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Relative Tor Functors for Level Modules with Respect to a Semidualizing Bimodule

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Abstract Let R and S be rings and ${}_SC_R$ a semidualizing bimodule. We investigate the relative Tor functors $\mathrm{Tor}_i^{\mathcal{ML}_C}(-,-)$ defined via C-level resolutions, and these functors are exactly the relative Tor functors $\mathrm{Tor}_i^{\mathcal{MF}_C}(-,-)$ defined by Salimi, Sather-Wagstaff, Tavasoli and Yassemi provided that $S=R$ is a commutative Noetherian ring. Vanishing of these functors characterizes the finiteness of $\mathcal{L}_C(S)$-projective dimension. Applications go in two directions. The first is to characterize when every S-module has a monic (or epic) C-level precover (or preenvelope). The second is to give some criteria for the isomorphism $\mathrm{Tor}_i^{\mathcal{ML}_C}(-,-)\cong\mathrm{Tor}_i^{\mathcal{MF}_C}(-,-)$ between the bifunctors.

Keywords Level · Proper resolution · Relative homology · Semidualizing bimodule

Mathematics Subject Classification (2010) 18G10 · 18G15 · 18G20

1 Introduction

Semidualizing modules originated from the study of various duality theories in commutative algebra. The study of such modules (under different names) are independently studied by Foxby, Golod and Vasconcelos (see [8, 10, 27]). Holm and White [13] extended this notion to arbitrary associative rings, while Christensen [5] and Kubik [16] extended it to

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semidualizing complexes and quasidualizing modules respectively. A semidualizing bimodule ${}_SC_R$ gives rise to several distinguished classes of modules. For instance, one has the class $\mathcal{P}_C(S)$ ($\mathcal{F}_C(S)$) of C-projective (C-flat) S-modules and the class $\mathcal{I}_C(R)$ of C-injective R-modules. Detailed definitions can be found in Section 2.

Recently, Salimi, Sather-Wagstaff, Tavasoli and Yassemi [19] investigated the relative Tor functors for C-flat R-modules over a commutative Noetherian ring R. These functors coincide with the classical Tor functors (see, for example [3, Chapter VI]) in case $C = R$. As an application, they showed how the vanishing of these functors characterizes the finiteness of $\mathcal{F}_C(R)$-projective dimension (see [19, Theorem 5.4]). To describe how Gorenstein homological algebra should work for general rings, the authors of [2] defined the notion of level modules, which is a natural generalization of flat modules. The main purpose of this paper is to further study the relative Tor functors for level modules with respect to a semidualizing bimodule.

In the following, we let R and S be rings and ${}_SC_R$ a semidualizing bimodule. All R- or S-modules are understood to be left R- or S-modules. Right R- or S-modules are identified with left modules over the opposite rings R^{op} or S^{op}. Let $\mathcal{L}_C(S) = \{C \otimes_R F \mid {}_RF \text{ is level}\}$. Such modules are called C-level S-modules. It is shown that every S-module M has a C-level precover (see Proposition 3.7(3)). This lets us give an appropriate definition of relative Tor functors for level modules with respect to a semidualizing bimodule (see Definition 3.10). The next result for relative Tor functors is our first main theorem which characterizes the finiteness of $\mathcal{L}_C(S)$-projective dimension. See Theorem 3.14.

Theorem 1.1 *Let ${}_SC_R$ be a faithfully semidualizing bimodule and n a non-negative integer. Then the following are equivalent for any S-module N:*

(1) *$\mathcal{L}_C(S)$-pd$(N) \leqslant n$;*
(2) *$\mathrm{Tor}_i^{\mathcal{M}\mathcal{L}_C}(M, N) = 0$ for each (finitely presented) S^{op}-module M and any $i > n$;*
(3) *$\mathrm{Tor}_{n+1}^{\mathcal{M}\mathcal{L}_C}(M, N) = 0$ for each (finitely presented) S^{op}-module M.*

Note that if $S = R$ is commutative Noetherian, then the above result was known in [19]. Thus Theorem 1.1 generalizes [19, Theorem 5.4], and provides many more possibilities for constructing an appropriate definition of relative Tor functors. Also, our proof of Theorem 1.1 is different from that in [19].

The first application of Theorem 1.1 is the next result characterizing when every S-module has a monic (or epic) C-level precover (or preenvelope). See Theorems 4.1, 4.2, 4.4 and Corollary 4.6.

Theorem 1.2 *The following are equivalent for any faithfully semidualizing bimodule ${}_SC_R$:*

(1) *Every S-module has a monic C-level precover if and only if $\mathrm{Tor}_1^{\mathcal{M}\mathcal{L}_C}(M, N) = 0$ for any S^{op}-module M and any S-module N.*
(2) *Every S-module has an epic C-level precover if and only if $\theta : \mathrm{Tor}_0^{\mathcal{M}\mathcal{L}_C}(M, N) \to M \otimes_S N$ is epic for any S^{op}-module M and any S-module N.*
(3) *Every S-module has a monic C-level preenvelope if and only if $\mathrm{Tor}_1^{\mathcal{M}\mathcal{L}_C}(M, (S_S)^+) = 0$ for each S^{op}-module M.*
(4) *Every S-module has an epic C-level preenvelope if and only if $\mathrm{Tor}_2^{\mathcal{M}\mathcal{L}_C}(M, N) = 0$ for each S^{op}-module M and any S-module N.*

As a consequence of Theorem 1.2, we give a necessary and sufficient condition for a semidualizing module C to be projective whenever R is commutative Noetherian (see Corollary 4.3).

Motivated by the fact that one can compute the homology $\mathrm{Tor}_i^R(M,N)$ in terms of a projective resolution of N or a flat resolution of N, Salimi, Sather-Wagstaff, Tavasoli and Yassemi [19] compared relative Tor functors $\mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(-,-)$ and $\mathrm{Tor}_i^{\mathcal{M}\mathcal{P}_C}(-,-)$ defined via C-flat and C-projective resolutions whenever C is a semidualizing module over a commutative Noetherian ring R. They proved that $\mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(-,-)\cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{P}_C}(-,-)$ for each integer i (see [19, Theorem A]). As the second application of Theorem 1.1, the next result characterizes when $\mathrm{Tor}_i^{\mathcal{M}\mathcal{L}_C}(-,-)\cong \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(-,-)$ for each integer i. See (4.11) for the proof.

Theorem 1.3 *The following are equivalent for any faithfully semidualizing bimodule ${}_SC_R$:*

(1) *R is right coherent;*
(2) *$\mathcal{L}_C(S)=\mathcal{F}_C(S)$;*
(3) *Every C-flat precover over S is a C-level precover;*
(4) *Every C-level precover over S is a C-flat precover;*
(5) *$Tor_i^{\mathcal{M}\mathcal{L}_C}(-,-)\cong Tor_i^{\mathcal{M}\mathcal{F}_C}(-,-)$ for each integer i.*

2 Preliminaries

Throughout this paper, $\mathsf{Mod}(R)$ is the class of (left) R-modules. For any R-module M, $\mathrm{id}_R(M)$ is the injective dimension of M and M^+ denotes $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$. $\mathcal{P}$, $\mathcal{F}$ and $\mathcal{C}$ stand for the classes of projective, flat and cotorsion R-modules, respectively.

Next we recall basic definitions and properties needed in the sequel. For more details the reader can consult [1, 5, 7], or [23].

Proper Resolutions. Let $\mathcal{X}$ be a class of R-modules and M an R-module. A *proper left $\mathcal{X}$-resolution* of M [11, 2.1] is a complex of the form $\mathbb{X}=\cdots\to X_n\to\cdots\to X_1\to X_0\to 0$ with each $X_i\in\mathcal{X}$, together with a morphism $X_0\to M$, such that $\mathbb{X}':=\cdots\to X_n\to\cdots\to X_1\to X_0\to M\to 0$ is also a complex (not necessarily exact), and such that the sequence

$$\mathrm{Hom}(X,\mathbb{X}')=\cdots\to\mathrm{Hom}(X,X_1)\to\mathrm{Hom}(X,X_0)\to\mathrm{Hom}(X,M)\to 0$$

is exact for every $X\in\mathcal{X}$. We refer to $\mathbb{X}':=\cdots\to X_n\to\cdots\to X_0\to M\to 0$ as an *augmented proper left $\mathcal{X}$-resolution* of M. Furthermore, the *proper left $\mathcal{X}$-dimension* of M, denoted by $\mathscr{L}\text{-dim}_{\mathcal{X}}M$, is defined as $\inf\{n$: there is an augmented proper left $\mathcal{X}$-resolution of M of the form $0\to X_n\to X_{n-1}\to\cdots\to X_0\to M\to 0\}$. The (*augmented*) *proper right $\mathcal{X}$-resoluion* and the *proper right $\mathcal{X}$-dimension* (denoted by $\mathscr{R}\text{-dim}_{\mathcal{X}}M$) of an R-module M are defined dually.

Covers and Envelopes. Let $\mathcal{X}$ be a class of R-modules and M an R-module. A homomorphism $\phi: M\to C$ with $C\in\mathcal{X}$ is called an $\mathcal{X}$-preenvelope of M [6] if for any homomorphism $f: M\to C'$ with $C'\in\mathcal{X}$, there is a homomorphism $g: C\to C'$ such that

$g\phi = f$. Moreover, if the only such g are automorphisms of C when $C' = C$ and $f = \phi$, the $\mathcal{X}$-preenvelope ϕ is called an *$\mathcal{X}$-envelope* of M. The class $\mathcal{X}$ is called (pre)enveloping if every R-module has an $\mathcal{X}$-(pre)envelope. Dually we have the definitions of an $\mathcal{X}$-precover, an $\mathcal{X}$-cover and a (pre)covering class.

If $\mathcal{X}$ is precovering (preenveloping), for any R-module M, one can iteratively take precovers (preenvelopes) to construct (augmented) proper left (right) $\mathcal{X}$-resolutions of M.

Cotorsion pair. Let $\mathcal{X}$ be a class of R-modules. For an R-module M, we write $M \in {}^{\perp_1}\mathcal{X}$ (resp. $M \in \mathcal{X}^{\perp_1}$) if $\mathrm{Ext}^1_R(M, X) = 0$ (resp. $\mathrm{Ext}^1_R(X, M) = 0$) for each $X \in \mathcal{X}$. Following Enochs [7], Hovey [14] and Salce [20], a *cotorsion pair* is a pair of classes $(\mathcal{A}, \mathcal{B})$ in $\mathsf{Mod}R$ such that $\mathcal{A}^{\perp_1} = \mathcal{B}$ and ${}^{\perp_1}\mathcal{B} = \mathcal{A}$. If we choose $\mathcal{A} = \mathsf{Mod}(R)$ for some ring R, the most obvious example of a cotorsion pair is $(\mathcal{P}, \mathsf{Mod}(R))$. Perhaps one of the most useful cotorsion pair is the flat cotorsion pair $(\mathcal{F}, \mathcal{C})$. Here $\mathcal{C}$ is the collection of all modules C such that $C \in \mathcal{F}^{\perp_1}$. Such modules are called cotorsion modules.

Complexes. A complex $\cdots \to X_{n+1} \xrightarrow{\partial^X_{n+1}} X_n \xrightarrow{\partial^X_n} X_{n-1} \to \cdots$ of R-modules will be denoted by X. We set $\mathrm{Z}_n(X) = \ker(\partial^X_n)$, $\mathrm{B}_n(X) = \mathrm{im}(\partial^X_{n+1})$ and $\mathrm{C}_n(X) = \mathrm{coker}(\partial^X_{n+1})$. The *nth homology module* of X is the module $\mathrm{H}_n(X) = \mathrm{Z}_n(X)/\mathrm{B}_n(X)$. If X and Y are both complexes, then by a morphism $\alpha : X \to Y$ we mean a sequence $\alpha_n : X_n \to Y_n$ such that $\alpha_{n-1}\partial^X_n = \partial^Y_n\alpha_n$ for each integer n. A quasi-isomorphism, indicated by the symbol "$\simeq$", is a morphism of complexes that induces an isomorphism in homology.

Let $\mathcal{X}$ be a class of R-modules. A complex L is *$Hom_R(\mathcal{X}, -)$ exact* if the complex $\mathrm{Hom}_R(X, L)$ is exact for each $X \in \mathcal{X}$. Dually, the complex L is *$Hom_R(-, \mathcal{X})$ exact* if $\mathrm{Hom}_R(L, X)$ is exact for each $X \in \mathcal{X}$.

We say that $\mathcal{X}$ is *closed under extensions* when, for every exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of R-modules, if $M', M'' \in \mathcal{X}$, then $M \in \mathcal{X}$. We say that $\mathcal{X}$ is *closed under cokernels of monomorphisms* when, for every exact sequence above, if $M', M \in \mathcal{X}$, then $M'' \in \mathcal{X}$. We say that $\mathcal{X}$ is *closed under kernels of epimorphisms* when, for every exact sequence above, if $M, M'' \in \mathcal{X}$, then $M' \in \mathcal{X}$.

Definition 2.1 ([22]) Let $\mathcal{X}$ be a class of R-modules and M an R-module. Then the *$\mathcal{X}$-injective dimension* of M, denoted by $\mathcal{X}$-id(M), is defined as $\inf\{n$: there is an exact sequence of the form $0 \to M \to X^0 \to X^1 \to \cdots \to X^n \to 0$ with each $X^i \in \mathcal{X}\}$. The $\mathcal{X}$-projective dimension $\mathcal{X}$-pd(M) of M is defined dually. Note that the modules of $\mathcal{X}$-projective dimension 0 are exactly the modules of $\mathcal{X}$.

Definition 2.2 ([13]) An (S, R)-bimodule ${}_SC_R$ is *semidualizing* if

(a1) ${}_SC$ admits a degreewise finite S-projective resolution.
(a2) C_R admits a degreewise finite R^{op}-projective resolution.
(b1) The homothety map ${}_SS_S \xrightarrow{S\gamma} \mathrm{Hom}_{R^{\mathrm{op}}}(C, C)$ is an isomorphism.
(b2) The homothety map ${}_RR_R \xrightarrow{\gamma_R} \mathrm{Hom}_S(C, C)$ is an isomorphism.
(c1) $\mathrm{Ext}^{\geqslant 1}_S(C, C) = 0$.
(c2) $\mathrm{Ext}^{\geqslant 1}_{R^{op}}(C, C) = 0$.

A semidualizing bimodule ${}_SC_R$ is *faithfully semidualizing* if it satisfies the following conditions for all modules ${}_SN$ and M_R.

(a) If $\mathrm{Hom}_S(C, N) = 0$, then $N = 0$.

(b) If $\mathrm{Hom}_{R^{\mathrm{op}}}(C, M) = 0$, then $M = 0$.

Some nice introductions to the basic theory of semidualizing modules can be found in [4, 21].

Definition 2.3 ([13]) The Auslander class $\mathcal{A}_C(R)$ ($\mathcal{A}_C(S^{\mathrm{op}})$) with respect to a semidualizing bimodule ${}_SC_R$ consists of all R-modules M (S^{op}-modules N) satisfying

(1) $\mathrm{Tor}^R_{\geqslant 1}(C, M) = 0$ ($\mathrm{Tor}^S_{\geqslant 1}(N, C) = 0$),
(2) $\mathrm{Ext}^{\geqslant 1}_S(C, C \otimes_R M) = 0$ ($\mathrm{Ext}^{\geqslant 1}_{R^{\mathrm{op}}}(C, N \otimes_S C) = 0$), and
(3) The natural evaluation homomorphism $\mu_M : M \to \mathrm{Hom}_S(C, C \otimes_R M)$ ($\mu'_N : N \to \mathrm{Hom}_{R^{\mathrm{op}}}(C, N \otimes_S C)$) is an isomorphism of R-modules (S^{op}-modules).

The Bass class $\mathcal{B}_C(S)$ ($\mathcal{B}_C(R^{\mathrm{op}})$)with respect to a semidualizing bimodule ${}_SC_R$ consists of all S-modules N (R^{op}-modules M) satisfying

1. $\mathrm{Ext}^{\geqslant 1}_S(C, N) = 0$ ($\mathrm{Ext}^{\geqslant 1}_{R^{\mathrm{op}}}(C, M) = 0$),
2. $\mathrm{Tor}^R_{\geqslant 1}(C, \mathrm{Hom}_S(C, N)) = 0$ ($\mathrm{Tor}^S_{\geqslant 1}(\mathrm{Hom}_{R^{\mathrm{op}}}(C, M), C) = 0$), and
3. The natural evaluation homomorphism $\nu_N : C \otimes_R \mathrm{Hom}_S(C, N) \to N$ ($\nu'_M : \mathrm{Hom}_{R^{\mathrm{op}}}(C, M) \otimes_S C \to M$) is an isomorphism of S-modules (R^{op}-modules).

Facts 2.4 Let ${}_SC_R$ be a semidualizing bimodule.

(1) The classes $\mathcal{A}_C(R)$ and $\mathcal{B}_C(S)$ are closed under extensions, direct summands, products and coproducts. Moreover, $\mathcal{A}_C(R)$ is closed under kernels of epimorphisms and $\mathcal{B}_C(S)$ is closed under cokernels of monomorphisms. See [13, Proposition 4.2 and Theorem 6.2].
(2) If C is a faithfully semidualizing module, then an R-module $M \in \mathcal{A}_C(R)$ if and only if $C \otimes_R M \in \mathcal{B}_C(S)$. See [26, Lemma 1.7(a)].
(3) If C is a faithfully semidualizing module, then an S-module $N \in \mathcal{B}_C(S)$ if and only if $\mathrm{Hom}_S(C, N) \in \mathcal{A}_C(R)$. See [26, Lemma 1.7(b)].

Definition 2.5 ([13])

Let ${}_SC_R$ be a semidualizing bimodule. We set
$\mathcal{F}_C(S) = \{C \otimes_R F \mid {}_RF \text{ is flat}\}$.
$\mathcal{P}_C(S) = \{C \otimes_R P \mid {}_RP \text{ is projective}\}$.
$\mathcal{I}_C(R) = \{\mathrm{Hom}_S(C, I) \mid {}_SI \text{ is injective}\}$.

Modules in $\mathcal{F}_C(S)$, $\mathcal{P}_C(S)$ and $\mathcal{I}_C(R)$ are called C-flat S-modules, C-projective S-modules and C-injective R-modules, respectively.

Definition 2.6 ([12]) Assume that R is an associative ring. A duality pair over R is a pair $(\mathcal{X}, \mathcal{Y})$, where $\mathcal{X}$ is a class of left R-modules and $\mathcal{Y}$ is a class of right R-modules, subject to the following conditions:

(1) For a left R-module M, one has $M \in \mathcal{X}$ if and only if $M^+ \in \mathcal{Y}$.

(2) $\mathcal{Y}$ is closed under direct summands and finite direct sums.

By an argument similar to the proof of [12, Proposition 2.4] or [21, Proposition 3.3.1], we have the following result in non-commutative setting.

Lemma 2.7 *Let ${}_SC_R$ be a semidualizing bimodule. Then $(\mathcal{B}_C(S), \mathcal{A}_C(S^{op}))$ is a duality pair.*

3 Relative Tor Functors for C-Level Modules

Recall that an S-module M over a ring S is said to be of type $\mathcal{FP}_\infty$ [2] if M has a projective resolution by finitely generated projectives. In what follows, we denote by $\mathcal{FP}_\infty(S)$ ($\mathcal{FP}_\infty(S^{op})$) the class of S-modules (S^{op}-modules) of type $\mathcal{FP}_\infty$. Thus we have the following definition.

Definition 3.1 ([2, Definition 2.6]) Let R be a ring and N an R-module.

(1) N is called *level* if $\mathrm{Tor}_1^R(M, N) = 0$ for all R^{op}-modules M in $\mathcal{FP}_\infty(R^{op})$.
(2) N is called FP_∞*-injective* (or *absolutely clean*) if $\mathrm{Ext}_R^1(M, N) = 0$ for all R-modules M in $\mathcal{FP}_\infty(R)$.

For convenience, we write $\mathcal{L}(R)$ and $\mathcal{FI}_\infty(R)$ for the classes of level and FP_∞-injective R-modules respectively. Dually, we can define the class of level R^{op}-modules (denoted by $\mathcal{L}(R^{op})$) and the class of FP_∞-injective R^{op}-modules (denoted by $\mathcal{FI}_\infty(R^{op})$).

Lemma 3.2 *Let ${}_SC_R$ be a semidualizing bimodule.*

(1) *$Hom_S(M, C) \in \mathcal{FP}_\infty(R^{op})$ whenever M is a finitely generated projective S-module.*
(2) *$Hom_{R^{op}}(M, C) \in \mathcal{FP}_\infty(S)$ whenever M is a finitely generated projective R^{op}-module.*

Proof We prove part (1); the proof of (2) is similar. Assume that M is a finitely generated projective S-module, then there is a split exact sequence $0 \to K \to S^n \to M \to 0$ of S-modules. Applying $\mathrm{Hom}_S(-, C)$ to the split exact sequence above, we have a split exact sequence $0 \to \mathrm{Hom}_S(M, C) \to \mathrm{Hom}_S(S^n, C) \to \mathrm{Hom}_S(K, C) \to 0$ of R^{op}-modules. Note that $\mathrm{Hom}_S(S^n, C) \cong C^n$ and $C \in \mathcal{FP}_\infty(R^{op})$. Then $\mathrm{Hom}_S(S^n, C) \in \mathcal{FP}_\infty(R^{op})$. So $\mathrm{Hom}_S(M, C) \in \mathcal{FP}_\infty(R^{op})$ by [2, Proposition 2.3]. This completes the proof. □

Proposition 3.3 *Let ${}_SC_R$ be a semidualizing bimodule.*

(1) *$\mathcal{L}(R) \subseteq \mathcal{A}_C(R)$;*
(2) *$\mathcal{FI}_\infty(S) \subseteq \mathcal{B}_C(S)$.*

Proof (1). Let M be a level R-module. Since $C \in \mathcal{FP}_\infty(R^{op})$, $\mathrm{Tor}_i^R(C, M) = 0$ for any $i \geqslant 1$. Note that ${}_SC$ admits a degreewise finite S-projective resolution by hypothesis. Then there exists a quasi-isomorphism $\alpha : \mathbb{L} \to C$ such that $\mathbb{L}_i$ is a finitely generated projective S-module for any $i \geqslant 0$ and $\mathbb{L}_i = 0$ for any $i < 0$. Since $\mathrm{Ext}_S^i(C, C) = 0$, $\mathrm{Hom}_S(\alpha, C) : \mathrm{Hom}_S(C, C) \to \mathrm{Hom}_S(\mathbb{L}, C)$ is a quasi-isomorphism. Note that each

$Z_i(\mathrm{Hom}_S(\mathbb{L}, C)) \in \mathcal{FP}_\infty(R^{op})$ for any $i \leqslant 0$ by Lemma 3.2(1) and [2, Proposition 2.3]. It follows that $\mathrm{Hom}_S(\alpha, C) \otimes_R 1 : \mathrm{Hom}_S(C, C) \otimes_R M \to \mathrm{Hom}_S(\mathbb{L}, C) \otimes_R M$ is a quasi-isomorphism. Since ${}_R R_R \xrightarrow{\gamma_R} \mathrm{Hom}_S(C, C)$ is an isomorphism, $\mathrm{Hom}_S(C, C) \otimes_R M \cong R \otimes_R M \cong M$. Thus $M \to \mathrm{Hom}_S(\mathbb{L}, C) \otimes_R M$ is a quasi-isomorphism. Note that $\mathrm{Hom}_S(\mathbb{L}, C) \otimes_R M \to \mathrm{Hom}_S(\mathbb{L}, C \otimes_R M)$ is a quasi-isomorphism by [4, A.2.10, p.170]. It follows that $M \to \mathrm{Hom}_S(\mathbb{L}, C \otimes_R M)$ is a quasi-isomorphism. Thus $\mathrm{H}_i(\mathrm{Hom}_S(\mathbb{L}, C \otimes_R M)) \cong M$ for any integer i. So we have $M \cong \mathrm{Hom}_S(C, C \otimes_R M)$ and $\mathrm{Ext}^i_S(C, C \otimes_R M) = 0$ for any $i \geqslant 1$, as desired.

(2). The proof is similar to that of part (1) by noting that Lemma 3.2(2) holds. □

Definition 3.4 Let ${}_S C_R$ be a semidualizing bimodule. An S-module is called *C-level* if it has the form $C \otimes_R F$ for some level R-module F. Similarly, an R-module is called *C-FP_∞-injective* if it has the form $\mathrm{Hom}_S(C, I)$ for some FP_∞-injective module ${}_S I$.

In what follows, we set $\mathcal{L}_C(S) = \{C \otimes_R F \mid {}_R F \text{ is level}\}$ and $\mathcal{FI}^\infty_C(R) = \{\mathrm{Hom}_S(C, I) \mid {}_S I \text{ is } FP_\infty\text{-injective}\}$. Dually, we can define the class of C-level R^{op}-modules (denoted by $\mathcal{L}_C(R^{op})$) and the class of C-FP_∞-injective S^{op}-modules (denoted by $\mathcal{FI}^\infty_C(S^{op})$).

Example 3.5 Assume that R is a commutative ring (not necessarily Noetherian) and $S = \mathbb{M}_2(R)$ is the 2×2 matrix ring over R. Let $C = R^2$. Then ${}_S C_R$ is a faithfully semidualizing bimodule and ${}_R C_R$ is not a semidualizing bimodule. We set $\mathcal{H} = \{M \mid M \cong C \otimes_R G^+ \text{ with } \mathrm{id}_R(G) < \infty\}$. Thus each module M in $\mathcal{H}$ has a finite resolution by modules in the class $\mathcal{L}_{{}_S C_R}(S)$ and no finite resolution by modules in the class $\mathcal{L}_{{}_R C_R}(R)$.

Proof Note that R and S are Morita equivalent rings by [1, Corollary 22.6]. It follows from [13, Example 2.1(b)] that ${}_S C_R$ is a semidualizing bimodule. However, ${}_R C_R$ is not a semidualizing bimodule since $\mathrm{Hom}_R(R^2, R^2) \ncong R$. We need to show that ${}_S C_R$ is a faithfully semidualizing bimodule. Note that $C \otimes_R N \cong N^2$ for any R-module N. It follows that $N = 0$ whenever $C \otimes_R N = 0$. Let M be an S^{op}-module such that $M \otimes_S C = 0$. It is easy to check that the sequence $0 \to C \to S \xrightarrow{\rho} C \to 0$ of S-modules is exact, where $\rho : S \to C$ is a homomorphism defined by $\rho(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a \\ c \end{pmatrix}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S$. Hence the sequence $M \otimes_S C \to M \otimes_S S \to M \otimes_S C \to 0$ is exact. Since $M \otimes_S C = 0$ by hypothesis, so is $M \otimes_S S$. So $M = 0$ and ${}_S C_R$ is a faithfully semidualizing bimodule by [13, Lemma 3.1].

Let G be an R-module with $\mathrm{id}_R(G) = n < \infty$. Then there exists an exact sequence $0 \to G \to E_0 \to E_{-1} \to \cdots \to E_{1-n} \to E_{-n} \to 0$ of R-modules with E_{-i} injective for $i = 0, 1, \cdots, n$. Thus $0 \to (E_{-n})^+ \to (E_{1-n})^+ \to \cdots \to (E_{-1})^+ \to (E_0)^+ \to G^+ \to 0$ is an exact sequence of R-modules with $(E_{-i})^+$ level for $i = 0, 1, \cdots, n$ by [2, Theorem 2.12]. Since $(E_{-i})^+ \in \mathcal{A}_C(R)$ for $i = 0, 1, \cdots, n$ by Proposition 3.3(1), $\mathrm{coker}((E_{-i})^+ \to (E_{1-i})^+) \in \mathcal{A}_C(R)$ for $i = 1, 2, \cdots, n$. Hence we have an exact sequence $0 \to C \otimes_R (E_{-n})^+ \to C \otimes_R (E_{1-n})^+ \to \cdots \to C \otimes_R (E_{-1})^+ \to C \otimes_R (E_0)^+ \to C \otimes_R G^+ \to 0$ of S-modules. So each module M in $\mathcal{H}$ has a finite resolution by modules in the class $\mathcal{L}_{{}_S C_R}(S)$. However each module M in $\mathcal{H}$ doesn't have a finite resolution by modules in the class $\mathcal{L}_{{}_R C_R}(R)$ by noting that ${}_R C_R$ is not a semidualizing bimodule. This completes the proof. □

Lemma 3.6 *Let ${}_SC_R$ be a semidualizing bimodule.*

(1) $M \in \mathcal{L}_C(S)$ *if and only if* $M \in \mathcal{B}_C(S)$ *and* $\mathrm{Hom}_S(C, M) \in \mathcal{L}(R)$.
(2) $N \in \mathcal{FI}_C^{\infty}(R)$ *if and only if* $N \in \mathcal{A}_C(R)$ *and* $C \otimes_R N \in \mathcal{FI}_{\infty}(S)$.

Proof We prove part (1); the proof of (2) is dual. Assume that $M \in \mathcal{L}_C(S)$. Then there exists a level R-module F such that $M \cong C \otimes_R F$. Note that $F \in \mathcal{A}_C(R)$ by Proposition 3.3(1). Then $M \in \mathcal{B}_C(S)$ and $\mathrm{Hom}_S(C, M) \cong \mathrm{Hom}_S(C, C \otimes_R F) \cong F \in \mathcal{L}(R)$. Conversely, assume that M is an R-module in $\mathcal{B}_C(S)$ and $\mathrm{Hom}_S(C, M) \in \mathcal{L}(R)$. It follows that $C \otimes_R \mathrm{Hom}_S(C, M) \cong M$. So $M \in \mathcal{L}_C(S)$. This completes the proof. □

Proposition 3.7 *Let ${}_SC_R$ be a semidualizing bimodule.*

(1) *Both* $\mathcal{L}_C(S)$ *and* $\mathcal{FI}_C^{\infty}(R)$ *are closed under direct products, coproducts and direct summands.*
(2) $(\mathcal{L}_C(S), \mathcal{FI}_C^{\infty}(S^{op}))$ *and* $(\mathcal{FI}_C^{\infty}(S^{op}), \mathcal{L}_C(S))$ *are duality pairs.*
(3) *Both* $\mathcal{L}_C(S)$ *and* $\mathcal{FI}_C^{\infty}(S^{op})$ *are covering and preenveloping.*

Proof (1). Since $\mathcal{B}_C(S)$ and $\mathcal{L}(R)$ are closed under direct products, coproducts and direct summands by [13, Proposition 4.2] and [2, Proposition 2.10], so is $\mathcal{L}_C(S)$ by Lemma 3.6(1). Similarly, we can prove $\mathcal{FI}_C^{\infty}(R)$ is closed under direct products, coproducts and direct summands.

(2). Let M be an S-module. Note that $\mathrm{Hom}_S(C, M) \in \mathcal{L}(R)$ if and only if $(\mathrm{Hom}_S(C, M))^+ \in \mathcal{FI}_{\infty}(R^{op})$ by [2, Theorem 2.12]. Since $(\mathrm{Hom}_S(C, M))^+ \cong M^+ \otimes_S C$ by [4, A.2.11, p.171], $\mathrm{Hom}_S(C, M) \in \mathcal{L}(R)$ if and only if $M^+ \otimes_S C \in \mathcal{FI}_{\infty}(R^{op})$. According to Lemma 2.7, $M \in \mathcal{B}_C(S)$ if and only if $M^+ \in \mathcal{A}_C(S^{op})$. Thus $M \in \mathcal{L}_C(S)$ if and only if $M^+ \in \mathcal{FI}_C^{\infty}(S^{op})$ by Lemma 3.6. So $(\mathcal{L}_C(S), \mathcal{FI}_C^{\infty}(S^{op}))$ is a duality pair by (1). Similarly, we can prove that $(\mathcal{FI}_C^{\infty}(S^{op}), \mathcal{L}_C(S))$ is a duality pair by noting that $M \in \mathcal{FI}_{\infty}(R^{op})$ if and only if $M^+ \in \mathcal{L}(R)$ (see [2, Theorem 2.12]).

(3). The result holds by (1), (2) and [12, Theorem 3.1]. □

Let $C = R$ and $S = R$ in Proposition 3.7, we get

Corollary 3.8 *Let R be a ring. Then both $\mathcal{L}(R)$ and $\mathcal{FI}_{\infty}(R)$ are covering and preenveloping.*

Remark 3.9 We note that the conclusion "$\mathcal{L}(R)$ is covering" has been proved by Bravo, Hovey and Gillespie in [2, Theorem 2.14].

Inspired by [19, Definition 3.5], we have the following

Definition 3.10 Assume that ${}_SC_R$ is a semidualizing bimodule. Let Q be a proper left $\mathcal{L}_C(S)$-resolution of an S-module N, and let G be a proper left $\mathcal{F}_C(S)$-resolution of an S-module N. For each $i \geqslant 0$ and any S^{op}-module M, set

$$\mathrm{Tor}_i^{\mathcal{M}\mathcal{L}_C}(M, N) := \mathrm{H}_i(M \otimes_S Q), \quad \mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(M, N) := \mathrm{H}_i(M \otimes_S G).$$

Remark 3.11 We note that $\mathrm{Tor}_i^{\mathcal{M}\mathcal{L}_C}(M,-)$ and $\mathrm{Tor}_i^{\mathcal{M}\mathcal{F}_C}(M,-)$ are homological functors for each integer i and any S^{op}-module M, independent of the choice of proper left $\mathcal{L}_C(S)$-resolutions and proper left $\mathcal{F}_C(S)$-resolutions respectively. We refer to [7, Section 8.2] and [11, 2.4] for a detailed discussion on this matter.

Lemma 3.12 *Let ${}_SC_R$ be a semidualizing bimodule.*

(1) *If M is an S-module in $\mathcal{B}_C(S)$, then every augmented proper left $\mathcal{L}_C(S)$-resolution of M is exact.*
(2) *If N is an S^{op}-module in $\mathcal{A}_C(S^{op})$, then every augmented proper right $\mathcal{FI}_C^{\infty}(S^{op})$-resolution of N is exact.*

Proof The proof is similar to that of Proposition 2.2 in [25]. □

Lemma 3.13 *Let ${}_SC_R$ be a faithfully semidualizing bimodule.*

(1) *The class $\mathcal{L}_C(S)$ is closed under extensions and kernels of epimorphisms.*
(2) *The class $\mathcal{L}_C(S)$ is closed under pure submodules and pure quotients.*

Proof (1). Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence of S-modules with $A'' \in \mathcal{L}_C(S)$. Assume that $A' \in \mathcal{L}_C(S)$. Then both A' and A'' are in $\mathcal{B}_C(S)$ by Lemma 3.6(1). Hence $A \in \mathcal{B}_C(S)$ and the sequence $0 \to \mathrm{Hom}_S(C, A') \to \mathrm{Hom}_S(C, A) \to \mathrm{Hom}_S(C, A'') \to 0$ is exact. Note that $\mathrm{Hom}_S(C, A')$ and $\mathrm{Hom}_S(C, A'')$ are in $\mathcal{L}(R)$ by Lemma 3.6(1). Then $\mathrm{Hom}_S(C, A) \in \mathcal{L}(R)$ by [2, Proposition 2.10(3)]. So $A \in \mathcal{L}_C(S)$ by Lemma 3.6(1). Conversely, we assume that $A \in \mathcal{L}_C(S)$. By the foregoing proof, we have an exact sequence $0 \to \mathrm{Hom}_S(C, A') \to \mathrm{Hom}_S(C, A) \to \mathrm{Hom}_S(C, A'') \to 0$ such that $\mathrm{Hom}_S(C, A)$ and $\mathrm{Hom}_S(C, A'')$ are in $\mathcal{L}(R)$. Applying [2, Proposition 2.10(3)] again, we get that $\mathrm{Hom}_S(C, A')$ is in $\mathcal{L}(R)$. So $A' \in \mathcal{L}_C(S)$ by Lemma 3.6(1), as desired.

(2). The result follows from Proposition 3.7(2) and [12, Theorem 3.1]. □

We are now in a position to prove the following theorem which contains Theorem 1.1 from the introduction.

Theorem 3.14 *Let ${}_SC_R$ be a faithfully semidualizing bimodule and n a non-negative integer. Then the following are equivalent for any S-module N:*

(1) *$\mathscr{L}$-$\dim_{\mathcal{L}_C(S)} N \leqslant n$;*
(2) *$\mathcal{L}_C(S)$-$pd(N) \leqslant n$;*
(3) *$N \in \mathcal{B}_C(S)$ and for any exact sequence $\cdots \to X_1 \to X_0 \to N \to 0$ of S-modules with each X_i in $\mathcal{L}_C(S)$, $\ker(X_{n-1} \to X_{n-2})$ is in $\mathcal{L}_C(S)$, where $X_{-1} = N$ and $X_{-2} = 0$.*
(4) *$\mathrm{Tor}_i^{\mathcal{M}\mathcal{L}_C}(M, N) = 0$ for each (finitely presented) S^{op}-module M and any $i > n$;*
(5) *$\mathrm{Tor}_{n+1}^{\mathcal{M}\mathcal{L}_C}(M, N) = 0$ for each (finitely presented) S^{op}-module M.*

Proof (1) $\Rightarrow$ (2). Note that there exists a $\mathrm{Hom}_S(\mathcal{L}_C(S), -)$ exact sequence $0 \to C \otimes_R F_n \to C \otimes_R F_{n-1} \to \cdots \to C \otimes_R F_1 \to C \otimes_R F_0 \to N \to 0$ of N with $F_i \in \mathcal{L}(R)$ for

$i = 0, 1, \cdots, n$ by (1). Then the sequence $0 \to \mathrm{Hom}_S(C, C \otimes_R F_n) \to \mathrm{Hom}_S(C, C \otimes_R F_{n-1}) \to \cdots \to \mathrm{Hom}_S(C, C \otimes_R F_1) \to \mathrm{Hom}_S(C, C \otimes_R F_0) \to \mathrm{Hom}_S(C, N) \to 0$ is exact. It follows from Proposition 3.3(1) that $\mathrm{Hom}_S(C, C \otimes_R F_i) \cong F_i \in \mathcal{A}_C(R)$ for any $i = 0, 1, \cdots, n$. Thus $\mathrm{Hom}_S(C, N) \in \mathcal{A}_C(R)$ by [13, Theorem 6.3], and hence $N \in \mathcal{B}_C(S)$ by Facts 2.4(3). So (2) holds by Lemma 3.12(1).

(2) $\Rightarrow$ (3). By (2), there is an exact sequence $0 \to C \otimes_R F_n \to \cdots C \otimes_R F_1 \to C \otimes_R F_0 \to N \to 0$ of S-modules with $F_i \in \mathcal{L}(R)$ for any $i = 0, 1, \cdots, n$. Hence every cokernel of this sequence is in $\mathcal{B}_C(S)$ by Lemma 3.6(1) and [13, Theorem 6.3]. Therefore, we have an exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to \mathrm{Hom}_S(C, N) \to 0$ of R-modules. Since $\mathrm{Ext}_R^{j\geqslant 1}(F_i, T) = 0$ for any $i = 0, 1, \cdots, n$ and each $T \in \mathcal{L}(R)^{\perp_1}$, $\mathrm{Ext}_R^{n+1}(\mathrm{Hom}_S(C, N), T) = 0$ for each $T \in \mathcal{L}(R)^{\perp_1}$ by dimension shifting.

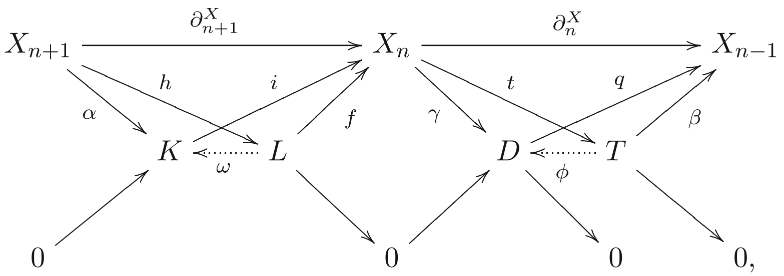
Let $\mathbb{X} : \cdots \to X_1 \to X_0 \to N \to 0$ be any exact sequence of S-modules with X_i in $\mathcal{L}_C(S)$ for any $i = 0, 1, 2, \cdots$. Since $N \in \mathcal{B}_C(S)$, so is $\mathrm{Z}_i(\mathbb{X})$ for any $i \geqslant 0$. Thus $\mathrm{Hom}_S(C, \mathbb{X})$ is exact with $\mathrm{Hom}_S(C, \mathbb{X})_i \in \mathcal{L}(R)$ for any $i = 0, 1, \cdots$. Note that $\mathrm{Ext}_R^{n+1}(\mathrm{Hom}_S(C, N), T) = 0$ for each $T \in \mathcal{L}(R)^{\perp_1}$ by the proof above. It follows that $\mathrm{Ext}_R^1(\mathrm{Z}_n(\mathrm{Hom}_S(C, \mathbb{X})), T) = 0$ for each $T \in \mathcal{L}(R)^{\perp_1}$. Since $(\mathcal{L}(R), \mathcal{L}(R)^{\perp_1})$ is a cotorsion pair by [2, Theorem 2.14], $\mathrm{Z}_n(\mathrm{Hom}_S(C, \mathbb{X}))$ is in $\mathcal{L}(R)$. Note that $\mathrm{Z}_n(\mathrm{Hom}_S(C, \mathbb{X})) \cong \mathrm{Hom}_S(C, \mathrm{Z}_n(\mathbb{X}))$. Then $\mathrm{Hom}_S(C, \mathrm{Z}_n(\mathbb{X}))$ is level. So $\mathrm{Z}_n(\mathbb{X}) \in \mathcal{L}_C(S)$ by Lemma 3.2(1), as desired.

(3) $\Rightarrow$ (1) holds by Lemma 3.12(1).

(1) $\Rightarrow$ (4) $\Rightarrow$ (5) are straightforward.

(5) $\Rightarrow$ (1). Let $\cdots \to X_n \xrightarrow{\partial_n^X} X_{n-1} \to \cdots \to X_1 \to X_0 \to N \to 0$ be an augmented proper left $\mathcal{L}_C(S)$-resolution of N. Since $M \otimes_S X_{n+2} \to M \otimes_S X_{n+1} \to M \otimes_S X_n$ is exact for any finitely presented S^{op}-module M by (5), $X_{n+2} \to X_{n+1} \to X_n$ is exact. Let $L = \mathrm{coker}(\partial_{n+2}^X)$ and $T = \mathrm{coker}(\partial_{n+1}^X)$. It is easy to check that there exist a monomorphism $f : L \to X_n$ and a homomorphism $\beta : T \to X_{n-1}$ such that $\partial_{n+1}^X = fh$ and $\partial_n^X = \beta t$, where h and t are natural maps. Thus $0 \to L \to X_n \to T \to 0$ is a pure exact sequence of S-modules by [15, Lemma 2.5]. So T and L are in $\mathcal{L}_C(S)$ by Lemma 3.13(2).

We are going to show that the complex $0 \to T \xrightarrow{\beta} X_{n-1} \to \cdots \to X_1 \to X_0 \to N \to 0$ is an augmented proper left $\mathcal{L}_C(S)$-resolution of N. Put $K = \ker(\partial_n^X)$ and $D = \ker(\partial_{n-1}^X)$. So we have the following commutative diagram:



where $\alpha : X_{n+1} \to K$ and $\gamma : X_n \to D$ are $\mathcal{L}_C(S)$-precover, i and q are inclusions. Note that $\ker(h) = \mathrm{im}(\partial_{n+2}^X) = \ker(\partial_{n+1}^X) = \ker(\alpha)$. Then there exists a monomorphism $\omega : L \to K$ such that $\alpha = \omega h$. Similarly, we have a homomorphism $\phi : T \to D$ such that $\gamma = \phi t$. Note that $\alpha : X_{n+1} \to K$ is a $\mathcal{L}_C(S)$-precover. Then $\omega : K \to L$ is a monic $\mathcal{L}_C(S)$-precover.

Let $X = \ker(\partial_n^X)/\mathrm{im}(\partial_{n+1}^X)$. It is easy to check that the sequences $0 \to L \overset{\omega}{\to} K \overset{u}{\to} X \to 0$ and $0 \to X \to T \overset{\phi}{\to} D$ are exact, where $u : K \to X$ is the natural map. We wish to show that $\phi_* : \mathrm{Hom}_S(F, T) \to \mathrm{Hom}_S(F, D)$ is monic for any C-level S-module F. By the exactness of $0 \to X \to T \overset{\phi}{\to} D$, it suffices to show that $\mathrm{Hom}_S(F, X) = 0$ for any C-level S-module F. Let $g : F \to X$ with $F \in \mathcal{L}_C(S)$. By pullback, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \overset{\rho}{\longrightarrow} & G & \overset{\mu}{\longrightarrow} & F & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & L & \overset{\omega}{\longrightarrow} & K & \overset{u}{\longrightarrow} & X & \longrightarrow & 0. \end{array}$$

Since L and F are in $\mathcal{L}_C(S)$, so is G by Lemma 3.13(1). Note that $\omega : L \to K$ is a monic $\mathcal{L}_C(S)$-precover. Then there exists $\rho' : G \to L$ such that $\theta = \omega\rho'$. Thus $\omega = \theta\rho = \omega\rho'\rho$, and hence $\rho'\rho = 1$. Consequently, there exists $\lambda : F \to K$ such that $g = u\lambda$. Since $\omega : L \to K$ is a monic $\mathcal{L}_C(S)$-precover, there exists $\tau : F \to L$ such that $\lambda = \omega\tau$. So $g = u\lambda = u\omega\tau = 0$, as desired.

Finally we show that the complex $0 \to T \overset{\beta}{\to} X_{n-1} \to \cdots \to X_1 \to X_0 \to N \to 0$ is $\mathrm{Hom}_S(F, -)$ exact for any C-level S-module F. It suffices to show that the following sequence

$$0 \longrightarrow \mathrm{Hom}_S(F,T) \xrightarrow{\beta_*} \mathrm{Hom}_S(F,X_{n-1}) \xrightarrow{(\partial_{n-1}^X)_*} \mathrm{Hom}_S(F,X_{n-2})$$

is exact. Since $\partial_{n-1}^X\beta = 0$, $(\partial_{n-1}^X)_*\beta_* = (\partial_{n-1}^X\beta)_* = 0$. Thus $\mathrm{im}(\beta_*) \subseteq \ker((\partial_{n-1}^X)_*)$. One easily checks that $\ker((\partial_{n-1}^X)_*) = \mathrm{im}((\partial_n^X)_*) \subseteq \mathrm{im}(\beta_*)$. So $\mathrm{im}(\beta_*) = \ker((\partial_{n-1}^X)_*)$. It is left to show that β_* is monic. Since $\beta t = \partial_n^X = q\gamma = q\phi t$ by the proof above, $\beta = q\phi$ by noting that t is epic. Hence $\beta_* = q_*\phi_*$. Note that ϕ_* is monic by the proof above. So β_* is monic. This completes the proof. $\square$

Note that $\mathcal{L}_C(R) = \mathcal{F}_C(R)$ if $R = S$ is a commutative Noetherian ring by [2, Corollary 2.9]. Thus we have the following

Corollary 3.15 *(*[19, Theorem 5.4]*) Let C be a semidualizing module over a commutative Noetherian ring R and n a non-negative integer. Then the following are equivalent for any R-module N:*

(1) *$\mathcal{F}_C(R)$-pd$(N) \leqslant n$;*
(2) *$\mathrm{Tor}_i^{\mathcal{MF}_C}(M, N) = 0$ for each R-module M and any $i > n$;*
(3) *$\mathrm{Tor}_{n+1}^{\mathcal{MF}_C}(M, N) = 0$ for each R-module M.*

Note that $\mathcal{F}_C(S)$ is closed under pure submodules, pure quotients and kernels of epimorphisms by [13, Lemmma 5.2(a) and Corollary 6.4]. As in the proof of Theorem 3.14, we can extend Corollary 3.15 to the case that R and S are rings and ${}_SC_R$ is a faithfully semidualizing bimodule.

Corollary 3.16 *Let ${}_SC_R$ be a faithfully semidualizing bimodule and n a non-negative integer. Then the following are equivalent for any S-module N:*

1. $\mathcal{F}_C(S)$*-pd*$(N) \leqslant n$;
2. $Tor_i^{\mathcal{MF}_C}(M, N) = 0$ *for each* S^{op}*-module M and any* $i > n$;
3. $Tor_{n+1}^{\mathcal{MF}_C}(M, N) = 0$ *for each* S^{op}*-module M.*

Proposition 3.17 *Let ${}_SC_R$ be a faithfully semidualizing bimodule and n a non-negative integer. Then the following are equivalent for any* S^{op}*-module M:*

1. $\mathscr{R}$*-dim*$_{\mathcal{FI}_C^{\infty}(S^{op})}M \leqslant n$;
2. $\mathcal{FI}_C^{\infty}(S^{op})$*-id*$(M) \leqslant n$;
3. $M \in \mathcal{A}_C(S^{op})$ *and for any exact sequence* $0 \to M \to X_0 \to X_{-1} \to \cdots$ *of* S^{op}*-modules with each* X_i *in* $\mathcal{FI}_C^{\infty}(S^{op})$*, coker*$(X_{-n+2} \to X_{-n+1})$ *is in* $\mathcal{L}_C(S)$*, where* $X_2 = 0$ *and* $X_1 = M$.

Proof The proof is dual to that of (1) ⇔ (2) ⇔ (3) in Theorem 3.14. □

We end this section with the following corollary.

Corollary 3.18 *Let ${}_SC_R$ be a faithfully semidualizing bimodule and n a non-negative integer. Then the following are equivalent for any* S^{op}*-module N:*

1. $\mathscr{R}$*-dim*$_{\mathcal{FI}_C^{\infty}(S^{op})}N \leqslant n$;
2. $\mathscr{L}$*-dim*$_{\mathcal{L}_C(S)}N^+ \leqslant n$;
3. $Tor_i^{\mathcal{ML}_C}(M, N^+) = 0$ *for each (finitely presented)* S^{op}*-module M and any* $i > n$;
4. $Tor_{n+1}^{\mathcal{ML}_C}(M, N^+) = 0$ *for each (finitely presented)* S^{op}*-module M.*

Proof (1) ⇔ (2) follows from Theorem 3.14 and Proposition 3.17 by noting that Proposition 3.7(2) holds.

(2) ⇔ (3) ⇔ (4) hold by Theorem 3.14. □

4 Applications

Some applications are given in this section. We start with the following theorem which characterizes when every S-module has a monic C-level precover. It contains Theorem 1.2(1) from the introduction.

Theorem 4.1 *Let ${}_SC_R$ be a faithfully semidualizing bimodule and n a non-negative integer. Then the following are equivalent:*

(1) $\mathcal{L}_C(S) = \mathsf{Mod}(S)$;
(2) *Every S-module has a monic C-level precover;*
(3) $Tor_1^{\mathcal{ML}_C}(M, N) = 0$ *for any* S^{op}*-module M and any S-module N.*

Proof (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). Let M be an S-module. Then M has a monic C-level precover $f : C\otimes_R F \to M$ by (2). Hence $f_* : \mathrm{Hom}_S(C, C\otimes_R F) \to \mathrm{Hom}_S(C, M)$ is an isomorphism. Since $C\otimes_R F \in \mathcal{B}_C(S)$ by Lemma 3.6(1), the sequence $0 \to \mathrm{Hom}_S(C, C\otimes_R F) \xrightarrow{f_*} \mathrm{Hom}_S(C, M) \to \mathrm{Hom}_S(C, \mathrm{coker}(f)) \to 0$ is exact. It follows that $\mathrm{Hom}_S(C, \mathrm{coker}(f)) = 0$. Since ${}_SC_R$ is a faithfully semidualizing bimodule by hypothesis, we get that $\mathrm{coker}(f) = 0$. So f is an isomorphism and M is C-level, as desired.

(1) $\Leftrightarrow$ (3) holds by Theorem 3.14. $\square$

Let N be an S-module and $\cdots \to X_1 \xrightarrow{f} X_0 \xrightarrow{\varepsilon} N \to 0$ an augmented proper left $\mathcal{L}_C(S)$-resolution of N. Applying $M\otimes_S -$, we obtain the deleted complex

$$\cdots \longrightarrow M\otimes_S X_1 \xrightarrow{1\otimes_S f} M\otimes_S X_0 \longrightarrow 0.$$

Then $\mathrm{Tor}_n^{\mathcal{M}\mathcal{L}_C}(M, N)$ is exactly the nth homology of the complex above. There is a canonical map

$$\theta : \mathrm{Tor}_0^{\mathcal{M}\mathcal{L}_C}(M, N) = \mathrm{coker}(1\otimes_S f) \to M\otimes_S N$$

defined by $\theta(\sum(m_i\otimes n_i + \mathrm{im}(1\otimes_S f))) = \sum(m_i\otimes \varepsilon(n_i))$ for any $\sum(m_i\otimes n_i) \in M\otimes_S X_0$. Next we have the following result which contains Theorem 1.2(2).

Theorem 4.2 *Let ${}_SC_R$ be a faithfully semidualizing bimodule and n a non-negative integer. Then the following are equivalent:*

(1) *Every S-module has an epic C-level precover;*
(2) *$\theta : Tor_0^{\mathcal{M}\mathcal{L}_C}(M, N) \to M\otimes_S N$ is an isomorphism for any S^{op}-module M and any S-module N;*
(3) *$\theta : Tor_0^{\mathcal{M}\mathcal{L}_C}(M, N) \to M\otimes_S N$ is an epimorphism for any S^{op}-module M and any S-module N.*

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are straightforward.

(3) $\Rightarrow$ (1). Let N be an S-module and $\cdots \to X_1 \xrightarrow{f} X_0 \xrightarrow{\varepsilon} N \to 0$ an augmented proper left $\mathcal{L}_C(S)$-resolution of N. Then $\theta : \mathrm{Tor}_0^{\mathcal{M}\mathcal{L}_C}(S, N) \to S\otimes_S N$ is an epimorphism. If we set $\pi : S\otimes_S X_0 \to \mathrm{Tor}_0^{\mathcal{M}\mathcal{L}_C}(S, N)$ be the natural map, then $1\otimes_S \varepsilon = \theta\pi$. Hence $1\otimes_S \varepsilon : S\otimes_S X_0 \to S\otimes_S N$ is epic. So $\varepsilon : X_0 \to N$ is an epic C-level precover. This completes the proof. $\square$

Corollary 4.3 *Let R be a commutative Noetherian ring and C a semidualizing module. Then the following are equivalent:*

(1) *C is projective;*
(2) *Every R-module has an epic C-flat precover;*
(3) *$\theta : Tor_0^{\mathcal{M}\mathcal{F}_C}(M, N) \to M\otimes_R N$ is an isomorphism for all R-modules M and N;*
(4) *$\theta : Tor_0^{\mathcal{M}\mathcal{F}_C}(M, N) \to M\otimes_R N$ is an epimorphism for all R-modules M and N.*

Proof Note that $\mathcal{L}_C(R) = \mathcal{F}_C(R)$ if R is a commutative Noetherian ring by [2, Corollary 2.9].

(1) $\Rightarrow$ (2). Since C is projective by (1), $\mathsf{Mod}(R) = \mathcal{B}_C(R)$ by [21, Corollary 4.1.6]. So (2) holds by Lemma 3.12(2).

(2) $\Rightarrow$ (1). It follows from (2) that R has an epic C-flat precover. Thus R is C-flat, and hence $R \in \mathcal{B}_C(R)$. So (1) holds by [21, Corollary 4.1.6].

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follow from Theorem 4.2. □

The following theorem characterizes when every S-module has a monic C-level preenvelope. It contains Theorem 1.2(3) from the introduction.

Theorem 4.4 *The following are equivalent for any faithfully semidualizing bimodule* ${}_SC_R$*:*

(1) $S \in \mathcal{FI}_C^{\infty}(S^{op})$;
(2) *Every* S*-module has a monic* C*-level preenvelope;*
(3) *Every* S^{op}*-module has an epic* C*-FP*$_\infty$*-injective cover;*
(4) $Tor_1^{\mathcal{ML}_C}(M, (S_S)^+) = 0$ *for each (finitely presented)* S^{op}*-module* M.

Proof (1) $\Rightarrow$ (2). Let M be an S-module. Then M has a C-level preenvelope $f : M \to F$ by Proposition 3.7(3). Consider the injective envelope $g : M \to E$ of M. There exists an exact sequence $S^{(I)} \to E^+ \to 0$ of S^{op}-modules. Thus we have an exact sequence $0 \to E^{++} \to (S^{(I)})^+$ of S-modules, and hence $(S^{(I)})^+ \in \mathcal{L}_C(S)$ by (1) and Proposition 3.7. Note that $0 \to E \to (S^{(I)})^+$ is split exact. It follows from Proposition 3.7(1) that $E \in \mathcal{L}_C(S)$. Thus there exists $h : F \to E$ such that $g = hf$ by noting that $f : M \to F$ is a C-level preenvelope. So f is monic since g is monic.

(2) $\Rightarrow$ (1). Note that $(S_S)^+$ has a monic C-level preenvelope $(S_S)^+ \to F$ by (2). Then $(S_S)^+ \in \mathcal{L}_C(S)$ by Proposition 3.7(1). So $S \in \mathcal{FI}_C^{\infty}(S^{op})$ by Proposition 3.7(2).

(1) $\Rightarrow$ (3). Let M be an S^{op}-module. It follows from Proposition 3.7(3) that M has a C-FP$_\infty$-injective cover $\alpha : L \to M$. There exists an exact sequence $\beta : S^{(I)} \to M \to 0$ of S^{op}-modules. Note that $S^{(I)} \in \mathcal{FI}_C^{\infty}(S^{op})$ by (1) and Proposition 3.7(1). Thus there exists $\gamma : S^{(I)} \to L$ such that $\beta = \alpha\gamma$ by noting that $\alpha : L \to M$ is a C-FP$_\infty$-injective cover. So α is epic since β is epic.

(3) $\Rightarrow$ (1). The proof is dual to that of (2) $\Rightarrow$ (1).

(1) $\Leftrightarrow$ (4) holds by Corollary 3.18. □

The following result characterizes when every S-module has an epic C-level preenvelope.

Theorem 4.5 *The following are equivalent for any faithfully semidualizing bimodule* ${}_SC_R$*:*

(1) *Every* S*-module has an epic* C*-level preenvelope;*
(2) *Every submodule of any* C*-level* S*-module is* C*-level;*
(3) *Every quotient of any* C*-FP*$_\infty$*-injective* S^{op}*-module is* C*-FP*$_\infty$*-injective;*
(4) *Every* S^{op}*-module has a monic* C*-FP*$_\infty$*-injective cover;*
(5) $\mathscr{R}\text{-dim}_{\mathcal{FI}_C^{\infty}(S^{op})} M \leqslant 1$ *for any* S^{op}*-module* M*;*
(6) $\mathscr{L}\text{-dim}_{\mathcal{L}_C(S)} M \leqslant 1$ *for any* S*-module* M.

Proof (1) $\Rightarrow$ (2). Assume that M is a C-level S-module. Let N be a submodule of M and $i : N \to M$ the inclusion. Note that N has an epic C-level preenvelope $f : N \to F$ by (1).

Then there exists $g : F \to M$ such that $i = gf$. Since i is the inclusion, f is monic. So f is an isomorphism and N is C-level, as desired.

(2) $\Rightarrow$ (1). Let M be an S-module. Then M has a C-level preenvelope $f : M \to F$ by Proposition 3.7(3). Note that $\mathrm{im}(f)$ is C-level by (2). So $M \to \mathrm{im}(f)$ is an epic C-level preenvelope.

(2) $\Rightarrow$ (3). Let $0 \to N \to M \to M/N \to 0$ be an exact sequence of S^{op}-modules with $M \in \mathcal{FI}_C^{\infty}(S^{op})$. Then $0 \to (M/N)^+ \to M^+ \to N^+ \to 0$ is an exact sequence of S-modules. Note that $M^+ \in \mathcal{L}_C(S)$ by Proposition 3.7(2). Then $(M/N)^+$ is C-level by (2). Applying Proposition 3.7(2) again, we get that M/N is a C-FP_∞-injective S^{op}-module, as desired.

(3) $\Rightarrow$ (4). Let M be an S^{op}-module. It follows from Proposition 3.7(3) that M has a C-FP_∞-injective cover $f : E \to M$. Note that $\mathrm{im}(f)$ is C-FP_∞-injective by (3). It is easy to check that $i : \mathrm{im}(f) \to M$ is a monic C-FP_∞-injective cover of M.

(4) $\Rightarrow$ (2). Assume that M is a C-level S-module. Let N be a submodule of M and $i : N \to M$ the inclusion. Then the sequence $0 \to (M/N)^+ \to M^+ \xrightarrow{\varphi} N^+ \to 0$ of S^{op}-modules is exact such that M^+ is C-FP_∞-injective by Proposition 3.7(2). Note that N^+ has a monic C-FP_∞-injective cover $f : E \to N^+$ by (4). Then there exists $g : M^+ \to E$ such that $\varphi = fg$. Since φ is epic, so is f. Thus f is an isomorphism, and hence N^+ is a C-FP_∞-injective S^{op}-module. So N is C-level by Proposition 3.7(2), as desired.

(3) $\Rightarrow$ (5) is trivial.

(5) $\Rightarrow$ (6). Let M be an S-module. Then $\mathscr{R}$-$\dim_{\mathcal{FI}_C^{\infty}(S^{op})} M^+ \leqslant 1$ by (5). Thus $M^+ \in \mathcal{A}_C(S^{op})$ by Proposition 3.17, and hence $M \in \mathcal{B}_C(S)$ by Lemma 2.7. Note that there exists an exact sequence $\cdots \to X_1 \to X_0 \to M \to 0$ of S-modules with each X_i in $\mathcal{L}_C(S)$ by Proposition 3.7(3) and Lemma 3.12(1). Let $K = \ker(X_0 \to M)$. Then the sequence $0 \to M^+ \to {X_0}^+ \to {X_1}^+ \to \cdots$ of S^{op}-modules is exact. It follows from Proposition 3.17 that $K^+ \in \mathcal{FI}_C^{\infty}(S^{op})$. Thus $K \in \mathcal{L}_C(S)$ by Proposition 3.2(2). So (6) holds by Theorem 3.14, as desired.

(6) $\Rightarrow$ (2). Let F be a C-level S-module and M a submodule of F. Then $\mathscr{L}$-$\dim_{\mathcal{L}_C(S)} F/M \leqslant 1$ by (6). Hence $F/M \in \mathcal{B}_C(S)$ by Theorem 3.14. Since $F \in \mathcal{B}_C(S)$ by Lemma 3.6(1), so is M by [13, Corollary 6.3]. It follows from Proposition 3.7(3) and Lemma 3.12(1) that M has an exact augmented proper left $\mathcal{L}_C(S)$-resolution. So we have an exact sequence $\cdots \to X_2 \to X_1 \to F \to F/M \to 0$ of S-modules with $X_i \in \mathcal{L}_C(S)$ for any $i \geqslant 1$. Applying Theorem 3.14 again, we get that M is a C-level S-module. This completes the proof. □

The following corollary contains Theorem 1.2(4) from the introduction.

Corollary 4.6 *The following are equivalent for any faithfully semidualizing bimodule* ${}_SC_R$*:*

(1) *Every* S*-module has an epic* C*-level preenvelope;*
(2) *Every* S^{op}*-module has a monic* C*-*FP_∞*-injective precover;*
(3) $Tor_2^{\mathcal{ML}_C}(M, N^+) = 0$ *for each (finitely presented)* S^{op}*-module* M *and any* S^{op}*-module* N*;*
(4) $Tor_2^{\mathcal{ML}_C}(M, N) = 0$ *for each (finitely presented)* S^{op}*-module* M *and any* S*-module* N*.*

Proof The result follows from Theorem 4.5 and Corollary 3.18. □

Recall that an R-module M is called FP-injective (or absolutely pure) [17, 24] if $\mathrm{Ext}_R^1(P, M) = 0$ for every finitely presented R-module P. Denote by $\mathcal{FI}(R^{op})$ the class of FP-injective R^{op}-modules. We let $\mathcal{FI}_C(S^{op}) = \{\mathrm{Hom}_{R^{op}}(C, I) \mid I \in \mathcal{FI}(R^{op})\}$ whenever ${}_SC_R$ is a semidualizing bimodule. Such modules are called C-FP-injective S^{op}-modules by [28, Definition 2.3].

Theorem 4.7 *The following are equivalent for any faithfully semidualizing bimodule ${}_SC_R$:*

(1) *R is right coherent;*
(2) *$\mathcal{FI}_C^{\infty}(S^{op}) = \mathcal{FI}_C(S^{op})$;*
(3) *Every C-FP-injective preenvelope over S^{op} is a C-FP_∞-injective preenvelope;*
(4) *$\mathcal{L}_C(S) = \mathcal{F}_C(S)$;*
(5) *Every C-flat precover over S is a C-level precover;*
(6) *Every C-flat cover over S is a C-level cover;*
(7) *Every C-level precover over S is a C-flat precover.*

Proof (1) $\Rightarrow$ (2). The proof is straightforward since $\mathcal{FI}_\infty(R^{op}) = \mathcal{FI}(R^{op})$ by [2, Corollary 2.9].

(2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) and (4) $\Rightarrow$ (7) are trivial.

(3) $\Rightarrow$ (1). To prove (1), it suffices to show that every FP_∞-injective R^{op}-module is FP-injective by [2, Corollary 2.9]. Let X be a FP_∞-injective R^{op}-module. Then there exists an exact sequence $0 \to X \overset{f}{\to} E \to L \to 0$ of R^{op}-modules with $E \in \mathcal{FI}(R^{op})$ and $L \in \mathcal{FI}(R^{op})^{\perp_1}$ by [18, Remark 2.8]. Note that X and E are in $\mathcal{B}_C(R^{op})$ by Proposition 3.3(2) and [28, Theorem 2.1]. Then L is $\mathcal{B}_C(R^{op})$ by Facts 2.4(1). Thus the sequence $0 \to \mathrm{Hom}_{R^{op}}(C, X) \overset{f_*}{\to} \mathrm{Hom}_{R^{op}}(C, E) \to \mathrm{Hom}_{R^{op}}(C, L) \to 0$ of S^{op}-modules is exact. Since $\mathrm{Ext}^1_{S^{op}}(\mathrm{Hom}_{R^{op}}(C, L), \mathrm{Hom}_{R^{op}}(C, Y)) \cong \mathrm{Ext}^1_{R^{op}}(L, Y) = 0$ for any $Y \in \mathcal{FI}(R^{op})$ by [28, Theorem 2.1] and [13, Theorem 6.4(b)], $f_* : \mathrm{Hom}_{R^{op}}(C, X) \to \mathrm{Hom}_{R^{op}}(C, E)$ is a C-FP-injective preenvelope. Thus f_* is a C-FP_∞-injective preenvelope by (3), and hence there exists $\varphi : \mathrm{Hom}_{R^{op}}(C, E) \to \mathrm{Hom}_{R^{op}}(C, X)$ such that $1 = \varphi f_*$. It follows from Proposition 3.7(1) that $\mathrm{Hom}_{R^{op}}(C, X)$ is C-FP-injective. Hence there exists $I \in \mathcal{FI}(R^{op})$ such that $\mathrm{Hom}_{R^{op}}(C, X) \cong \mathrm{Hom}_{R^{op}}(C, I)$. Thus $X \cong \mathrm{Hom}_{R^{op}}(C, X) \otimes_S C \cong \mathrm{Hom}_{R^{op}}(C, I) \otimes_S C \cong I$. So X is FP-injective, as desired.

(1) $\Rightarrow$ (4) holds by [2, Corollary 2.11].

(5) $\Rightarrow$ (6). Let M be an S-module and $f : X \to M$ a C-flat cover of M. Then f is a C-level precover by (5). Assume that $g : X \to X$ is a homomorphism such that $f = fg$. Since f is a C-flat cover, g is an automorphism. So f is a C-level cover, as desired.

(6) $\Rightarrow$ (1). By [2, Corollary 2.11], it suffices to show that every level R-module is flat. Let M be a level R-module. Then there exists an exact sequence $0 \to K \to F \overset{f}{\to} M \to 0$ of R-modules with F flat and K cotorsion such that f is a flat cover by [9, Theorem 4.1.1(a)]. Note that M is in $\mathcal{A}_C(R)$ by Proposition 3.3. It follows that K is in $\mathcal{A}_C(R)$ by noting that $F \in \mathcal{A}_C(R)$. Thus the sequence $0 \to C \otimes_R K \to C \otimes_R F \overset{1 \otimes_R f}{\to} C \otimes_R M \to 0$ of S-modules is exact. Since $\mathrm{Ext}^1_S(C \otimes_R G, C \otimes_R K) \cong \mathrm{Ext}^1_R(G, K) = 0$ for any flat R-module G by [13, Theorem 6.4(a)], $1 \otimes_R f : C \otimes_R F \to C \otimes_R M$ is a C-flat precover. Hence there exists $\varphi : C \otimes_R F \to C \otimes_R F$ such that $1 \otimes_R f = (1 \otimes_R f)\varphi$. We will show

that φ is an isomorphism. Note that we have the following commutative diagram with exact rows such that the columns are isomorphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_S(C, C\otimes_R K) & \longrightarrow & \mathrm{Hom}_S(C, C\otimes_R F) & \xrightarrow{(1\otimes_R f)_*} & \mathrm{Hom}_S(C, C\otimes_R M) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \mu_K} & & \uparrow{\scriptstyle \mu_F} & & \uparrow{\scriptstyle \mu_M} & & \\ 0 & \longrightarrow & K & \longrightarrow & F & \xrightarrow{f} & M & \longrightarrow & 0. \end{array}$$

Since μ_F and μ_M are isomorphisms, we have

$$\begin{aligned} f &= \mu_M^{-1} \circ (1 \otimes_R f)_* \circ \mu_F \\ &= \mu_M^{-1} \circ (1 \otimes_R f)_* \circ \varphi_* \circ \mu_F \\ &= \mu_M^{-1} \circ (1 \otimes_R f)_* \circ \mu_F \circ \mu_F^{-1} \circ \varphi_* \circ \mu_F \\ &= f \circ \mu_F^{-1} \circ \varphi_* \circ \mu_F. \end{aligned}$$

Since f is a flat cover, $\mu_F^{-1} \circ \varphi_* \circ \mu_F$ is an isomorphism. Hence φ_* is an isomorphism. Applying $\mathrm{Hom}_S(C, -)$ to the exact sequence $0 \to \ker(\varphi) \to C \otimes_R F \xrightarrow{\varphi} C \otimes_R F$ of S-modules, we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_S(C, \ker(\varphi)) \longrightarrow \mathrm{Hom}_S(C, C\otimes_R F) \xrightarrow{\varphi_*} \mathrm{Hom}_S(C, C\otimes_R F).$$

Note that $\mathrm{Hom}_S(C, \ker(\varphi)) = 0$ since φ_* is an isomorphism. It follows from [13, Proposition 3.1] that $\ker(\varphi) = 0$. Thus $\varphi : C \otimes_R F \to C \otimes_R F$ is a monomorphism. Applying $\mathrm{Hom}_S(C, -)$ to the exact sequence $0 \to C \otimes_R F \xrightarrow{\varphi} C \otimes_R F \to \mathrm{coker}(\varphi) \to 0$ of S-modules, we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_S(C, C\otimes_R F) \xrightarrow{\varphi_*} \mathrm{Hom}_S(C, C\otimes_R F) \longrightarrow \mathrm{Hom}_S(C, \mathrm{coker}(\varphi)) \longrightarrow 0.$$

Since φ_* is an isomorphism, $\mathrm{Hom}_S(C, \mathrm{coker}(\varphi)) = 0$. Thus $\mathrm{coker}(\varphi) = 0$ by [13, Proposition 3.1], and hence φ is an isomorphism. So $1 \otimes_R f : C \otimes_R F \to C \otimes_R M$ is a C-flat cover. Consequently, $1 \otimes_R f$ is a C-level cover by (6). Thus there exists $\psi : C \otimes_R M \to C \otimes_R F$ such that $1 = (1 \otimes_R f) \circ \psi$. It follows from [13, Proposition 5.1] that $C \otimes_R M$ is C-flat. Hence there exists a flat R-module L such that $C \otimes_R M \cong C \otimes_R L$. Thus $M \cong \mathrm{Hom}_S(C, C \otimes_R M) \cong \mathrm{Hom}_S(C, C \otimes_R L) \cong L$. So M is flat, as desired.

(7) $\Rightarrow$ (4). Let M be a C-level S-module. Then $1_M : M \to M$ is a C-level precover. Thus $1_M : M \to M$ is a C-flat precover by (7). So M is C-flat. This completes the proof. □

Corollary 4.8 *The following are equivalent for any faithfully semidualizing bimodule ${}_SC_R$:*

(1) *R is right coherent and left perfect;*
(2) *$\mathcal{L}_C(R) = \mathcal{P}_C(R)$;*
(3) *Every C-projective precover over R is a C-level precover.*

Proof The results hold by Theorem 4.7 and [28, Theorem 6.1]. □

Corollary 4.9 *The following are equivalent for any faithfully semidualizing bimodule* ${}_SC_R$*:*

(1) *R is right Noetherian;*
(2) $\mathcal{FI}_C^{\infty}(S^{op}) = \mathcal{I}_C(S^{op})$;
(3) *Every* C*-injective preenvelope over* S^{op} *is a* C*-*FP_{∞}*-injective preenvelope;*

Proof The results hold by Theorem 4.7 and [28, Proposition 5.1]. □

The next result parallels [19, Construction 3.9].

Construction 4.10 For each integer i, there is a natural transformation of bifunctors $\varepsilon_i(-,-) : \mathrm{Tor}_i^{\mathcal{MF}_C}(-,-) \to \mathrm{Tor}_i^{\mathcal{ML}_C}(-,-)$. To construct $\varepsilon_i(-,-)$, let $X \to N$ be a proper left C-level resolution of an S-module N. Choose a proper left C-flat resolution $F \to N$ and a morphism $f : F \to X$ lifting the identity on N. For each S^{op}-module M, let $\varepsilon_i(M,N) : \mathrm{Tor}_i^{\mathcal{MF}_C}(M,N) \to \mathrm{Tor}_i^{\mathcal{ML}_C}(M,N)$ be the natural homomorphism induced by the morphism of complexes $1 \otimes_S f : M \otimes_S F \to M \otimes_S X$.

We now finish this paper by giving the proof of Theorem 1.3 as follows.

4.11 Proof of Theorem 1.3. (1) ⇔ (2) ⇔ (3) ⇔ (4) follow from Theorem 4.5.

(2) ⇒ (5) holds by Construction 4.1.

(5) ⇒ (2). Let N be a C-level S-module. Then $\mathrm{Tor}_1^{\mathcal{ML}_C}(M,N) = 0$ for any S^{op}-module M. Thus $\mathrm{Tor}_1^{\mathcal{MF}_C}(M,N) = 0$ for any S^{op}-module M by (5). It is easily checked that N is C-flat and the proof is similar to that of (5) ⇒ (1) in Theorem 3.14. This completes the proof. □

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References

1. Anderson, F.W., Fuller, K.R.: Rings and Categories of Modules. Springer-Verlag, New York (1974)
2. Bravo, D., Hovey, M., Gillespie, J.: The stable module category of a general ring, preprint. arXiv:1405.5768
3. Cartan, H., Eilenberg, S.: Homological Algebra. Princeton Univ Press, Princeton (1956)
4. Christensen, L.W.: Gorenstein Dimensions, Lecture Notes in Math **1747** (2000)
5. Christensen, L.W.: Semi-dualizing complexes and their Auslander categories. Trans. Amer. Math. Soc **353**, 1839–1383 (2001)
6. Enochs, E.E.: Injective and flat covers, envelopes and resolvents. Israel J. Math **39**, 189–209 (1981)
7. Enochs, E.E., Jenda, O.M.G.: Relative Homological Algebra. Walter de Gruyter, Berlin-New York (2000)
8. Foxby, H.B.: Gorenstein modules and related modules. Math. Scand **31**, 267–284 (1972)
9. Göbel, R., Trlifaj, J.: Approximations and Endomorphism Algebras of Modules. Walter de Gruyter, New York (2006)
10. Golod, E.S.: G-dimension and generalized perfect ideals, Algebraic geometry and its applications, Collection of articles, Turdy Mat. Inst. Steklov **165**, 62–66 (1984)
11. Holm, H.: Gorenstein derived functors. Proc. Amer. Math. Soc **132**(7), 1913–1923 (2004)

12. Holm, H., Jørgensen, P.: Cotorsion pairs induced by duality pairs. J. Commut. Algebra **1**, 621–633 (2009)
13. Holm, H., White, D.: Foxby equivalence over associative rings. J. Math. Kyoto Univ **47**(4), 781–808 (2007)
14. Hovey, M.: Cotorsion pairs and model categories. Contemp. Math **436**, 277–296 (2007)
15. Hu, J.S., Zhang, D.D., Ding, N.Q.: Gorenstein right derived functors of $-\otimes-$ with respect to semidualizing modules, Comm. Algebra **42**(7), 3205–3219 (2014)
16. Kubik, B.: Quasidualizing modules. J. Commut. Algebra **6**, 209–229 (2014)
17. Maddox, B.H.: Absolutely pure modules. Proc. Amer. Math. Soc **18**, 155–158 (1967)
18. Mao, L.X., Ding, N.Q.: FP-projective dimensions. Comm. Algebra **33**, 1153–1170 (2005)
19. Salimi, M., Sather-Wagstaff, S., Tavasoli, E., Yassemi, S.: Relative Tor functors with respect to a semidualizing module. Algebr. Represent. Theory **17**, 103–120 (2014)
20. Salce, L.: Cotorsion theories for abelian groups. Symposia Math. **23**, 11–32 (1979)
21. Sather-Wagstaff, S.: Semidualizing modules, http://www.ndsu.edu/pubweb/ssatherw/DOCS/sdmhist.html
22. Sather-Wagstaff, S., Sharif, T., White, D.: Comparison of relative cohomology theories with respect to semidualizing modules. Math. Z **264**, 571–600 (2010)
23. Sather-Wagstaff, S., Sharif, T., White, D.: AB-contexts and stability for Gorenstein flat modules with respect to semidualizing modules. Algebr. Represent. Theory **14**(3), 403–428 (2011)
24. Stenström, B.: Coherent rings and FP-injective modules . J. London Math. Soc **2**, 323–329 (1970)
25. Takahashi, R., White, D.: Homological aspects of semidualizing modules. Math. Scand **106**(1), 5–22 (2010)
26. Tang, X.: FP-injectivity relative to a semidualizing bimodule. Publ. Math. Debrecen **80**, 311–326 (2012)
27. Vasconcelos, W.V.: Divisor Theory in Module Categories, North-Holland Math. Stud., vol. 14. North-Holland Publishing Co., Amsterdam (1974)
28. Zhang, D.D., Ouyang, B.Y.: Semidualizing bimodules and related modules. J. Algebra Appl. **10**(6), 1261–1282 (2011)