Models for wave propagation in two-dimensional random composites: A comparative study

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This paper provides a set of benchmark results on existing theoretical models for wave propagation in two-dimensional composite materials. This comparative study is motivated to investigate the reason why results from an accurate ultrasonic measurement often significantly contradict theoretical predictions. Eight different models are evaluated with their numerically calculated effective wave speeds and coherent attenuations. For computational simplicity, the problem of horizontal shear wave propagation in an elastic matrix containing parallel circular cylinders is considered. Numerical calculations are conducted for different composites in wide ranges of material properties, volume concentration, and frequency. Some of the numerical results are compared with experimental data. Judgments are made based on fundamental theoretical and physical criteria as well as relative agreements in the numerical results, and then possible causes of failures are discussed. The effect of microstructure, potentially as a major source of the observed disagreements between models, is also discussed.

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I. INTRODUCTION

For last several decades, authors in theoretical physics, applied mathematics, and engineering have proposed numerous models for multiple scattering and propagation of waves in random inhomogeneous media. Reviews on the models and their mathematical backgrounds can be found in Refs. 1–3. In spite of the abundance in literature, comprehensive exposure of the subject to engineering investigators is severely limited, possibly due to the complex physical processes involved in the multiple-scattering phenomenon and accordingly difficult mathematical treatments. While every model proves its validity through a comparison with some experimental data or with the classical theories of Foldy,4 Lax,5 or Waterman and Truell,6 significant disagreements between experimental and theoretical results are often encountered. In most cases, experimental errors are blamed to be responsible for the disagreements. Meanwhile, it is also found that there are little agreements between different theoretical models. These pose a need of a large scale comparative evaluation study on the existing models. However, due to the mathematical complexity in the theories, an analytical study on the validity range of a model is a challenging task. The major difficulty in such an evaluation study is the lack of an exact reference solution, either from an analysis or an experiment. A numerical approach such as the Monte Carlo simulation7,8 and the semianalytical technique9 may be used. However, these numerical simulations are, by nature, limited to simple cases. On the other hand, experiments often suffer from the multiple-scattering noise that causes troubles identifying coherent signals. For these reasons, few works on the evaluation of multiple-scattering models have been reported so far.

Anson and Chiver10 examined 12 different theoretical models for their wave speeds in the long wavelength limit and broadly concluded that except for four models that failed to satisfy test conditions considered, all of the remaining models are in a qualitative agreement with the experimental results. In their study, comparisons were made only for wave speeds, mainly in the long wavelength limit. Romack and Weaver7 conducted Monte Carlo simulations for one- and two-dimensional wave propagations in a random lattice composed of a large number of simple oscillators and concluded that the multiple scattering is considered in the same degree of accuracy both in the quasicrystalline approximation (QCA) and the coherent potential approximation (CPA). The Monte Carlo simulations for multiple scattering of electromagnetic waves have been performed to compare the QCA and the combined QCA-CPA methods.8 Berryman11 examined three single scattering approximations—the average T-operator approximation, the CPA, and the differential effective medium, for the coefficients in the Biot’s poroelasticity equations. Martin1 theoretically compared several multiple scattering models.

In this paper, a comparative study is conducted for eight existing models for predicting the dynamic effective properties (the effective wave speed and coherent attenuation) of two-dimensional random composites. Models considered include the models of Waterman and Truell,6 Lloyd and Berry,12 Varadan et al.,13 Kanaun and Levin,14 Sabina and Willis,15 Kim,16 Beltzer and Brauner,17 and Yang and Mal.18 A brief review with comments is given for each model. The models are tested for weak scattering, quasistatic wave speed, and low frequency attenuation. The effects of micro-

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structure on the wave speed are also considered. For composites having quite different dynamic characteristics, the calculated wave speeds and attenuation factors are compared for high fiber volume concentrations and in a wide frequency range. The judgments are based on some pertinent theoretical and physical criteria. When a sound physical criterion is not available, relative coincidence among the calculated results is used; however, no generalization is made beyond the test conditions.

II. MODELS

A. Multiple-scattering theories

Foldy\(^4\) and Lax\(^5\) developed a probabilistic theory for multiple scattering of waves by randomly distributed point scatterers in which the scattered field is ensemble averaged with the probability distributions for their absolute and relative positions. These configuration averages result in an infinite hierarchy of integral equations in which the average scattered field with \(n\) scatterers is factored in terms of the average scattered field with \(n+1\) scatterers fixed. To truncate the hierarchy, Foldy\(^4\) assumed that the average exciting field near a fixed scatterer is approximately equal to the average total field near that scatterer; thus, the first-order single scattering is considered. Lax\(^5\) proposed the QCA in which the average exciting field with two scatterers fixed is approximately equal to the average exciting field with one scatterer fixed. Since the total scattering field is described as a collection of singly and doubly scattered fields in this approximation, the probability distribution for the relative positions of two scatterers, that is, the pair-correlation function (PCF) suffices in the analysis. The QCA has been widely used in the analyses of finite-size scatterer systems.

Many authors attempted to extend the theory of Foldy and Lax to the case in which scatterers are finite sized and thus spatial correlations among the scatterers are important. Waterman and Truell\(^6\) obtained a formula for the effective wavenumber for three-dimensional finite-sized scatterers using the Foldy’s approximation and therefore, ignoring the correlation between scatterers. A two-dimensional version of the Waterman–Truell formula is

\[
k^2_v = k^2_1 - 4in_nf(0) - \frac{4n^2}{k^2_1} \left[ f'(0) - f'(\pi) \right],
\]

where \(k_1\) and \(k_v\) are the wavenumbers associated with the wave speeds in the matrix and the effective medium, \(i\) is the imaginary unit, \(n_f\) is the number of scatterers in the unit area, and \(f(\theta)\) is the directional scattering amplitude; \(f(0)\) and \(f(\pi)\) are the forward and backward scattering amplitudes. In their almost forgotten paper, Lloyd and Berry\(^12\) pointed out that the second-order term \((n_f^2)\) in the Waterman–Truell formula is incorrect and proposed a corrected one for three-dimensional problems. Recently, Linton and Martin\(^10\) confirmed the Lloyd–Berry formula and also derived the formula for two dimensions.

\[
k^2_v = k^2_1 - 4in_0f(0) + \frac{8n^2}{\pi k^2_1} \int_0^\pi \cot(\theta/2) \frac{d}{d\theta} [f'(\theta)]^2 d\theta. \quad (2)
\]

These two formulas are extensions of Foldy’s formula \(k^2_v = k^2_1 - 4in_0f(0)\) to finite-sized scatterers. Fikioris and Waterman, starting from the earlier work, made the so-called hole corrections in which the volume around the fixed scatterer where other scatterers cannot reside is excluded from the integral for average excitation field. This meets the requirement of the impenetrability between scatterers but it is valid at a low concentration of small scatterers.

Bose and Mal,\(^21\) using the wave function expansion technique, derived an integral for averaged wave field with two scatterers fixed and then applied the Lax’s QCA and extinction theorem\(^5\) to obtain a secular (or dispersion) equation for the unknown effective wavenumber,

\[
|\delta_{mm} - 8v_2T_mF_{n-m}| = 0, \quad \text{for} \quad m, n \in \{-\infty, \infty\}, \quad (3)
\]

where \(\delta_{mm}\) is the Kronecker delta and \(v_2\) is the fiber volume concentration,

\[
T_m = \frac{\mu_2k_2J_m(k_2a)J_m(k_2a) - \mu_1k_1J_m(k_1a)J_m(k_1a)}{\mu_1k_1J_m(k_2a)H_n(k_2a) - \mu_2k_2J_m(k_1a)H_n(k_1a)}, \quad (4)
\]

\[
F_n = \frac{k_1J_n(2k_1a)H^n_n(2k_1a) - k_2J_n(2k_2a)H^n_n(2k_2a)}{2a(k^2_1 - k^2_v)} + \int_1^\infty (g(x)-1)H^n_n(2k_1ax)J_n(2k_vax)xdx. \quad (5)
\]

In Eq. (4), \(\mu_1\) and \(\mu_2\) are the shear moduli of the matrix and the fiber, respectively, \(a\) is the fiber radius, \(k_2\) is the wavenumber associated with the wave speed in the fiber, \(J_n(z)\) is the Bessel function of order \(n\), and \(H_n(z)\) is the \(n^\text{th}\) Hankel function of order \(n\). \(g(x)\) in Eq. (5) is the PCF for impenetrable scatterers (hard disks). Bose and Mal assumed an exponentially decaying PCF, while Varadan et al.\(^13\) used an exact PCF to improve the accuracy of the QCA at high frequencies and high volume concentrations. The exact PCFs can be obtained from the Monte Carlo simulation.\(^3,22,23\) In this theoretical context, the Bose and Mal formulation with the exact PCF, that is, the method of Varadan et al.,\(^13\) appears to be most accurate.

B. Effective medium theories

The effective medium theories are commonly based on the following hypothesis:\(^1,13\) Every inclusion in the composite behaves as an isolated inclusion embedded in a homogeneous medium with the effective properties of the composite. The field that acts on this inclusion is a plane wave propagating in the effective medium. While this hypothesis is hard to justify theoretically, it offers a simple way to calculate the effective properties; it reduces the original multiple-scattering problem defined in the host medium to a single scattering problem defined in the effective medium. These theories have been often very successful in many occasions.\(^24-26\)

A few variants of the effective medium theory exist, in which the roles of the host medium and inclusions are treated
in different ways. A simplest one is to embed both the matrix and inclusions in the effective medium having yet-unknown properties (self-consistent embedding) and then apply the self-consistency condition that the ensemble average of the total scattering vanishes in the effective medium. The average total scattering field is approximated as the average of the single scattering fields of the inclusion and matrix weighted by their volume concentrations. The conditions for the effective medium (the effective density \( \rho_e \) and the effective shear modulus \( \mu_e \)) of a two-dimensional composite medium are obtained:

\[
\sum_{p=1,2} v_p (\rho_e - \rho_p) \langle \Lambda^p_\rho \rangle_{\Omega_p} = 0, \\
\sum_{p=1,2} v_p (\mu_e - \mu_p) i k_e \cdot \langle \Lambda^p_\mu \rangle_{\Omega_p} = 0,
\]

where \( \Omega_p \) is the representative volume element of the \( p \)th kind of material. \( \langle \cdot \rangle_{\Omega_p} \) denotes the volume average of a physical quantity over \( \Omega_p \), and

\[
\langle \Lambda^p_\rho \rangle_{\Omega_p} = \frac{1}{\Omega_p} \int_{\Omega_p} u(r) e^{-i k_e r} d\Omega, \\
\langle \Lambda^p_\mu \rangle_{\Omega_p} = \frac{1}{\Omega_p} \int_{\Omega_p} \nabla u(r) e^{-i k_e r} d\Omega.
\]

Note that adding Eqs. (6) and (7) yields

\[
\sum_{p=1,2} v_p \langle \tilde{f}_p(0) \rangle_{\Omega_p} = 0,
\]

where the overbar means that the forward scattering amplitudes are obtained in the effective medium. Equation (10) gives the physical notion that the effective medium in this theory is defined such that the average forward scattering amplitudes in the effective medium vanishes, that is, a medium in which there is no scattering of the mean field by the constituents, which fulfills the presumed self-consistency. This approach is analogous to the CPA in the solid-state physics, and has been used also in the electromagnetic wave propagation problems. Berge et al. and Berryman used a similar approach for quasistatic properties of three-dimensional composites. Note that Eqs. (6) and (7) are in the symmetric form, indicating that this model treats the matrix and the fibers equally and thus assumes implicitly a microstructure in which the roles of the matrix and the fibers are not distinguishable (an aggregate or granular structure). Authors have criticized that the roles of the matrix and inclusions should be treated differently. Various aspects of this issue have been discussed.

Another effective medium model is the scheme proposed by Kanaun and Levin. In this scheme, the average field inside the inclusions is first estimated by solving the scattering problem for a single inclusion in the effective medium and then the inclusions having the average field are embedded in the original host medium and the scattered field by these inclusions is averaged. Finally, the self-consistent condition that the effective field is the plane wave propagating in the effective medium is applied to obtain the following equations for the effective shear modulus and density:

\[
\rho_e = \rho_1 + v_2 (\rho_2 - \rho_1) \langle \Lambda^e_\rho \rangle_{\Omega_e}, \\
\mu_e = \mu_1 - v_2 (\mu_2 - \mu_1) \langle \Lambda^e_\mu \rangle_{\Omega_e}/k_e^2.
\]

Note that this model is a dynamic generalization of the static self-consistent theory of Hill and Budiansky. Kanaun and Levin also showed that these equations are reduced to the two-dimensional version of Sabina and Willis’ self-consistent equations under an appropriate approximation to the field inside the inclusion:

\[
\rho_e = \rho_1 + v_2 h^2(\mu_e - \mu_1)[1 + \tilde{M}(\rho_2 - \rho_1)]^{-1}, \\
\mu_e = \mu_1 - v_2 h^2(\mu_2 - \mu_1)[1 + \tilde{S}(\mu_2 - \mu_e)]^{-1},
\]

where \( h(k_e) = 2 J_1(k_e a)/k_e a, \tilde{S} = i \pi H_1(k_e a) J_0(k_e a)/2 \mu_e, \) and \( \tilde{M} = (1 - i \pi H_1(k_e a) J_1(k_e a))/\rho_e \) are the average of the effective wave over the inclusion and the average convolution operators, respectively.

C. Dynamic generalized self-consistent model

A dynamic generalization of the generalized self-consistent model for the static effective properties has been set out by Yang and Mal. As in the static version, a concentric cylindrical fiber embedded in an infinite medium possessing unknown effective properties is considered. The ratio between two radii in the concentric cylinder is determined by the volume concentrations of constituents. The model uses the self-consistency that the incident wave propagating in the effective medium is equal to the effective wave field. This self-consistency is implemented in the dispersion relation of Waterman and Truell. Note that adding Eqs. (6) and (7) yields

\[
\sum_{p=1,2} v_p \langle \tilde{f}_p(0) \rangle_{\Omega_p} = 0,
\]

which means that the forward scattering amplitude vanishes when the unknown surrounding medium has the effective properties. This condition is similar to Eq. (10) and is found in many other effective medium theories. With Eq. (16) and the effective static density \( \rho_e = \nu_1 \rho_1 + \nu_2 \rho_2 \), the effective wavenumber is calculated. This model has attracted significant attention for the success of its static counterpart.

D. Differential approach

Beltzer and Brainerd proposed an incremental realization of the effective medium concept in a way similar to the differential effective medium theory for the effective static moduli. Since it is difficult to derive a set of differential equations for dynamic moduli and density, the method instead numerically implements the homogenization process.
The attenuation is calculated from the single scattering by a newly added small amount of inclusions in the medium homogenized in the previous step:

\[ \alpha(v_2 + \Delta v_2) = \alpha(v_2) + \frac{\Delta v_2}{\Omega_2} \gamma^\text{ext}, \]

where \( \Delta v_2 \) is the increment of volume concentration, \( \Omega_2 \) is the volume of the single inclusion, and \( \gamma^\text{ext} \) is the extinction cross section of the fiber in the energy-absorbing medium.\(^\text{37}\)

The current step is completed by calculating the frequency-dependent wave speed using the Kramers–Kronig relation:\(^\text{17}\)

\[
\frac{1}{c(\omega)} - \frac{1}{c(0)} = \frac{2\omega^2}{\pi} P \int_0^\infty \frac{\alpha(\sigma)}{\sigma^2(\sigma^2 - \omega^2)} d\sigma,
\]

where \( c(0) \) is the wave speed at \( \omega = 0 \) and the integral is a principal value integral. Note that since this model does not yield \( c(0) \), one should specify it as an input parameter. In the numerical calculations, \( c(0) \) is calculated with Hashin and Rosen’s\(^\text{38}\) effective shear modulus which corresponds to the lower bound. Note that the differential scheme assumes an isolated microstructure.\(^\text{36}\)

Finally, it should be noted that all these theories are approximate and this fact may be one fundamental reason for discrepancies among them which are shown in Sec. III.

### III. RESULTS AND DISCUSSION

Table I summarizes assumptions and microstructures, abbreviations, equations used in computations, and symbols in figures. Numerical calculations are performed for two-phase composites having much different combinations of constituent properties. The mechanical properties of constituent materials are listed in Table II and characteristics of the composites are in Table III. The effective wave speed and coherent wave attenuation are calculated for frequencies up to \( k_1a = 3.5 \) and for volume concentrations up to 47%. While any of these theories has been proven to be applicable at a frequency above \( k_1a = 1 \), there are many cases in which these theories successfully predict wave speeds and attenuations at high frequencies (e.g., Ref. 26). Therefore, it will be interesting to see how these models are compared in the high frequency region. In this study, the maximum frequency range is where the wavelength is comparable to the size (diameter) of the inclusions. At higher frequencies where the wavelength is much shorter than the size of scatters, the wave speed and attenuation reach their geometric optics limits that are constant values. The coherent dynamic response of composites at such high frequencies is negligible due to the high attenuation and thus is not experimentally measurable, thereby losing its practical meaning.

The directional scattering amplitude in Eqs. (1) and (2) is calculated from \( f(\theta) = 2T_0 + \sum_{m=1} T_m \cos(m\theta) \), and the effective wave speed and attenuation are obtained directly from the effective wave number. The determinant in Eq. (3) is truncated to a finite order (increasing with frequency) and the roots (the effective wave numbers) of the complex nonlinear equation are searched numerically. Equations (6) and (7) are a system of nonlinear equations for complex effective density and shear modulus. The multidimensional Newton–Raphson method is used to calculate these parameters. The same procedure is used for computations of Eqs. (11) and (12) and Eqs. (13) and (14). Equation (16) is solved using two-dimensional (real and imaginary parts of \( k_1 \)) Newton–Raphson method. Equations (17) and (18) are calculated incrementally with a small \( \Delta v_2 \). The extinction cross section is calculated using the method of Kim,\(^\text{37}\) and then the integral is evaluated using the Simpson rule with a high frequency limit \( k_1a = 20 \) above which the integrand is assumed to be

### TABLE II. Elastic properties of constituents.

<table>
<thead>
<tr>
<th>Materials</th>
<th>Density (kg/m(^3))</th>
<th>Shear modulus (GPa)</th>
<th>Phase velocity (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum</td>
<td>2720</td>
<td>38.7</td>
<td>3772</td>
</tr>
<tr>
<td>Steel</td>
<td>7800</td>
<td>80.9</td>
<td>3220</td>
</tr>
<tr>
<td>Graphite</td>
<td>1310</td>
<td>21.0</td>
<td>4004</td>
</tr>
<tr>
<td>Titanium</td>
<td>4510</td>
<td>41.4</td>
<td>3030</td>
</tr>
<tr>
<td>SiC (SCS-6)</td>
<td>3200</td>
<td>182.0</td>
<td>7542</td>
</tr>
<tr>
<td>Epoxy</td>
<td>1260</td>
<td>1.98–0.06i</td>
<td>1254</td>
</tr>
</tbody>
</table>

### TABLE III. Characteristics of composites.

<table>
<thead>
<tr>
<th>Composites</th>
<th>( p_2/p_1 )</th>
<th>( m_2/m_1 )</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel/aluminum</td>
<td>2.9</td>
<td>2.1</td>
<td>( p_2 &gt; p_1, ; m_2 &gt; m_1 )</td>
</tr>
<tr>
<td>Graphite/aluminum</td>
<td>0.48</td>
<td>0.54</td>
<td>( p_2 &lt; p_1, ; m_2 &lt; m_1 )</td>
</tr>
<tr>
<td>SiC/titanium</td>
<td>0.71</td>
<td>0.4</td>
<td>( p_2 &lt; p_1, ; m_2 &gt; m_1 )</td>
</tr>
<tr>
<td>Graphite/epoxy(^a)</td>
<td>1.04</td>
<td>10.6</td>
<td>( p_2 = p_1, ; m_2 &gt; m_1 )</td>
</tr>
</tbody>
</table>

\(^a\)Real part of the shear modulus of epoxy is used.
constant. All computations were validated by comparing with those numerical results presented in the original papers. The numerical calculations are performed within the accuracy of the double precision floating point.

A. Weak scattering

When the scattering of each fiber is weak and the fiber volume concentration is low, the multiple-scattering effect will be small. The condition for the weak scattering is that the properties of constituents are similar, that is, \( \rho_2/\rho_1 \sim 1 \) and \( \mu_2/\mu_1 \sim 1 \). Then, the total scattering field can be approximated as the sum of the single scattering field from every scatterer (the single scattering approximation). In this weak scattering limit, one may expect that all of these different multiple-scattering models would yield a single coincident result. This criterion will examine the soundness of fundamental assumptions in each model and the model’s capability of predicting the multiple-scattering effect in more complicated cases. A model which is incapable of correctly predicting this weak multiple-scattering effect would not predict correctly or even approximately a stronger multiple-scattering effect. Therefore, the weak scattering problem can be a good benchmark problem.

Figure 1 shows the normalized wave speeds \( (c_2/c_1) \) and the normalized attenuation coefficients \( 4\pi \text{Im}[k]/\text{Re}[k] \) versus the normalized frequency \( (k_1a) \), predicted by the different models. The material properties are assumed \( \rho_2/\rho_1=1.2 \) and \( \mu_2/\mu_1=1.1 \) and the fiber volume concentration is 0.07. The wave speeds from five models coincide strikingly well [Fig. 1(a)]—a definitive indication that this coincident wave speed is the true effective wave speed at least for this weak scattering case and for this particular composite. The SW2, YM, and WT2 models visibly deviate from the true wave speed. These models may attribute their failures to different causes. The SW2 model assumes a constant field inside the scatterer, which makes the effective wave speed to approach the wave speed in the matrix and the attenuation to vanish \( (c_2→c_1 \text{ and Im}[k]→0) \). As pointed out earlier by Lloyd and Berry and readdressed recently by Linton and Martin, the second-order terms in the dispersion relation of WT2 model is incorrect, which would lead to this inaccuracy. This fact implies an important conclusion that a model that is built based on the Waterman and Truell model will suffer from the same problem either in an explicit or implicit manner. The three-phase geometry in the YM model introduces excessive fluctuations in the dispersion curve and also in the attenuation shown below as the result of spurious wave motions in the annulus that represents the matrix phase (the structure effect), which obviously disappears in the static limit where wavelength is greater than any of the geometric parameters.

Figure 1(b) shows that four models predict very close attenuations, even though the agreement is not as good as in the wave speed. Consistently as in the wave speed, the SW2, YM, and WT2 models show attenuations more or less different from the attenuation agreed by the other four. The attenuation by the LB2 model is closer to the agreed one, demonstrating that the model is correct under the assumption of uncorrelated scatterers.

B. Wave speed in long wavelength limit

Figure 2 shows the wave speeds in a steel/aluminum composite in the low frequency limit \( (k_1a=0.05) \), predicted by different models. The thick lines are the upper and lower bounds of the effective wave speed. These are calculated with the average static density \( \rho_5=\rho_1(1-v_2)+\rho_2v_2 \) and the variational bounds for the axial shear modulus of Hashin. While the bound solution is not unique, Hashin’s variational bounds are tight and have a clear physical meaning. The interpretation of these bounds is closely related to the role of constituents: when a stiffer phase forms a more continuous phase, the effective stiffness will be close to the upper bound, and vice versa. While the assembly of impenetrable circular disks has its full packing limit at the fiber volume concentration 0.785, numerical calculations are shown up to 1. It is interesting to note that the models for point scatterers (WT2 and LB2) may be used to predict the wave speed at a high volume concentration, for example, up to 0.5 for LB2. In other words, the size and shape (e.g., noncircularity) of scatterers are unimportant for the volume concentration below 0.5 in the quasistatic limit. All other models for finite-sized scatterer fall in the upper and lower bounds. The YM, BB,
and VVM models predict the wave speed coincident with the lower bound, while wave speeds by the effective medium models (KM and KL) are close to the lower bound at low volume concentrations and to the upper bound at higher volume concentrations. The small departure of the BB model at high volume concentrations is possibly due to numerical errors (the volume increment may have been coarse).

C. Low frequency attenuation

Figure 3 shows the attenuation in the steel/aluminum composite versus the volume concentration calculated at frequency $k_1a=0.2$. Many interesting facts are observed.

At this frequency, the wavelength is about 15 times larger than the diameter of scatterers. So, the propagating wave interacts with a few scatterers at the same time and sees the average response of the matrix-scatterer assemblage. In this low frequency range, the coherent attenuation of propagating waves will be simply proportional to the degree of disorder or the spatial randomness. In view of the microstructure of a two-phase mixture, the spatial fluctuation in material properties will be maximum at a volume concentration around $v_2=0.5$, where the attenuation is accordingly expected to be maximum. Another consequence is the endpoint constraint that the coherent attenuation should vanish at $v_2=0$ and 1.0 where no macroscopic randomness exists. It is seen that only the effective medium models (SW2, KM, and KL) satisfy this constraint. The iterative homogenization process of building up the effective properties in the BB model does not exactly satisfy the constraint at $v_2=1.0$.

The VVM model predicts negative attenuations in the entire range of frequency, violating the passivity requirement—the material as a passive linear system. This is a quite surprising result since it is expected\(^1\) that QCA would give the same level of accuracy as CPA (the effective medium model). Moreover, the exact PCF is used to overcome this defect in the model without the exact one. The YM model produces small attenuation that decreases with volume concentration. This is the result of the geometric requirement for constructing the three-phase-model (again the structure effect). The outer radius of the concentric circle is determined by $v_2=(r_2/r_1)^2$. For a given size of the fiber ($r_2$), $r_1$ should decrease with the increase in volume concentration. Since the scattered power in the low frequency region is proportional to the fourth power of the scatterer size, the model predicts more incident power scatters at a low volume concentration than at a higher volume concentration. The SW2, WT2, and LB2 models will not be further examined hereafter.

D. Effects of microstructure

Figure 4 shows calculated and measured wave speeds in a unidirectional graphite/epoxy composite with a fiber volume concentration 53% and a fiber diameter 6 $\mu$m. For conducting ultrasonic measurements at different frequencies (2.5, 5.0, 7.5, and 10 MHz), composite plates with various thicknesses (2.5–8 mm) were prepared. The pulse-echo method using a single narrow band shear wave contact transducer with a radius of 3.2 mm was employed in which axi- ally polarized shear wave pulses propagate in the direction perpendicular to the fiber axis. Both the pulse overlap\(^{39}\) and phase spectrum\(^{40}\) techniques were used to accurately determine the wave speed from measured ultrasonic signals.

Microscopic study revealed that the arrangement of fibers varied from one location to the other. Among numerous different fiber arrangements, three distinctive were observed: First, most of fibers are isolated each other [Fig. 5(a)]; second, fibers are locally agglomerated forming several clusters of fibers [Fig. 5(b)]; third, the fibers are in contact, forming long chains and some chains are bridging across the sample thickness [Fig. 5(c)]. To study the effects of the microstructure on the wave speed, the measurements were performed at 28 different locations on the same plate.
samples. They interpreted that the assumed microstructure in predicting the ultrasonic speeds in the artificial sandstone that the self-consistent theory can lead to a meaningless conclusion. Berge et al. showed, and repeated about ten times at each location. The measured wave speeds were then averaged. Note that no data were discarded or selected as they were apart from or close to predetermined theoretical results.

In Fig. 4, open rhombuses are the wave speeds for seven representative locations (out of 28). Note that each open rhombus represents an average of ten wave speeds for one location and also that all 28 wave speeds fall in the range covered by the seven; only seven are shown for a better presentation. The absolute measurement error range for each frequency (and plate thickness) is calculated by averaging error ranges of all 28 locations and is indicated in Fig. 4 with a vertical bar next to the solid rhombus that represents the average of 28 wave speeds. It is seen that the variability range of the wave speed about 200 m/s is not from the measurement error, which is about 100 m/s, but it is thought to be due to the local variation in the microstructure. Calculated lower (1961 m/s) and upper (2673 m/s) bound wave speeds are also shown. As mentioned earlier, the VVM and YM models predict the lower bound while the effective medium models (KM and KL) predict a wave speed in the middle between the lower and upper bounds. In general, the measured wave speeds are closer to the lower bound, indicating that fibers in these composite samples are mostly isolated in the matrix. At the location where the wave speed is high, fibers are possibly more in contact with each other so that they form many channels of fiber clusters along the wave propagation direction [Fig. 5(c)]. A direct comparison between the ultrasonic result and the microstructure analysis was quite difficult because the ultrasonic waves cover a finite area while microscopy can be done for only few cross sections in that area. However, a few good correlations were observed that the wave speed is faster in the section with more contacting fibers. This is somewhat analogous to the critical phenomenon of the insulator-conductor transition in electrical properties of this composite, while the transition in the elastic property is very small. This experimental result demonstrates that the overall properties of a composite are sensitive to the microstructural arrangement of constituents. Therefore, a blind comparison between experimental and theoretical results ignoring the composite’s microstructure can lead to a meaningless conclusion. Berge et al. showed that the self-consistent theory (effective medium model) best predicts the ultrasonic speeds in the artificial sandstone samples. They interpreted that the assumed microstructure in the self-consistent theory is compatible with the actual microstructure of the sample materials.

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exhibits excessive undulation. The wave speed and attenuation in this composite converge quickly to their geometric optics limits after the wave speed rises rapidly and the attenuation reaches a peak at the low frequency where the rigid-body resonance of particles occurs. This is a common characteristic of composites with heavy inclusions, which determines the shape of the dispersion and attenuation. It is thought that the relative good agreement comes from this strong characteristic of the single fiber scattering that dominates over the multiple-scattering effect at this volume concentration.

The wave speed and attenuation in a graphite/aluminum composite having a fiber volume concentration of 0.47 are shown in Fig. 8. This material exhibits a weak dispersion and oscillatory attenuation. Near zero (YM model) and negative (VVM model) attenuations are seen in $k_1a<0.75$.

IV. CONCLUSIONS

Many interesting facts are found in this study. First of all, the results for the weak scattering critically compare the soundness of fundamental assumptions in the models considered. Whenever the point scattering approximation is relevant, the LT2 model is more accurate than the WT2 model. This needs to be emphasized because many authors take the WT2 model as a reference for validating their results and also attempt to extend the model to more general cases. The layered three-phase geometry in the YM model produces undesirable structure effects in the finite frequency regime. The approximate consideration of the double scattering through the QCA using the exact PCF appears to be not accurate enough when the constituent densities differ by more than two times. The two effective medium models (KM and KL), possibly due to the common hypothesis and the similar self-consistency condition, predict values close to each other, and do not exhibit an apparent failure in all cases considered. The assumed granular microstructure enables these models to adapt to geometrically more complex microstructures such as clustered fibers and aggregated mixture of phases. Since the overall properties are sensitive to the composite’s microstructure, the microstructure should first be investigated before choosing a model for comparing experimental results. In all cases considered, the disagreement in attenuation is more pronounced than in wave speed. Recalling that the attenuation is more sensitive to the composite’s microstructure, the disagreements should be, to a large degree, due to the different microstructures assumed in these different models. While the subject of the wave propagation in inhomogeneous media has long been investigated, the present evaluation study reveals numerous fundamental questions to be answered. The indirect method such as the effective medium theory might be a choice for further studies on this subject.


Jin-Yeon Kim: Evaluation of multiple-scattering models