Psi_p(y) = 1 - \Psi_r(y)$, since the spanwise vortices are made up only of prograde and retrograde vortices. A previous study of ZPG TBLs showed that plots of $\Psi_r$ versus $y/\delta$ at different Reynolds numbers almost collapsed onto a universal curve [5]. When an APG is applied, the fractions are similar to those of a ZPG, for which $\Psi_r = 0.25–0.3$ at the top of
the log layer \((y/\delta = 0.2)\). Finally, \(\Psi_r\) reaches approximately 0.28 at the boundary layer. The presence in APG TBLs of outer streaky structures does not affect the population fractions. We confirmed that there is a quantitative relationship between the prograde and retrograde spanwise vortices and found evidence of instantaneous flow fields. Furthermore, the inner-scaled population fractions are shown in Figure 8(b). In contrast to the results of the previous study of ZPG TBLs, there are significant disparities between the fractions for increasing \(\beta\) in the log layer, which arise from the variance of the velocity scale under an APG [27].

3.4. Contributions to the mean shear

The contributions of the prograde and retrograde spanwise vortex cores to the total mean shear of the APG flows were determined [5]. Here, the total mean shear, which is the sum of the mean viscous shear and the shear contributions because of turbulent stresses, is given by

\[
\tau(y) = \mu \frac{\partial U}{\partial y} - \rho \langle uv \rangle.
\]  

(5)

The contributions of the prograde (retrograde) spanwise vortex cores to \(\tau(y)\) can be computed as stress fractions of the form defined by Wu and Christensen [5],

\[
S_{\rho(r)}(y) = \frac{\left[ \frac{1}{M} \sum_{i=1}^{M} \left( \mu \frac{\partial u}{\partial y} - \rho \langle uv \rangle \right)_{(x_i, y)} \right] \cdot I_{\rho(r)}(x_i, y)}{\tau(y)},
\]  

(6)

which gives the mean shear contained within each vortex core at a given wall-normal location [5]. Here, the number of events \(M\) is \(M = N_e N_x\), where \(N_e\) is the number of realizations and \(N_x\) is the number of streamwise grid points in each velocity realization of that ensemble. The indicator function, \(I_{\rho(r)}\), is given by

\[
I_{\rho(r)} = \begin{cases} 
1, & \text{if } (x_i, y) \text{ is within an identified prograde (retrograde) core} \\
0, & \text{otherwise}.
\end{cases}
\]  

(7)

Figure 9(a) shows the mean shear contributions of the prograde spanwise vortex cores, \(S_p\), as a function of \(y/\delta\). Since, as shown in the previous study, the contributions decrease
with increasing $Re_{\tau}$, $S_p$ and $S_r$ are expected to have larger values in our low-$Re$ DNS database than in the previous study. When we compare these equilibrium flows, it can be seen that the contributions of the flows have similar profiles for $y < 0.6\delta$. The contributions reach about 15% as a local maximum at the bottom of the log layer and decline with $y$-location in the log layer. For $0.2\delta < y < 0.6\delta$, the contributions are almost constant, irrespective of the magnitude of the APG, and the vortex population densities and Reynolds shear stress for the APG flows are higher than those for the ZPG flow. These findings show that the effects of the hairpin vortices on the total mean shear are proportional to the vortex population densities for $0.2\delta < y < 0.6\delta$. Near the top of the boundary layer, $S_p$ increases again because of the local Reynolds shear stress, for which positive values are predominant within the cores. However, the rate of increase with increasing $\beta$ is slow. In the wake region, the population densities of spanwise vortices are larger for the APG flows than for the ZPG flow, but their contributions to the total mean shear are small. The retrograde vortex cores contribute much less to the mean shear than the prograde vortex cores, especially close to the wall where $S_r$ is almost zero. The trend in the contributions is similar to that of the population density; however, the contributions do increase in the wake region with decreasing $\beta$.

Figures 9(c) and 9(d) show the contributions of the prograde and retrograde spanwise vortex cores and the locally induced motions. Here “cores and locally induced motion” refers to the swirling motion of each vortex core with sides $L$ larger than the determined extent of the vortex core in the $x$–$y$ plane. The contributions of the prograde vortex cores and the induced motions reach values of approximately 50% at the bottom of the log layer and also display a region of relative constancy. Thus, the head of each hairpin vortex and its induced motions contribute significantly to the total mean shear in the log layer. Figure 9(d) shows the contributions of the retrograde vortex cores and the induced motions. Although the $S_p$ or $S_r$ of the vortex cores account for a small proportion of the total mean shear, the sum of $S_p$ and $S_r$, which includes the induced motions, is higher than 50% in the log layer regardless of $\beta$. It is unlikely that only the hairpin vortex cores make significant contributions to the total mean shear. However, the induced motions include Q2/Q4 events.
induced by hairpin-like structures and are predominant with respect to the total mean shear. The motions induced by the vortex cores and the cores themselves can generate significant shear even in APG flows.

3.5. Two-point correlations between streamwise velocities

The coherent structures in the inclined cross-stream plane (y–z plane) of ZPG TBLs were reported by Ganapathisubramani et al. [28] and Hutchins et al. [6] and used to deduce the 3-D spatial nature of the TBLs. In DNS of low-Re APG TBLs, Skote et al. [16] found that the flow structures in the y–z plane provide cross-sectional evidence of outer streaky structures. In the present study, the coherent structures in the outer region of the y–z plane were examined statistically to provide a comprehensive analysis of their 3-D nature. In this section, to quantify the spatial coherence and understand the statistical significance of various features within the velocity fields, the two-point correlations of the fluctuating velocity components are discussed. The two-point correlations between any two quantities \( R_{AB} \) are defined as

\[
R_{AB}(r_z, y_{ref}) = \frac{\langle A(z, y_{ref})B(z + r_z, y_{ref}) \rangle}{\sigma_A(y_{ref})\sigma_B(y)} ,
\]

where \( y_{ref} \) is the reference wall-normal location at which the correlation is computed. \( \sigma_A(y_{ref}) \) and \( \sigma_B(y) \) are the root mean squares of \( A \) at \( y_{ref} \) and \( B \) at location \( y \) respectively, and \( r_z \) is the in-plane spanwise separation between \( A \) and \( B \). Figure 10 shows the two-point correlations between streamwise velocity fluctuations, \( R_{uu} \), at \( y = 0.2\delta \) in the y–z plane. The average spanwise dimension of the correlations for \( \beta = 1.68 \) is wider than that for \( \beta = 0 \). The 0.15 contours show that the average spanwise extent of the correlations is approximately 0.45\( \delta \) for \( \beta = 1.68 \) and, by contrast, the average width is approximately 0.38\( \delta \) for \( \beta = 0 \). The average wall-normal dimension of the correlations for \( \beta = 1.68 \) is wider than that for \( \beta = 0 \). The spanwise distance between the positive peak (marked by “+”) and the negative peak (marked by “−”) decreases for the APG flow.

Figure 11 shows the 0.4 contours of \( R_{uu} \) in the y–z plane as a function of \( y_{ref} \). The proximity of this contour to the wall \( (R_{uu} = 0.4) \) is actually a statistical measure of how
well fluctuations at $y_{ref}$ are correlated with fluctuations near the wall [6]. These correlations were examined in more detail to investigate the effects of APGs on the spatial coherence of velocity fluctuations. For clarity, the maximum or minimum $y$ ordinate has been extracted for each of the contours and is shown in Figure 12(b). Hutchins et al. [6] showed that with outer scaling for ZPG TBLs, the locations $y_{min}$ collapse onto a common curve. However, beyond the log layer, the angle for a strong APG flow becomes larger than that for a ZPG flow. This result implies that the detached regime is strongly influenced by the APG, which corresponds to a decorrelation between the fluctuations at $y_{ref}$ and those at the wall. There are also significant decorrelations at $y_{max}$ locations of the velocity fluctuations in wake

![Figure 11](image1)

Figure 11. The 0.4 contours of two-point correlations between the streamwise velocities when (a) $\beta = 0$ and (b) $\beta = 1.68$.

![Figure 12](image2)

Figure 12. (a) Definitions (b) wall-normal, and (c) spanwise extents of the 0.4 contour, and (d) spanwise distance between positive and negative peaks of the two-point correlations $R_{uu}$ obtained in the spanwise-wall-normal plane.
flows subjected to an APG. In the attached regime, the inclination angle of $y_{min}$ in the log layer is similar for ZPG and APG flows. Hence, different treatments are required for the two regimes to predict accurately the turbulence statistics of APG flows with turbulence models. Moreover, the average spanwise width of the contour at $y_{ref}$ where the width has a maximum $\Delta z$ is shown in Figure 12(c). Beyond the log layer, the spanwise width is almost constant for this strong APG flow; however, the width continues to increase with a slow rate of growth rather than beneath the log layer for the ZPG flow. The trend in this invariant extent in the spanwise direction is similar to that in the findings for the wall-normal direction. The width grows almost linearly with distance from the wall in the log layer. This result is similar to the results for conditionally averaged negative-$u$ events reported in the $x-z$ plane [17] with respect to the average spanwise width of the coherent structures. The linear variation of the spanwise scales with wall-normal location in the log layer is also consistent with Townsend's attached eddy hypothesis. Figure 12(d) illustrates the spanwise distance between the positive and negative correlation peaks $\Delta z_{peak}$ as a function of $y_{ref}$. The variations of $\Delta z_{peak}$ are insensitive to the APG flows.

3.6. **Stochastic estimation for low-momentum regions**

LMRs were investigated statistically in the $y-z$ plane. LSE was employed to obtain quantitative structural information about the LMRs in the cross-stream plane. It is well known that LSE is a robust conditionally averaged approximation that provides satisfactory results for various turbulent fields [29]. Moreover, the average size of the LMRs has been quantified by using an LSE of conditionally averaged structures resulting from an event of negative fluctuating streamwise velocity by Tomkins and Adrian [4]. The linear estimate of the conditional average is given by

$$\langle u_i(x + r)|u_1(x) \rangle = L_{i1}u_1(x),$$

with $i = 1, 2, 3$ and

$$L_{i1} = \frac{\langle u_i(x + r)u_1(x) \rangle}{\langle u_1(x)^2 \rangle}.$$  \hspace{1cm} (10)

Figure 13 shows $\langle u_i(x + r)|u_1(x) \rangle$ of the estimate at $y = 0.2\delta$, where the contour and the velocity vectors are normalized by the friction velocity. There is a negative-$u$ region confined by positive-$u$ regions as well as a pair of counter-rotating $v-w$ motions swirling from side to side of the LMR. When the APG is applied, the ejection motion ($u < 0$ and $v > 0$) around the reference point becomes stronger. Furthermore, these results show that a taller LMR arises for $\beta = 1.68$ than for $\beta = 0$. These results confirm that the fluctuating velocity components around the LMR are intensified in all directions in an APG boundary layer, which augments the turbulence intensities. Moreover, these results and other conventional planes of view enable the 3-D velocity fluctuations of LMRs in TBL flows to be deduced.

3.7. **Two-point correlations between streamwise vortex and velocity**

The two-point correlations between streamwise vortex and velocity fluctuations were calculated to obtain structural information about the hairpin vortices in the $y-z$ plane. These correlations are also useful for the estimation of the conditionally averaged velocity field for a given vortex core and show the average velocity field associated with a vortex core.
Figure 13. A linear stochastic estimate for a low-momentum region \((u < -u_t)\) at \(y = 0.2\delta\) when (a) \(\beta = 0\) and (b) \(\beta = 1.68\). Contours show the conditional averaged streamwise velocity for increments of \(0.15u_t\) (negative contours are dashed).

located at the reference point [30]. In this section, the streamwise vortical structures are identified via the swirling-strength parameter \(\Lambda_{ci,x}\) acquired from the 2-D velocity gradient tensor in the \(y-z\) plane \((D_{y,z})\) [24]. Since the absolute value of \(\Lambda_{ci,x}\) was taken to obtain the correlation, \(R_{\Lambda v}\) and \(R_{\Lambda w}\) have the same sign as the wall-normal and spanwise velocities respectively. Thus, the statistically dominant fluid motions around the vortex core can be determined from the correlations.

Figure 14 shows the contours of the two-point correlations between the swirling-strength parameter and wall-normal velocity fluctuations \(R_{\Lambda v}\) at \(y = 0.2\delta\). In the \(y-z\) plane, \(R_{\Lambda v}\) is positive in the right-hand side of \(r_z/\delta = 0\) and negative in the left-hand side of \(r_z/\delta = 0\) for the counter-clockwise (CCW) rotating vortex, while positive in the left-hand side of \(r_z/\delta = 0\) and negative in the right-hand side of \(r_z/\delta = 0\) for the clockwise (CW) rotating vortex. The symmetry of the structures is a consequence of the statistical symmetry of

Figure 14. Two-point correlations between the wall-normal velocity and the swirling strength at \(y = 0.2\delta\) when (a) \(\beta = 0\) and (b) \(\beta = 1.68\). Contour levels are from \(-0.32\) to \(0.32\) with increments of \(0.04\).
Figure 15. Two-point correlations between the spanwise velocity and the swirling strength at $y = 0.2\delta$ when (a) $\beta = 0$ and (b) $\beta = 1.68$. Contour levels are from $-0.32$ to $0.32$ with increments of $0.04$.

the given event with respect to the reflection about $r_z/\delta = 0$. The spatial extent of both the positive and negative $R/L\Lambda_1v$ regions increases with increasing $\beta$, since the spanwise extents of the conditional eddies for the Q2 event increase because of the strong APG (see [17, Figure 23]). The correlations between the swirling-strength parameter and spanwise velocity fluctuations $R/L\Lambda_1w$ at $y = 0.2\delta$ are shown in Figure 15. For the CCW rotating vortex, the positive and negative $R/L\Lambda_1w$ are predominant below the vortex core and above the reference point respectively. As expected, the sign of $R/L\Lambda_1w$ for the CW rotating vortex is opposite to the sign of $R/L\Lambda_1w$ for the CCW rotating vortex. The spatial extent of $R/L\Lambda_1w$ above the vortex core is larger than that below the core irrespective of the pressure gradients and rotational direction. This result shows that the streamwise vortices induce fluid motions in the spanwise direction and that their zone of influence is enlarged when the APG is applied.

3.8. Stochastic estimation of the streamwise swirl

The conditionally averaged velocity field for the streamwise-oriented swirling event $\langle u_i(x + r) | \Lambda_{ci,x}(x) \rangle$ at $y = 0.2\delta$ was estimated in the $y-z$ plane using the stochastic estimation [29, 30]. The flow structures for $\beta = 0$ and $\beta = 1.68$ are shown in Figure 16, and the estimations associated with positive and negative swirling-strength parameters correspond to the CCW and CW rotating vortices respectively. The velocity vector was normalized by the friction velocity and the vector grid spacing was uniformized and down-sampled. A circular pattern of the velocity vectors represents the core of a vortex centered at the event location. Since most of hairpin-like vortices are present in the form of asymmetry, the estimation does not show a pair of counter-rotating streamwise vortices irrespective of the pressure gradients [31]. When a strong APG is applied, the conditionally averaged velocities of the APG flow normalized by the friction velocity are larger than those of the ZPG flow. The APG enhances not only in-plane velocity components but also streamwise
velocity components. There is a clear upward motion for both APG and ZPG flows where the conditionally averaged streamwise velocity is negative, which is consistent with the ejection (Q2) event. Strong ejection occurs over a wider range of the APG flow. This upward motion has a LMR lifted to the outer region, which may lead to make a large-scale outer structure [17]. In contrast, a downward motion is present where the conditionally averaged streamwise velocity is positive; this downward motion is consistent with the sweep (Q4) event. Strong sweep motions of the APG flow increase the transportation of the outer turbulence toward the wall in a wider spanwise range. These spatial imprints show the averaged spanwise motions around the hairpin vortex or hairpin packet in the $y-z$ plane of both APG and ZPG boundary layers.

4. Summary and conclusions
The instantaneous flow fields of the DNS of Lee and Sung [15] were used to elucidate the turbulence statistics and coherent structures of TBLs subjected to APGs. Equilibrium APG flows and a ZPG flow ($m = 0, -0.15, \text{ and } -0.2$) were chosen. We found in the APG TBLs that there are more prograde spanwise vortices present than retrograde vortices. The retrograde ones are mainly located near the prograde vortex cluster, especially in the outer region. The spanwise vortices are aggregated in the wake region as well as in the log layer. The distributions of vortical structures were obtained by analyzing the population trends of the spanwise vortices. Strong swirling motions are more frequently observed in the APG TBLs than in the ZPG TBL. The p.d.f.s of the retrograde spanwise vortices are almost zero and are sensitive to the wall-normal location near the wall. Few retrograde spanwise vortices are present near the wall because of the large viscous mean shear. In the wake region, the densities of the spanwise vortices increase significantly. These findings provide statistical evidence for the presence of outer vortical structures that might affect
the maximum Reynolds shear stress in an APG boundary layer. Our inspection of the contributions to total mean shear showed that the vortex cores and the induced motions generate significant shear. The coherent structures in the spanwise-wall-normal plane were examined statistically to provide a comprehensive analysis of the 3-D nature of the hairpin packets. The coherent velocity structures for a strong APG flow are longer and wider at the top of the log layer than those for the ZPG flow. The wall-normal extent of the correlations shortens rapidly with increases in the reference height. This behavior corresponds to a decorrelation between the velocity fluctuations in a strong wake flow. A pair of counter rotating $\nu-\omega$ motions swirls from side to side of the low-momentum regions. The APG significantly affects the coherent velocity structures, in that the velocity fluctuations are stronger and the extent of negative-$u$ events is enhanced for strong APG flow. Finally, structural information about a hairpin vortex in the cross-stream plane was obtained by examining the correlations and LSE of the conditionally averaged flow field. The outer turbulence is transported toward the wall and the zone of influence is widened for a strong APG flow. The conditionally averaged velocity field for the streamwise-oriented swirling event shows a circular pattern, and ejection and sweep motions are enhanced because of the pressure gradient.

Acknowledgments
This work was supported by the Creative Research Initiatives (Center for Opto-Fluid-Flexible Body Interaction) of MEST/NRF.

References


