Distributed Proportional-Spatial Derivative Control Design for 3-Dimensional Parabolic PDE Systems

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Abstract—This paper addresses the problem of distributed proportional-spatial derivative (P-sD) control design for a class of linear distributed parameter systems (DPs) modeled by 3-dimensional parabolic partial differential equations (PDEs). Based on the Lyapunov direct method and the technique of integration by parts, the P-sD control design is developed in terms of standard linear matrix inequality (LMI) such that the closed-loop 3-dimensional PDE system is exponentially stable. Finally, numerical simulation result illustrates its effectiveness.

Keywords: Robust control; Proportional-spatial derivative (P-sD) control; Spatially distributed processes; Exponential stability; Linear matrix inequalities (LMIs).

I. INTRODUCTION

A significant research area, the control for distributed parameter systems (DPSs), has received a lot of attention over the past few decades. DPSs, modeled by partial differential equations (PDEs), can be applied to describe many industrial processes, such as thermal diffusion processes, fluid, and heat exchangers, etc [1]-[4]. With space distribution nature, the study of control design for parabolic PDE systems has theoretical and practical importance.

Many researchers have paid attention to the study on the stability control of PDE systems in the past few years, and many control design strategies have been presented [5]-[11]. More recently, [12] proposed distributed fuzzy proportional-spatial derivative (P-sD) control design for a class of nonlinear PDE systems, and [13] proposed distributed H∞ P-sD control design for a class of nonlinear PDE systems. However, the control design approach [5]-[13] for exponential stability dealing with 1-dimensional PDE systems, there are few studies devoted to these properties of high dimensional PDE systems.

This paper deals with the problems of the distributed P-sD control design for exponential stability of a linear 3-dimensional second order PDE system. Based on the Lyapunov direct method and the technique of integration by parts, a distributed P-sD controller is used to guarantee the stability of the PDE system. The condition is presented by a linear matrix inequality (LMI). Finally, a numerical example is given to illustrate the effectiveness of the proposed methods.

The remainder of this paper is organized as follows. The problem formulation and preliminaries are given in Section II. The P-sD control is designed in Section III. Section IV presents an example to illustrate the effectiveness of the proposed method. Finally, Section V offers the concludes.

Notations: The following notations will be used in this paper. ||·|| and ⟨·, ·⟩ℜn denote the Euclidean norm and inner product for vectors, respectively. A symmetric matrix $M > (≥, ≤, ≤, ≤, ≤) 0$ means that it is positive definite (semi-positive definite, negative definite, semi-negative definite, respectively). $H^0 ≜ L_2([l_1, l_2]; ℜ^n)$ is an infinite-dimensional Hilbert space of $n$-dimensional square integrable vector functions $θ(x, t) ∈ ℜ^n$, $x ∈ [l_1, l_2] ⊂ ℜ, ∀ t ≥ 0$ with the inner product and norm:

$$⟨θ_1(·, t), θ_2(·, t)⟩ = ∫_l_1^l_2 ⟨θ_1(x, t), θ_2(x, t)⟩ℜ^2 dx$$

and

$$∥θ_1(·, t)∥_2 = (⟨θ_1(·, t), θ_1(·, t)⟩)^{1/2}$$

where $θ_1(·, t)$ and $θ_2(·, t)$ are any two elements of $H^0$. For a space-varying matrix function $S(x) = [s_ij(x)]_{ij=1}^n$, let $∥S(x)∥_{F} ≜ \sqrt{∑_{i=1}^m ∑_{j=1}^n s_{ij}^2(x)}$, $x ∈ [l_1, l_2]$. The superscript ‘T’ and symbol ‘∗’ is used as:

$$\left[ \begin{array}{c|c} P + [M + N + ∗] & X \\ \hline Q & X^T \end{array} \right] ≜ \left[ \begin{array}{c} P + [M + N + M^T + N^T] \\ X \end{array} \right] X^T$$

II. PRELIMINARIES AND PROBLEM FORMULATION

We consider linear distributed parameter systems described by 3-dimensional parabolic PDEs of the following form:

$$T_t(x, y, z, t) = Θ_1 T_{xx}(x, y, z, t) + Θ_2 T_{yy}(x, y, z, t) + Θ_3 T_{zz}(x, y, z, t) + Θ_4 T_{xy}(x, y, z, t) + Θ_5 T_{yz}(x, y, z, t) + Θ_6 T_{zx}(x, y, z, t) + \beta(x, y, z, t) + F(x, y, z, t)$$

subject to the homogeneous Neumann boundary conditions:

$$\begin{cases} T(l_1, y, z, t) = T(l_2, y, z, t) = 0 \\ T(x, l_1, z, t) = T(x, l_2, z, t) = 0 \\ T(x, y, l_1, t) = T(x, y, l_2, t) = 0 \end{cases}$$

and the initial condition:

$$T(x, y, z, 0) = T_0(x, y, z)$$
where $T(x, y, z, t) \in \mathbb{R}^{n \times p \times q}$ is the state, the subscripts $x$ and $t$ stand for the partial derivatives with respect to $x$, $t$, respectively, $(x, y, z) \in \Omega \subset \mathbb{R}^{n \times p \times q}$ and $t \in [0, \infty)$. The position and time, respectively, and $u(x, y, z, t) \in \mathbb{R}^{n \times p \times q}$ is the control input. $\Theta_i \in \mathbb{R}^{n \times p \times q}$, $i = 1, 2, \ldots, 9$, $A \in \mathbb{R}^{n \times p \times q}$, $G \in \mathbb{R}^{n \times m}$, are known matrices.

Choose $\mathcal{H}^{n \times p \times q}$ as the state space and the trajectory segment $T(\cdot, \cdot, \cdot, t) = \{T(x, y, z, t), (x, y, z) \in \Omega\}$ as the state. Define the spatial differential operator $\mathcal{A}$ in $\mathcal{H}^{n \times p \times q}$ as

$$\mathcal{A}T(x, y, z) \triangleq \Theta_1 T_{xx}(x, y, z) + \Theta_2 T_{xy}(x, y, z) + \Theta_3 T_{yy}(x, y, z) + \Theta_4 T_{zz}(x, y, z)$$

with its domain $\mathcal{D}(\mathcal{A}) = \{T(x, y, z) \in \mathcal{H}^{n \times p \times q} : \tilde{T}_x(x, y, z), \tilde{T}_y(x, y, z), \tilde{T}_z(x, y, z), \tilde{T}_t(x, y, z)\}$ are absolutely continuous, $T_{xx}(x, y, z), T_{xy}(x, y, z), T_{yy}(x, y, z), T_{zz}(x, y, z) \in \mathcal{H}^{n \times p \times q}$. $\forall (x, y, z) \in \mathcal{D}(\mathcal{A})$, $T_x$ is defined as

$$T_x(x, y, z) |_{t=0} = \tilde{T}_x(x, y, z) \big|_{t=0} = \tilde{T}_t(x, y, z) \big|_{t=0} = 0.$$}

Let $\mathcal{H}$ be control input space and the trajectory segment $u(\cdot, \cdot, \cdot, t) = \{u(x, y, z, t), (x, y, z) \in \Omega\}$ as the control input. Then the state-space description of the model (1)-(3) can be rewritten as the following nonlinear abstract differential equation on the Hilbert space $\mathcal{H}$ [14]:

$$T_t(t) = \mathcal{A}T(t) + AT(t) + G_1u(t), t \geq 0, T(0) = T_0$$

where $\mathcal{A} \triangleq \mathcal{A}(\cdot, \cdot, \cdot)$. Since the operator $\mathcal{A}$ generates a $C_0$-semigroup on $\mathcal{H}^{n \times p \times q}$, by applying Theorem 1.4 of Chapter 6 in [14], we can easily prove the local existence of the unique mild solution to the system (5) with $u(t) = 0$. Using (5), we get

$$T_t(x, y, z, t) = \mathcal{A}T(x, y, z, t) + AT(x, y, z, t) + G_1u(x, y, z, t)$$

(6)

This study considers the following P-sD controller:

$$u(x, y, z, t) = K_1T_t(x, y, z, t) + K_2T_y(x, y, z, t) + K_3T_z(x, y, z, t)$$

(7)

where $K_i, i \in \{1, 2, 3, 4\}$ are real $m \times n$ matrices to be determined. The controller structure is shown in Fig. 1, in which the notation $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ means first-order spatial differentiators.

Substituting (4) into (1) leads to the following closed-loop systems:

$$T_t(x, y, z, t) = \mathcal{A}T(x, y, z, t) + AT(x, y, z, t) + GK_1T_t(x, y, z, t) + GK_2T_y(x, y, z, t) + GK_3T_z(x, y, z, t)$$

(8)

where $\hat{A} \triangleq A + GK_3$.

For simplicity, when $u(x, y, z, t) \equiv 0$, the PDE system (1)-(3) is referred to as an unforced PDE system. We introduce the following definition:

**Definition 1.** Given a constant $\rho > 0$. The unforced PDE system of (6)-(7) (i.e., $u(x, y, z, t) \equiv 0$) is said to be $\rho$-exponentially stable (ES) or ES with decay rate $\rho$, if there exists a constant $\sigma > 0$ such that the following inequality holds:

$$||T(\cdot, \cdot, \cdot, t)||^2 \leq \sigma e^{-(2\rho t)} ||T_0(\cdot, \cdot, \cdot, 0)||^2, \forall t \geq 0. \tag{9}$$

### III. CONTROL DESIGN

The aim of this section is to present a robust distributed P-sD feedback control design method to exponentially stabilize the parabolic PDE system (1)-(3) with nonlinear perturbation. Consider the following Lyapunov functional for the system (6)-(7),

$$V(t) = \int_\Omega T^T(x, y, z, t) PT(x, y, z, t)dx dy dz \tag{10}$$

where $P > 0$ is a real $n \times n$ matrix to be determined.

The time derivative of $V(t)$ along the solution of the closed-loop system (6)-(7) is given by

$$\dot{V}(t) = 2 \int_\Omega T^T(x, y, z, t) PT(x, y, z, t)dx dy dz$$

(11)

$$+ 2 \int_\Omega T^T(x, y, z, t) G_1^T \Theta_1 T_t dx dy dz$$

From the definition of operator $\mathcal{A}$ in (6) and considering (11), we can obtain

$$\int_\Omega T^T(x, y, z, t) \mathcal{A}T(x, y, z, t)dx dy dz$$

$$= \int_\Omega T^T(x, y, z, t) \Theta_1 T_{xx} dx dy dz$$

(12)
By integrating by parts and taking into account (3), we can find that

\[
\begin{aligned}
\iiint T(x, y, z, t) \mathcal{P}_1 T_x(x, y, z, t) dxdydz & = \iint_\Omega \int_\mathbb{R}^3 T(x, y, z, t) \mathcal{P}_2 T_x(x, y, z, t) dxdydz \\
& + \iiint_\Omega \int_\mathbb{R}^3 T(x, y, z, t) \mathcal{P}_3 T_x(x, y, z, t) dxdydz \\
& = \iiint_\Omega \int_\mathbb{R}^3 T(x, y, z, t) \mathcal{P}_3 T_x(x, y, z, t) dxdydz
\end{aligned}
\]

(13)

In the same way, we get

\[
\begin{aligned}
\iiint T(x, y, z, t) \mathcal{P}_2 T_y(x, y, z, t) dxdydz & = -\iiint T(x, y, z, t) \mathcal{P}_3 T_y(x, y, z, t) dxdydz \\
& + \iiint T(x, y, z, t) \mathcal{P}_4 T_y(x, y, z, t) dxdydz
\end{aligned}
\]

(14)

and

\[
\begin{aligned}
\iiint T(x, y, z, t) \mathcal{P}_3 T_z(x, y, z, t) dxdydz & = -\iiint T(x, y, z, t) \mathcal{P}_4 T_z(x, y, z, t) dxdydz \\
& + \iiint T(x, y, z, t) \mathcal{P}_5 T_z(x, y, z, t) dxdydz
\end{aligned}
\]

(15)

By integrating by parts,

\[
\begin{aligned}
\iiint T(x, y, z, t) \mathcal{P}_3 T_x(x, y, z, t) dxdydz & = \iint_\Omega \int_\mathbb{R}^3 T(x, y, z, t) \mathcal{P}_4 T_x(x, y, z, t) dxdydz \\
& - \iiint T(x, y, z, t) \mathcal{P}_6 T_x(x, y, z, t) dxdydz \\
& = -\iiint T(x, y, z, t) \mathcal{P}_6 T_x(x, y, z, t) dxdydz
\end{aligned}
\]

In the same way, we get

\[
\begin{aligned}
\iiint T(x, y, z, t) \mathcal{P}_4 T_y(x, y, z, t) dxdydz & = -\iiint T(x, y, z, t) \mathcal{P}_7 T_y(x, y, z, t) dxdydz \\
& + \iiint T(x, y, z, t) \mathcal{P}_8 T_y(x, y, z, t) dxdydz
\end{aligned}
\]

(17)

and

\[
\begin{aligned}
\iiint T(x, y, z, t) \mathcal{P}_5 T_z(x, y, z, t) dxdydz & = -\iiint T(x, y, z, t) \mathcal{P}_6 T_z(x, y, z, t) dxdydz \\
& + \iiint T(x, y, z, t) \mathcal{P}_7 T_z(x, y, z, t) dxdydz
\end{aligned}
\]

(18)

From the definition of operator \( \mathcal{A} \), and considering (12)-(18), we can obtain

\[
\begin{aligned}
\iiint T(x, y, z, t) \mathcal{A} T(x, y, z, t) dxdydz & = -\iiint T(x, y, z, t) \mathcal{P}_1 T_x(x, y, z, t) dxdydz \\
& - \iiint T(x, y, z, t) \mathcal{P}_2 T_y(x, y, z, t) dxdydz \\
& - \iiint T(x, y, z, t) \mathcal{P}_3 T_z(x, y, z, t) dxdydz
\end{aligned}
\]

(19)

Using (11)-(19), we have

\[
\begin{aligned}
\dot{V}(t) + 2\rho V(t) & = -\iiint [T(x, y, z, t)]^T \mathcal{P}_1 [T_x(x, y, z, t)] dxdydz \\
& - \iiint [T(x, y, z, t)]^T \mathcal{P}_2 [T_y(x, y, z, t)] dxdydz \\
& - \iiint [T(x, y, z, t)]^T \mathcal{P}_3 [T_z(x, y, z, t)] dxdydz
\end{aligned}
\]

(20)

where

\[
\begin{aligned}
\mathcal{T}(t) & \triangleq \begin{bmatrix} T^T(x, t) & T^T_x(x, t) & T^T_y(x, t) & T^T_z(x, t) \end{bmatrix}^T \\
\mathcal{P} & \triangleq \begin{bmatrix} -\rho & -\rho & -\rho & -\rho \\
-\rho & -\rho & -\rho & -\rho \\
-\rho & -\rho & -\rho & -\rho \\
-\rho & -\rho & -\rho & -\rho \end{bmatrix}
\end{aligned}
\]

(21)

(22)

where \( \mathcal{P}_{ii} = [\mathcal{P} + \mathcal{A} \mathcal{P} + \mathcal{P} \mathcal{A}] + 2\rho \mathcal{P} \). In this case, the gain matrix of a suitable controller can be constructed as follows

\[
\begin{aligned}
\mathcal{K}_i & = \mathcal{N}_i \mathcal{Q}^{-1}, i = 1, 2, 3, 4.
\end{aligned}
\]

(23)

Proof. Assume that the LMI in (22) hold for \( \mathcal{Q} \in \mathbb{R}^{m \times m} > 0 \) and \( \mathcal{N}_i \in \mathbb{R}^{m \times m}, i \in [1, 2, 3, 4] \), setting

\[
\begin{aligned}
\mathcal{Q} & = P^{-1}, \mathcal{N}_i = \mathcal{K}_i, i = 1, 2, 3, 4.
\end{aligned}
\]

(24)

pre- and post-multiplying both side of (22) by the matrix \( \text{diag}(\mathcal{Q}^{-1}, \mathcal{Q}^{-1}, \mathcal{Q}^{-1}, \mathcal{Q}^{-1}) \), respectively, with the commutative law of matrix multiplication[15], we can obtain

\[
\begin{aligned}
\mathcal{Q} & = \text{diag}(\mathcal{Q}^{-1}, \mathcal{Q}^{-1}, \mathcal{Q}^{-1}, \mathcal{Q}^{-1}) \mathcal{P} \text{diag}(\mathcal{Q}^{-1}, \mathcal{Q}^{-1}, \mathcal{Q}^{-1}, \mathcal{Q}^{-1})
\end{aligned}
\]

(19)

Since \( \text{diag}(\mathcal{Q}^{-1}, \mathcal{Q}^{-1}, \mathcal{Q}^{-1}, \mathcal{Q}^{-1}) > 0 \), we can get the following inequality from (22) and (24):

\[
\mathcal{P} < 0
\]

(25)
Substituting the inequality (25) into (20) yields
\[ \dot{V}(t) + 2\rho V(t) < 0 \] (26)
for all non-zero \( \tilde{c}(\cdot, t) \). Integration of (26) from 0 to \( t \) derives
\[ V(t) \leq V(0) \exp(-2\rho t) \] (27)
Since \( P(x) > 0 \) is a spatially continuous matrix function of \( x \)
defined on \([l_1, l_2]\), it is easily observed that \( V(t) \) given by (10)
satisfies the following inequality:
\[ p_1 \|T(\cdot, \cdot, \cdot, t)\|_2^2 \leq V(t) \leq p_2 \|T(\cdot, \cdot, \cdot, t)\|_2^2 \] (28)
where \( p_1 \equiv \min_{\kappa \in [1, 2, \ldots, npq]} \lambda_\kappa(P) \) and \( p_2 \equiv \max_{\kappa \in [1, 2, \ldots, npq]} \lambda_\kappa(P) \) are two positive constants. Using (28), we can get the following relation
\[ p_1 \|T(\cdot, \cdot, \cdot, t)\|_2^2 \leq V(t) \leq p_2 \|T(0, \cdot, \cdot, \cdot)\|_2^2 \exp(-2\rho t) \] (29)
Therefore, we have
\[ \|T(\cdot, \cdot, \cdot, t)\|_2^2 \leq p_2 p_1^{-1} \|T(0, \cdot, \cdot, \cdot)\|_2^2 \exp(-2\rho t), t \geq 0 \] (30)
Thus, from (30) and Definition 1, the system (1)-(3) is \( \rho \)-ES. Moreover, from (24), we have (23). The proof is complete.

Theorem 1 presents an LMI-based condition for the existence of a distributed controller (7) for the exponential stability of the 3-dimensional parabolic PDE system (1)-(3). Explicit expressions of a desired controller are proposed when the LMI (22) is feasible.

Remark 1. Notice that the result in this paper is different from previous ones on the P-sD control of 1-dimensional PDE systems in [12] and [13]. This paper provides the P-sD control for exponential stability of the 3-dimensional PDE systems.

Remark 2. The main results of this paper are also easy to extended to the PIDE system subject to the homogenous Dirichlet boundary conditions i.e.,
\[
\begin{align*}
T(l_1, y, z, t) &= T(l_2, y, z, t) = 0 \\
T(x, 0, z, t) &= T(x, l_3, z, t) = 0 \\
T(x, y, 0, t) &= T(x, y, l_2, t) = 0
\end{align*}
\]

IV. Numerical simulation

In this section, in order to illustrate the effectiveness and benefits of the proposed results, we consider the control problem of (1) with the following parameters:
\[
\begin{align*}
\Theta_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \Theta_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}, \Theta_3 &= \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}, \\
\Theta_4 &= \Theta_5 = \Theta_6, \Theta_7 &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \\
\Theta_8 &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \Theta_9 &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, A &= \begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix}, \\
G &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{align*}
\] (31)

Fig. 2 shows the trajectories of open-loop \( \|T(\cdot, \cdot, \cdot, t)\|_2 \) starting from the initial conditions. It is easily observed from Fig. 2 that the equilibrium \( T(\cdot, \cdot, \cdot, t) = 0 \) of the system is not stable.

Fig. 3 shows the trajectories of close-loop \( \|T(\cdot, \cdot, \cdot, t)\|_2 \) after applying the P-sD controller (7) with the obtained control gains matrices given in (32) to the system. The trajectories of closed-loop \( \|T(\cdot, \cdot, \cdot, t)\|_2 \) are shown in Fig. 3, which implies that the proposed P-sD controller (5) with the obtained control gain matrices given in (32) can stabilize the PDE system (1)-(3). Moreover, the trajectories of \( \|u(\cdot, \cdot, \cdot, t)\|_2 \) is shown in Fig.4.

V. Conclusions

In this paper, we have addressed the problem of designing a P-sD state-feedback controller for a class of linear spatially distributed processes modeled by 3-dimensional second-order parabolic PDEs. Distributed P-sD state feedback control design has been developed and the stability criterion was first obtained in terms of LMI based on directly the original PDE model. Finally, a numerical example illustrates the effectiveness and validity of the presented method.
Fig. 4. Trajectories of control inputs $||u(\cdot, \cdot, t)||_2$

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