ON FUZZY $h$-IDEALS OF HEMIRINGS

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Abstract The concept of quasi-coincidence of a fuzzy interval value in an interval valued fuzzy set is considered. In fact, this concept is a generalized concept of the quasi-coincidence of a fuzzy point in a fuzzy set. By using this new concept, the authors define the notion of interval valued $(\in, \in \lor q)$-fuzzy $h$-ideals of hemirings and study their related properties. In addition, the authors also extend the concept of a fuzzy subgroup with thresholds to the concept of an interval valued fuzzy $h$-ideal with thresholds in hemirings.

Key words Hemirings, $h$-ideals, interval valued $(\in, \in \lor q)$-fuzzy $h$-ideals, interval valued fuzzy $h$-ideals with thresholds.

1 Introduction

The theory of fuzzy sets was first developed by Zadeh[1] and has been applied to many branches in mathematics. The fuzzification of algebraic structures was initiated by Rosenfeld[2] and he introduced the notion of fuzzy subgroups. The readers are referred to [3] for some basic definitions and results on fuzzy sets and fuzzy algebras, not given in this paper. In 1975, Zadeh[4] introduced the concept of interval valued fuzzy subset, where the values of the membership functions are intervals of numbers instead of the numbers. Later on, Biswas[5] defined the interval valued fuzzy subgroups with the same nature of Rosenfeld’s fuzzy subgroups. A new kind of fuzzy subgroup, that is, the $(\in, \in \lor q)$-fuzzy subgroup, was then introduced by Bhakat and Das[6] by using the combined notions of “belongingness” and “quasicoincidence” of fuzzy points and fuzzy sets proposed by Pu and Liu in [7]. In fact, the $(\in, \in \lor q)$-fuzzy subgroup is a significant generalization of Rosenfeld’s fuzzy subgroup. The concept has been further studied in [8–11]. In particular, the generalizations of Rosenfeld’s fuzzy group and also Bhakat and Das’s fuzzy subgroup were given in [12].

The fuzzy algebraic structures play a prominent role in mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces etc., for example, the reader is referred to [13–15]. In algebra, we notice that the ideals of semirings play a crucial role in the structure theory, but they do not in general coincide with the usual ring ideals if $S$ is a ring and, for this reason, their usage is somewhat limited when we try to obtain some analogous ring theorems for semirings. Indeed, many results in rings apparently have no analogues in semirings by using only ideals. In this aspect, Henriksen defined in [16] a more restricted class of ideals in

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semirings, which is called the class of $k$-ideals with the property that if the semiring $S$ is a ring then a complex in $S$ is a $k$-ideal if and only if it is a ring ideal. Another more restricted class of ideals was considered by Iizuka\cite{17} in hemirings. However, in an additively commutative semiring $S$, the ideals of a semiring coincide with the usual “ideals” of a ring, provided that the semiring is a hemiring. We now call this kind of ideal an $h$-ideal of the hemiring $S$. The properties of the $h$-ideals and also the $k$-ideals of hemirings were thoroughly investigated by D. R. La Torre in \cite{18} and by using the $h$-ideals and $k$-ideals, D. R. La Torre established some analogous ring theorems for hemirings.

Recently, Zhan et al.\cite{19} gave a characterization of an $h$-hemiregular hemiring in terms of the fuzzy $h$-ideals and some useful properties of fuzzy $h$-ideals. Some other important results were also shown in \cite{20–28}.

In this paper, the concept of quasi-coincidence of a fuzzy interval value in an interval valued fuzzy set is given. This concept is a generalization of the quasi-coincidence of a fuzzy point in a fuzzy set. By using this new idea, the notion of interval valued ($\in, \in \vee q$)-fuzzy $h$-ideals of hemirings is provided. In fact, this concept is a generalization of a fuzzy $h$-ideal of hemirings. In particular, we extend the concept of a fuzzy subgroup with thresholds to the concept of an interval valued fuzzy $h$-ideal with thresholds in hemirings.

2 Preliminaries

Recall that a semiring is an algebraic system $(S, +, \cdot)$ consisting of a non-empty set $S$ together with two binary operations on $S$, called addition and multiplication (denoted in the usual manner) such that $(S, +)$ and $(S, \cdot)$ are semigroups and the following distributive laws

\[ a(b + c) = ab + ac \quad \text{and} \quad (a + b)c = ac + bc \]

are satisfied for all $a, b, c \in S$.

By a zero of a semiring $(S, +, \cdot)$, we mean an element $0 \in S$ such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. A semiring with zero and a commutative semigroup $(S, +)$ is called a hemiring.

A left ideal of a semiring is a subset $A$ of $S$ closed with respect to the addition and such that $SA \subseteq A$. A left ideal $A$ of a hemiring $S$ is called a left $h$-ideal if for any $x, z \in S$ and $a, b \in A$ from $x + a + z = b + z$ it follows $x \in A$. A right $h$-ideal is defined analogously.

A fuzzy set $F$ of a semiring $S$ is called a fuzzy left ideal if for all $x, y \in S$ we have (i) $F(x + y) \geq \min\{F(x), F(y)\}$; (ii) $F(xy) \geq F(y)$. A fuzzy right ideal is defined analogously.

**Definition 1** A fuzzy left ideal $F$ of a hemiring $S$ is called a fuzzy left $h$-ideal if for all $a, b, x, z \in S$, $x + a + z = b + z \implies F(x) \geq \min\{F(a), F(b)\}$.

A fuzzy right $h$-ideal is defined similarly.

For any fuzzy set $F$ of $S$ and $t \in (0, 1]$, the set $U(F; t) = \{x \in S | F(x) \geq t\}$ is called a level subset of $S$.

**Theorem 1** A fuzzy set $F$ in a hemiring $S$ is a fuzzy left (resp. right) $h$-ideal of $S$ if and only if each non-empty level subset $U(F; t)(\neq \emptyset)$ is a left (resp. right) $h$-ideal of $S$.

A fuzzy set $F$ of a hemiring $S$ of the form

\[ F(y) = \begin{cases} t \ (\neq 0), & \text{if } y = x, \\ 0, & \text{if } y \neq x \end{cases} \]

is said to be fuzzy point with support $x$ and value $t$ and is denoted by $U(x; t)$. A fuzzy point $U(x; t)$ is said to belong to (resp. be quasi-coincident with) a fuzzy set $F$, written as $U(x; t) \in F$. 
(resp. \( U(x; t)qF \)) if \( F(x) \geq t \) (resp. \( F(x) + t > 1 \)). If \( U(x; t) \in F \) or (resp. and) \( U(x; t)qF \), then we write \( U(x; t) \in \forall q \) (resp. \( \in \forall q \)) \( F \). The symbol \( \in \forall q \) means \( \in \forall q \) does not hold. Using the notion of “belongingness (\( \in \))” and “quasi-coincidence (\( q \))” of fuzzy points with fuzzy subsets, the concept of \((\alpha, \beta)-\)fuzzy subsemigroup, where \( \alpha \) and \( \beta \) are any two of \( \{\epsilon, \eta, \in \forall q, \in \wedge q\} \) with \( \alpha \neq \in \wedge q \), was introduced in [6]. It is noteworthy that the most viable generalization of Rosenfeld's fuzzy subgroup is the notion of \((\epsilon, \in \forall q)-\)fuzzy subgroup. For information of \((\epsilon, \in \forall q)-\)fuzzy subgroup, the reader is referred to [8].

By an interval number \( \tilde{a} \), we mean an interval \([a^-, a^+]\), where \( 0 \leq a^- \leq a^+ \leq 1 \). The set of all interval numbers is denoted by \( D[0, 1] \). The interval \([a, a]\) can be identified by the number \( a \in [0, 1] \).

For interval numbers \( \tilde{a}_i = [a^-_i, a^+_i] \in D[0, 1] \), \( i \in I \), we define
\[
\begin{align*}
\text{rmax}\{\tilde{a}_i, b_i\} &= [\max(a^-_i, b^-_i), \max(a^+_i, b^+_i)], \\
\text{rmin}\{\tilde{a}_i, b_i\} &= [\min(a^-_i, b^-_i), \min(a^+_i, b^+_i)], \\
\text{rinf}_{a_i} &= \left( \bigwedge_{i \in I} a^-_i, \bigwedge_{i \in I} a^+_i \right), \\
\text{rsup}_{a_i} &= \left( \bigwedge_{i \in I} a^-_i, \bigwedge_{i \in I} a^+_i \right)
\end{align*}
\]
and put
1) \( \tilde{a}_1 \preceq \tilde{a}_2 \iff a^-_1 \leq a^-_2 \) and \( a^+_1 \leq a^+_2 \),
2) \( \tilde{a}_1 \preceq \tilde{a}_2 \iff a^-_1 = a^-_2 \) and \( a^+_1 = a^+_2 \),
3) \( \tilde{a}_1 < \tilde{a}_2 \iff \tilde{a}_1 \preceq \tilde{a}_2 \) and \( \tilde{a}_1 \neq \tilde{a}_2 \),
4) \( k\tilde{a} = [ka^-, ka^+] \), whenever \( 0 \leq k \leq 1 \).

It is clear that \((D[0, 1], \leq, \vee, \wedge)\) is a complete lattice with \( 0 = [0, 0] \) as the least element and \( 1 = [1, 1] \) as the greatest element.

By an interval valued fuzzy set \( F \) on \( X_{[4]} \), we mean the set
\[
F = \{(x, [\mu^-_F(x), \mu^+_F(x)]) \mid x \in X\},
\]
where \( \mu^-_F \) and \( \mu^+_F \) are two fuzzy subsets of \( X \) such that \( \mu^-_F(x) \leq \mu^+_F(x) \) for all \( x \in X \). Putting \( \mu^-_F(x) = [\mu^-_F(x), \mu^+_F(x)] \), we see that \( F = \{(x, \mu^-_F(x)) \mid x \in X\} \), where \( \mu^-_F : X \rightarrow D[0, 1] \).

### 3 Interval Valued \((\epsilon, \in \forall q)-\)Fuzzy h-Ideals

Based on the results in [8–11], we can extend the concept of quasi-coincidence of fuzzy point in a fuzzy set to the concept of quasi-coincidence of a fuzzy interval value in an interval valued fuzzy set as follows.

An interval valued fuzzy set \( F \) of a hemiring \( S \) of the form
\[
\hat{\mu}_F(y) = \begin{cases} 
\hat{I} (\neq 0), & \text{if } y = x, \\
0, & \text{if } y \neq x
\end{cases}
\]
is said to be fuzzy interval value with support \( x \) and interval value \( \hat{I} \) and is denoted by \( U(x; \hat{I}) \).

A fuzzy interval value \( U(x; \hat{I}) \) is said to belong to (resp. be quasi-coincident with) an interval valued fuzzy set \( F \), written as \( U(x; \hat{I}) \in F \) (resp. \( U(x; \hat{I})qF \)) if \( \hat{\mu}_F(x) \geq \hat{I} \) (resp. \( \hat{\mu}_F(x) + \hat{I} > 1 \)). If \( U(x; \hat{I}) \in F \) or (resp. and) \( U(x; \hat{I})qF \), then we write \( U(x; \hat{I}) \in \forall q \) (resp. \( \in \wedge q \)) \( F \). The symbol \( \in \forall q \) does not hold.

In what follows, \( S \) is a hemiring unless otherwise stated. Also we emphasize the following properties: \( [\mu^-_F(x), \mu^+_F(x)] < 0.5 \) or \( 0.5 \leq [\mu^-_F(x), \mu^+_F(x)] \), for all \( x \in S \).
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We first extend the concept of fuzzy h-ideals to the concept of interval valued fuzzy h-ideals of S as follows:

**Definition 2** An interval valued fuzzy set F of S is said to be an interval valued fuzzy left (resp. right) ideal of S if for all x, y ∈ S, the following properties hold:

(F1) \( \tilde{\mu}_F(x + y) \geq \min\{\tilde{\mu}_F(x), \tilde{\mu}_F(y)\} \);
(FFF1) \( \tilde{\mu}_F(xy) \geq \tilde{\mu}_F(y) \) (resp. \( \tilde{\mu}_F(xy) \geq \tilde{\mu}_F(x) \)).

Furthermore, F is called an interval valued fuzzy left (resp. right) h-ideal of S if F is an interval valued fuzzy left (resp. right) ideal of S and

(FFF1) for all \( a, b, x, z \in S \),

\[ x + a + z = b + z \rightarrow \tilde{\mu}_F(x) \geq \min\{\tilde{\mu}_F(a), \tilde{\mu}_F(b)\}. \]

Let F be an interval valued fuzzy set. For every \( t \in (0, 1] \), the set \( U(F; \tilde{t}) = \{x \in S | \tilde{\mu}_F(x) \geq t\} \) is called the level subset of F.

Now, we characterize interval valued fuzzy h-ideals by their level h-ideals.

**Theorem 2** An interval valued fuzzy set F of S is an interval valued fuzzy left (resp. right) h-ideal of S if and only if for any \( 0 < \tilde{t} \leq 1 \), \( U(F; \tilde{t}) \neq \emptyset \) is a left (resp. right) h-ideal of S.

**Proof** The proof is similar to that of Theorem 1 and is therefore omitted here.

We now introduce the following concept:

**Definition 3** An interval valued fuzzy set F of S is said to be an interval valued \((\epsilon, \in \forall q)\)-fuzzy h-ideal of S if for all \( t, r \in (0, 1] \) and \( x, y \in S \),

(F2) \( U(x; \tilde{t}) \in F \) and \( U(y; \tilde{r}) \in F \) imply \( U(x + y; \min\{\tilde{t}, \tilde{r}\}) \in qF \);
(FFF2) \( U(x; \tilde{r}) \in F \) implies \( U(xy; \tilde{r}) \in qF \) (resp. \( U(y; \tilde{r}) \in qF \)).

Moreover, F is called an interval valued \((\epsilon, \in \forall q)\)-fuzzy left (resp. right) h-ideal of S if F is an interval valued \((\epsilon, \in \forall q)\)-fuzzy left (resp. right) ideal of S and

(FFF2) \( U(a; \tilde{t}) \in F \) and \( U(b; \tilde{r}) \in F \) imply \( U(x; \min\{\tilde{t}, \tilde{r}\}) \in qF \), for all \( a, b, x, z \in S \) with \( x + a + z = b + z \).

Observe that if F is an interval valued fuzzy h-ideal of S according to Definition 2, then F is an interval valued \((\epsilon, \in \forall q)\)-fuzzy h-ideal of S according to Definition 3. However, we notice that the converse statement is not true in general.

**Example 1** The interval valued fuzzy set

\[ \tilde{\mu}_F(x) = \begin{cases} 1, & \text{if } x \in \{4\}, \\ 0.5, & \text{if } x \in \{2\} - \{4\}, \\ 0, & \text{otherwise}, \end{cases} \]

defined on a hemiring \( (N_0, +, \cdot) \), where \( N_0 \) is the set of all non-negative integers, is an interval valued \((\epsilon, \in \forall q)\)-fuzzy left h-ideal of \( N_0 \).

**Theorem 3** The conditions of (F2), (FF2) and (FFF2) in Definition 3, are equivalent to the following conditions respectively:

(F3) \( \tilde{\mu}_F(x + y) \geq \min\{\tilde{\mu}_F(x), \tilde{\mu}_F(y), 0.5\} \), for all \( x, y \in S \);
(FF3) \( \tilde{\mu}_F(xy) \geq \min\{\tilde{\mu}_F(y), 0.5\} \) (resp. \( \tilde{\mu}_F(xy) \geq \min\{\tilde{\mu}_F(x), 0.5\} \)), for all \( x, y \in S \);
(FFF3) for all \( a, b, x, z \in S \), \( x + a + z = b + z \) implies

\[ \tilde{\mu}_F(x) \geq \min\{\tilde{\mu}_F(a), \tilde{\mu}_F(b), 0.5\}. \]

**Proof** (F2) \( \implies \) (F3): Suppose that \( x, y \in S \), we consider the following cases:

(a) \( \min\{\tilde{\mu}_F(x), \tilde{\mu}_F(y)\} < 0.5 \); (b) \( \min\{\tilde{\mu}_F(x), \tilde{\mu}_F(y)\} \geq 0.5 \).
Case (a): Assume $\overline{\mu}_F(x + y) < \min\{\overline{\mu}_F(x), \overline{\mu}_F(y), 0.5\}$. Then, $\overline{\mu}_F(x + y) < \min\{\overline{\mu}_F(x), \overline{\mu}_F(y)\}$. Choose $t$ such that $\overline{\mu}_F(x + y) < t < \min\{\overline{\mu}_F(x), \overline{\mu}_F(y)\}$. Then $U(x; t) \in F$ and $U(y; t) \in F$, but $U(x + y; t) \in \overline{\nu}qF$, this contradicts (F2).

Case (b): Assume that $\overline{\mu}_F(x + y) < 0.5$. Then $U(x; 0.5) \in F$ and $U(y; 0.5) \in F$, but $U(x + y; 0.5) \in \overline{\nu}qF$, a contradiction. Hence (F3) holds.

(FF2) $\implies$ (FF3): Suppose that $x, y \in S$. Now, we consider the following two cases:

(a) $\overline{\mu}_F(y) < 0.5$; (b) $\overline{\mu}_F(y) \geq 0.5$.

Case (a): Assume that $\overline{\mu}_F(y) = \tilde{t} < 0.5$ and $\overline{\mu}_F(xy) = \tilde{r} < \overline{\mu}_F(x)$. Choose $s$ such that $\tilde{r} < \tilde{s} < \tilde{t}$ and $\tilde{r} + \tilde{s} < 1$. Then $U(xy; \tilde{s}) \in F$, but $U(y; \tilde{s}) \in \overline{\nu}qF$, which contradicts (FF2). So

$$\overline{\mu}_F(xy) \geq \overline{\mu}_F(y) = \min\{\overline{\mu}_F(y), 0.5\}.$$ 

Case (b): Let $\overline{\mu}_F(y) \geq 0.5$. If $\overline{\mu}_F(xy) < \min\{\overline{\mu}_F(y), 0.5\}$, then $U(y; 0.5) \in F$, but $U(xy; 0.5) \in \overline{\nu}qF$, which contradicts (FF2). Hence (FF3) holds.

(FF2) $\implies$ (F3): Let $a, b, x, z \in S$ be such that $x + a + z = b + z$. We consider the following cases:

(a) $\min\{\overline{\mu}_F(a), \overline{\mu}_F(b)\} < 0.5$; (b) $\min\{\overline{\mu}_F(a), \overline{\mu}_F(b)\} \geq 0.5$.

Case (a): Assume $\overline{\mu}_F(x) < \min\{\overline{\mu}_F(a), \overline{\mu}_F(b)\}$. Then, we have $\overline{\mu}_F(x) < \min\{\overline{\mu}_F(a), \overline{\mu}_F(b)\}$. Choose $t$ such that $\overline{\mu}_F(x) < t \leq \min\{\overline{\mu}_F(a), \overline{\mu}_F(b)\}$. Then $U(a; t) \in F$ and $U(b; t) \in F$, but $U(x; t) \in \overline{\nu}qF$, which contradicts (FF2).

Case (b): Assume $\overline{\mu}_F(x) < 0.5$. Then $U(a; 0.5) \in F$ and $U(b; 0.5) \in F$, but $U(x; 0.5) \in \overline{\nu}qF$, a contradiction. Hence (F3) holds.

(F3) $\implies$ (F2): Let $U(x; t) \in F$ and $U(y; \tilde{t}) \in F$. Then $\overline{\mu}_F(x) \geq \tilde{t}$ and $\overline{\mu}_F(y) \geq \tilde{r}$. Now, we have

$$\overline{\mu}_F(x + y) \geq \min\{\overline{\mu}_F(x), \overline{\mu}_F(y), 0.5\} \geq \min\{\tilde{t}, \tilde{r}, 0.5\}.$$ 

If $\min\{\tilde{t}, \tilde{r}\} > 0.5$, then $\overline{\mu}_F(x + y) \geq 0.5$, which implies that $\overline{\mu}_F(x + y) + \min\{\tilde{t}, \tilde{r}\} > 1$. If $\min\{\tilde{t}, \tilde{r}\} \leq 0.5$, then $\overline{\mu}_F(x + y) \geq \min\{\tilde{t}, \tilde{r}\}$. Therefore, $U(x + y; \min\{\tilde{t}, \tilde{r}\}) \in \overline{\nu}qF$.

(FF3) $\implies$ (FF2): Suppose that $U(y; t) \in F$. Then $\overline{\mu}_F(y) \geq t$. For every $x \in S$, we have $\overline{\mu}_F(xy) \geq \min\{\overline{\mu}_F(y), 0.5\} \geq \min\{t, 0.5\}$, which implies that $\overline{\mu}_F(xy) \geq t$ or $\overline{\mu}_F(xy) \geq 0.5$, according to $t \leq 0.5$ or $t > 0.5$. Therefore, $U(xy; t) \in \overline{\nu}qF$.

(FF3) $\implies$ (F2): The proof of this part is similar to (F3) $\implies$ (F2).

By Definition 3 and Theorem 3, we immediately deduce the following corollary:

**Corollary 1** An interval valued set $F$ of $S$ is an interval valued $(\varepsilon, \in \overline{\nu}q)$-fuzzy left (resp. right) $h$-ideal of $S$ if and only if the conditions (F3), (FF3), and (FFF3) in Theorem 3 hold.

**Theorem 4** For any subset $A$ of $S$, the characteristic function $\chi_A$ of $A$ is an interval valued $(\varepsilon, \in \overline{\nu}q)$-fuzzy left $h$-ideal of $S$ if and only if $A$ is a left $h$-ideal of $S$.

**Proof** Assume that $A$ is a left $h$-ideal of $S$. Then it is easy to check that $\chi_A$ is an interval valued $(\varepsilon, \in \overline{\nu}q)$-fuzzy left $h$-ideal of $S$.

Conversely, assume that $\chi_A$ is an interval valued $(\varepsilon, \in \overline{\nu}q)$-fuzzy left $h$-ideal of $S$. Then for every $x, y \in A$, we have $\chi_A(x + y) \geq \min\{\chi_A(x), \chi_A(y), 0.5\} = 0.5$, and so $x + y \in A$. Also, for every $x \in A$ and $r \in S$, we have $\chi_A(rx) \geq \min\{\chi_A(x), 0.5\} = 0.5$, which implies $rx \in A$, and so $RA \subseteq A$. Hence $A$ is a left ideal of $S$. Finally, let $a, b, x, z \in S$ be such that $x + a + z = b + z$. Then $\chi_A(x) \geq \min\{\chi_A(a), \chi_A(b), 0.5\} = 0.5$, which implies $x \in A$. Therefore, $A$ is a left $h$-ideal of $S$.

Let $F$ and $G$ be interval valued fuzzy sets in $S$. Then the 0.5-product of $F$ and $G$ is defined by

$$F \overline{\circ_0.5} G(x) = \sup_{x + a + b = x + \overline{b_1} + \overline{b_2} + z}(\min\{\overline{\mu}_F(a_1), \overline{\mu}_F(a_2), \overline{\mu}_G(b_1), \overline{\mu}_G(b_2), 0.5\})$$.
and \((F \circ_{0.5} G)(x) = 0\) if \(x\) cannot be expressed as \(x + a_1b_1 + z = a_2b_2 + z\).

We now obtain the following theorem for the interval valued \((\in, \in \lor)\)-fuzzy left \(h\)-ideals of a hemiring \(S\).

**Theorem 5** If \(F\) and \(G\) are interval valued \((\in, \in \lor)\)-fuzzy left \(h\)-ideals of \(S\), then so is \(F \cap G\), where \(F \cap G\) is defined by \(\tilde{\mu}_{F \cap G}(x) = \min\{\tilde{\mu}_F(x), \tilde{\mu}_G(x)\}\), for all \(x \in S\). Moreover, if \(F\) and \(G\) are an interval valued \((\in, \in \lor)\)-fuzzy right \(h\)-ideal and an interval valued \((\in, \in \lor)\)-fuzzy left \(h\)-ideal of \(S\), respectively, then \(F \circ_{0.5} G \subseteq F \cap G\).

**Proof** For \(x, y \in S\), we have

\[
\tilde{\mu}_{F \cap G}(x + y) = \min\{\tilde{\mu}_F(x + y), \tilde{\mu}_G(x + y)\}
\]

\[
\geq \min\{\min\{\tilde{\mu}_F(x), \tilde{\mu}_F(y), 0.5\}, \min\{\tilde{\mu}_G(x), \tilde{\mu}_G(y), 0.5\}\}
\]

\[
= \min\{\min\{\tilde{\mu}_F(x), 0.5\}, \min\{\tilde{\mu}_G(x), 0.5\}\}
\]

\[
= \min\{\tilde{\mu}_{F \cap G}(x), \tilde{\mu}_{F \cap G}(y), 0.5\},
\]

\[
\tilde{\mu}_{F \cap G}(xy) = \min\{\tilde{\mu}_F(xy), \tilde{\mu}_G(xy)\}
\]

\[
\geq \min\{\min\{\tilde{\mu}_F(y), 0.5\}, \min\{\tilde{\mu}_G(y), 0.5\}\}
\]

\[
= \min\{\min\{\tilde{\mu}_F(y), 0.5\}, \min\{\tilde{\mu}_G(y), 0.5\}\}
\]

\[
= \min\{\tilde{\mu}_{F \cap G}(y), 0.5\}.
\]

Hence, \(F \cap G\) is an interval valued \((\in, \in \lor)\)-fuzzy left ideal of \(S\).

Let \(a, b, x, z \in S\) be such that \(x + a + z = b + z\). Then

\[
\tilde{\mu}_{F \cap G}(x) = \min\{\tilde{\mu}_F(x), \tilde{\mu}_G(x)\}
\]

\[
\geq \min\{\min\{\tilde{\mu}_F(a), 0.5\}, \min\{\tilde{\mu}_G(a), 0.5\}\}
\]

\[
= \min\{\min\{\tilde{\mu}_F(a), 0.5\}, \min\{\tilde{\mu}_G(a), 0.5\}\}
\]

\[
= \min\{\tilde{\mu}_{F \cap G}(a), 0.5\}.
\]

Therefore, \(F \cap G\) is an interval valued \((\in, \in \lor)\)-fuzzy left \(h\)-ideal of \(S\).

Now, let \(F\) and \(G\) be an interval valued \((\in, \in \lor)\)-fuzzy right \(h\)-ideal and an interval valued \((\in, \in \lor)\)-fuzzy left \(h\)-ideal of \(S\), respectively. Then, the proof is obvious if \((F \cap G)(x) = 0\). If otherwise, for every \(a_i, b_i \in S\), \(i = 1, 2\) satisfying \(x + a_1b_1 + z = a_2b_2 + z\), we have

\[
\tilde{\mu}_F(x) \geq \min\{\tilde{\mu}_F(a_1b_1), \tilde{\mu}_F(a_2b_2), 0.5\}
\]

\[
\geq \min\{\min\{\tilde{\mu}_F(a_1), 0.5\}, \min\{\tilde{\mu}_F(a_2), 0.5\}, 0.5\}
\]

\[
= \min\{\tilde{\mu}_F(a_1), \tilde{\mu}_F(a_2), 0.5\},
\]

since \(F\) is an interval valued \((\in, \in \lor)\)-fuzzy right \(h\)-ideal of \(S\), and

\[
\tilde{\mu}_G(x) \geq \min\{\tilde{\mu}_G(a_1b_1), \tilde{\mu}_G(a_2b_2), 0.5\}
\]

\[
\geq \min\{\min\{\tilde{\mu}_G(b_1), 0.5\}, \min\{\tilde{\mu}_G(b_2), 0.5\}, 0.5\}
\]

\[
= \min\{\tilde{\mu}_G(b_1), \tilde{\mu}_G(b_2), 0.5\},
\]

since \(G\) is also an interval valued \((\in, \in \lor)\)-fuzzy left \(h\)-ideal of \(S\). Thus, we have

\[
(F \circ_{0.5} G)(x) = \sup_{x + a_1b_1 + z = a_2b_2 + z} \left\{ \min\{\tilde{\mu}_F(a_1), \tilde{\mu}_F(a_2), \tilde{\mu}_G(b_1), \tilde{\mu}_G(b_2), 0.5\} \right\}
\]

\[
= \sup_{x + a_1b_1 + z = a_2b_2 + z} \left\{ \min\{\tilde{\mu}_F(a_1), \tilde{\mu}_F(a_2), 0.5\}, \min\{\tilde{\mu}_G(b_1), \tilde{\mu}_G(b_2), 0.5\} \right\}
\]

\[
\leq \inf\{\tilde{\mu}_F(x), \tilde{\mu}_G(x)\}
\]

\[
= (F \cap G)(x).
\]
Consequently, $F \circ_{0.5} G \subseteq F \cap G$.

We now proceed to characterize the interval valued $(\varepsilon, \in \lor q)$-fuzzy $h$-ideals by their level $h$-ideals.

**Theorem 6** Let $F$ be an interval valued $(\varepsilon, \in \lor q)$-fuzzy left $h$-ideal of $S$. Then for all $0 < \tilde{t} \leq 0.5$, $U(F; \tilde{t})$ is an empty set or a left $h$-ideal of $S$. Conversely, if $F$ is an interval valued fuzzy set of $S$ such that $U(F; \tilde{t})(\neq \emptyset)$ is a left $h$-ideal of $S$ for all $0 < \tilde{t} \leq 0.5$, then $F$ is an interval valued $(\varepsilon, \in \lor q)$-fuzzy left $h$-ideal of $S$.

**Proof** Let $F$ be an interval valued $(\varepsilon, \in \lor q)$-fuzzy left $h$-ideal of $S$, $0 < \tilde{t} \leq 0.5$, and $x, y \in U(F; \tilde{t})$. Then $\bar{\mu}_F(x) \geq \tilde{t}$ and $\bar{\mu}_F(y) \geq \tilde{t}$. Now we have

$$\bar{\mu}_F(x + y) \geq \text{rmin}\{\bar{\mu}_F(x), \bar{\mu}_F(y), 0.5\} \geq \text{rmin}\{\tilde{t}, 0.5\} = \tilde{t}.$$  

This implies that $x + y \in U(F; \tilde{t})$. Now, for every $x \in U(F; \tilde{t})$ and $r \in S$, we have

$$\bar{\mu}_F(rx) \geq \text{rmin}\{\bar{\mu}_F(x), 0.5\} \geq \text{rmin}\{\tilde{t}, 0.5\} = \tilde{t}.$$  

This implies that $rx \in U(F; \tilde{t})$, and so $RU(F; \tilde{t}) \subseteq U(F; \tilde{t})$. Hence, $U(F; \tilde{t})$ is a left ideal of $S$.

Conversely, let $F$ be an interval valued fuzzy set of $S$ such that $U(F; \tilde{t})(\neq \emptyset)$ is a left $h$-ideal of $S$ for all $0 < \tilde{t} \leq 0.5$. Then, for every $x, y \in S$, we can write

$$\bar{\mu}_F(x) \geq \text{rmin}\{\bar{\mu}_F(x), \bar{\mu}_F(y), 0.5\} = \tilde{s}_o,$$

and so $x \in U(F; \tilde{t})$. Therefore, $U(F; \tilde{t})$ is indeed a left $h$-ideal of $S$.

Finally, let $a, b, x, z \in S$ such that $x + a + z = b + z$. Then we can write

$$\bar{\mu}_F(a) \geq \text{rmin}\{\bar{\mu}_F(a), \bar{\mu}_F(b), 0.5\} = \tilde{s}_o, \quad \bar{\mu}_F(b) \geq \text{rmin}\{\bar{\mu}_F(a), \bar{\mu}_F(b), 0.5\} = \tilde{s}_o.$$  

Thus $a, b, x, z \in U(F; \tilde{s}_o)$, and so $x \in U(F; \tilde{s}_o)$ or $\bar{\mu}_F(x) \geq \tilde{s}_o$. Now, we have $\bar{\mu}_F(x) \geq \text{rmin}\{\bar{\mu}_F(a), \bar{\mu}_F(b), 0.5\}$. This completes the proof.

**Theorem 7** Let $F$ be an interval valued fuzzy set of $S$. Then $U(F; \tilde{t})(\neq \emptyset)$ is a left ideal of $S$ for all $0.5 < \tilde{t} \leq 1$ if and only if the following conditions hold:

1. $\text{rmax}\{\bar{\mu}_F(x + y), 0.5\} \geq \text{rmin}\{\bar{\mu}_F(x), \bar{\mu}_F(y)\}$;
2. $\text{rmax}\{\bar{\mu}_F(xy), 0.5\} \geq \bar{\mu}_F(y)$.

Moreover, $U(F; \tilde{t})(\neq \emptyset)$ is a left $h$-ideal of $S$ for all $0.5 < \tilde{t} \leq 1$ if and only if $F$ satisfies the above conditions and satisfies the following condition:

1. $\text{rmax}\{\bar{\mu}_F(x), 0.5\} \geq \text{rmin}\{\bar{\mu}_F(a), \bar{\mu}_F(b)\}$.

**Proof** Assume that $U(F; \tilde{t})(\neq \emptyset)$ is a left $h$-ideal of $S$. Suppose that for some $x, y \in S$, $\text{rmax}\{\bar{\mu}_F(x + y), 0.5\} \leq \text{rmin}\{\bar{\mu}_F(x), \bar{\mu}_F(y)\} = \tilde{t}$. Then $0.5 < \tilde{t} \leq 1$, $\bar{\mu}_F(x + y) < \tilde{t}$, and
$x, y \in U(F; \tilde{t})$. Since $x, y \in U(F; \tilde{t})$ and $U(F; \tilde{t})$ is a left $h$-ideal, we have $x + y \in U(F; \tilde{t})$ or $\tilde{\mu}_F(x + y) \geq \tilde{t}$, which contradicts $\tilde{\mu}_F(x + y) < \tilde{t}$. Hence, (F4) holds.

If there exist $x, y \in S$ such that $\max\{\tilde{\mu}_F(xy), 0.5\} < \tilde{\mu}_F(y) = \tilde{t}$, then $0.5 < \tilde{t} < 1$, $\tilde{\mu}_F(xy) < \tilde{t}$, and $y \in U(F; \tilde{t})$. Since $y \in U(F; \tilde{t})$, we have $xy \in U(F; \tilde{t})$ or $\tilde{\mu}_F(xy) \geq \tilde{t}$, which is a contradiction. Hence (FF4) holds.

Finally, there exist $a, b, x, z \in S$ such that $x + a + z = b + z$ and $\max\{\tilde{\mu}_F(x), 0.5\} < \min\{\tilde{\mu}_F(a), \tilde{\mu}_F(b)\} = \tilde{t}$. Then $0.5 < \tilde{t} \leq 1$, $\tilde{\mu}_F(x) < \tilde{t}$, and $a, b \in U(F; \tilde{t})$. Since $a, b \in U(F; \tilde{t})$, it follows that $x \in U(F; \tilde{t})$, and so $\tilde{\mu}_F(x) \geq \tilde{t}$, which is a contradiction. Hence (FFF4) holds.

Conversely, suppose that conditions (F4), (FF4), and (FFF4) hold. Now, we show that $U(F; \tilde{t})$ is a left $h$-ideal of $S$. For this purpose, we assume $0.5 < \tilde{t} < 1$, $x, y \in U(F; \tilde{t})$, and $a \in S$. Then

$$0.5 < \tilde{t} \leq \min\{\tilde{\mu}_F(x), \tilde{\mu}_F(y)\} \leq \max\{\tilde{\mu}_F(x + y), 0.5\} < \tilde{\mu}_F(x + y),$$

and so $x + y \in U(F; \tilde{t})$ and $ax \in U(F; \tilde{t})$. Hence, $U(F; \tilde{t})$ is a left ideal of $S$. Now, assume that $0.5 < \tilde{t} \leq 1$. Let $x, z \in S$ and $a, b \in U(F; \tilde{t})$ be such that $x + a + z = b + z$. Then $0.5 < \tilde{t} \leq \min\{\tilde{\mu}_F(a), \tilde{\mu}_F(b)\} \leq \max\{\tilde{\mu}_F(x), 0.5\} < \tilde{\mu}_F(x)$, which implies $x \in U(F; \tilde{t})$. Therefore, $U(F; \tilde{t})$ is a left $h$-ideal of $S$.

Let $F$ be an interval valued fuzzy set of a hemiring $S$ and $J = \{t \in (0, 1) \mid U(F; \tilde{t})$ is an empty set or a left $h$-ideal of $S\}$. In particular, if $J = (0, 1)$, then $F$ is an ordinary interval valued fuzzy left $h$-ideal of $S$ (Theorem 2); if $J = (0, 0.5)$, $F$ is an interval valued $(\in, \in \vee q)$-fuzzy left $h$-ideal of $S$ (Theorem 6).

In [12], Yuan et al. gave the definition of a fuzzy subgroup with thresholds which is a generalization of the Rosenfeld’s fuzzy subgroup, and also the Bhkat and Das’s fuzzy subgroup. Based on the method in [12], we can extend the fuzzy subgroup with thresholds to the fuzzy left $h$-ideals with thresholds in the following way:

**Definition 4** Let $\alpha, \beta \in [0, 1]$ and $\tilde{\alpha} < \tilde{\beta}$, then an interval valued fuzzy set $F$ of $S$ is called an interval valued fuzzy left $h$-ideal with thresholds $(\tilde{\alpha}, \tilde{\beta})$ of $S$ if for all $x, y \in S$, the following conditions hold:

\begin{align*}
(F5) & \quad \max\{\tilde{\mu}_F(x + y), \tilde{\alpha}\} \geq \min\{\tilde{\mu}_F(x), \tilde{\mu}_F(y), \tilde{\beta}\}, \\
(FF5) & \quad \max\{\tilde{\mu}_F(xy), \tilde{\alpha}\} \geq \min\{\tilde{\mu}_F(y), \tilde{\beta}\}.
\end{align*}

Moreover, $F$ is called an interval valued fuzzy left $h$-ideal with thresholds of $S$ if $F$ satisfies the above conditions and satisfies the following additional condition:

\begin{align*}
(FFF5) & \quad \text{Let } a, b, x, z \in S \text{ be such that } x + a + z = b + z. \text{ Then } \max\{\tilde{\mu}_F(x), \tilde{\alpha}\} \geq \min\{\tilde{\mu}_F(a), \tilde{\mu}_F(b), \tilde{\beta}\}.
\end{align*}

Now, we characterize the interval valued fuzzy left $h$-ideals with thresholds by using their level left $h$-ideals.

**Theorem 8** An interval valued fuzzy set $F$ of $S$ is an interval valued fuzzy left $h$-ideal with thresholds $(\tilde{\alpha}, \tilde{\beta})$ of $S$ if and only if $U(F; \tilde{t})(\neq \emptyset)$ is a left $h$-ideal of $S$ for all $\tilde{\alpha} < \tilde{t} \leq \tilde{\beta}$.

**Proof** The proof is similar to that of Theorems 6 and 7. □

4 Conclusions

In this paper, we consider a new kind of interval valued fuzzy left $h$-ideals of hemirings and investigate some of their related properties. Our results can be further applied to other algebraic structure. In our future research, we will focus on their applications in information sciences and general systems.
References