Global conservative and dissipative solutions of a coupled Camassa-Holm equations

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I. INTRODUCTION

The well-known Camassa-Holm equation,3

\[ u_t - u_{txx} + ku_x + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \quad t > 0, x \in \mathbb{R}, \]

(1.1)

has attracted much attention in the past decade. It models the propagation of unidirectional surface waves over a flat bed with \( u(t, x) \) representing the fluid velocity at time \( t \) in the horizontal direction. Equation (1.1) was actually found much earlier by Fuchssteiner and Fokas16 as an abstract bi-Hamiltonian partial differential equation with infinitely many conservation laws. It admits bi-Hamiltonian structure3 and is completely integrable. It attracted a lot of attention after Ca-
massa and Holm derived it as a model for shallow water waves and discovered that it is formally integrable, in the sense that there is an associated Lax pair, and that its solitary waves are soli-
tons, i.e., they retain their shape and speed after the interaction with waves of the same type. The mathematical properties of Eq. (1.1) have also been studied further in many works, see e.g., Refs. 1, 2, 6, 7, 9–14, 22, 27, 28, 30, 32, and 34 and others. In Ref. 1, Bressan and Constantin developed a new approach in the analysis of Eq. (1.1). Based on the introduction of suitable Lagrangian variables, they transformed the equation into a semilinear hyperbolic system. Then a semigroup of conservative solutions was provided, globally defined forward and backward in time.

A similar approach, based on the introduction of suitable Lagrangian variables and a suitable modification in the definition of the semilinear hyperbolic system, a continuous semigroup of dissipative solutions of Eq. (1.1) was constructed forward in time in Ref. 2.

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Equation (1.1) admits many integrable multicomponent generalizations,\(^\text{25}\) the most popular of which is
\[
m_t + um_x + 2mu_x + \sigma \rho \rho_x = 0,
\]
\[
\rho_t + (\rho u)_x = 0,
\]  
(1.2)
where \(m = u - u_{xx}\) and \(\sigma = \pm 1\). Constantin and Ivanov\(^\text{8}\) derived this system in the context of shallow water theory. Here, \(u\) can be interpreted as the horizontal fluid velocity and \(\rho\) is related to the water elevation in the first approximation.\(^\text{8, 26}\) They showed that while small initial data develop into global solutions, for some initial data wave breaking occurs. They also discussed the solitary wave solutions. In Vlasov plasma models, system (1.2) describes the closure of the kinetic moments of the single-particle probability distribution for geodesic motion on the symplectomorphisms. While in the large-deformation diffeomorphic approach to image matching, system (1.2) is summoned in a type of matching procedure called metamorphosis (see Ref. 23 and the references therein). This system appeared originally in Ref. 31. Based on the deformation of bi-Hamiltonian structure of the hydrodynamic type, Chen et al.\(^\text{4}\) obtained system (1.2) when \(\sigma = -1\). They showed that it has the peakon and multikink solitons and is integrable in the sense that it has Lax pair. The mathematical properties of system (1.2) have been studied further in many works, see, e.g., Refs. 5, 15, 19–21, 24, 29, 35, and 36. In Ref. 23, Holm and Ivanov generalized the Lax pair formulation of system (1.2) to produce an integrable multicomponent family, \(CH(n, k)\), of equations with \(n\) components and \(1 \leq |k| \leq n\) velocities. They determined their Lie-Poisson Hamiltonian structures and gave numerical examples of their soliton solution behavior.

Recently, Fu and Qu\(^\text{17}\) proposed a coupled Camassa-Holm equations,
\[
m_t = 2mu_x + m_x u + (mv)_x + nv_x,
\]
\[
n_t = 2nv_x + n_x v + (nu)_x + mu_x,
\]  
(1.3)
with \(m = u - u_{xx}\), \(n = v - v_{xx}\), which has peakon solitons in the form of a superposition of multipeakons and may as well be integrable. They investigated local well-posedness and blow-up solutions of system (1.3) by means of Kato’s semigroup approach to nonlinear hyperbolic evolution equation and obtained a criterion and condition on the initial data guaranteeing the development of singularities in finite time for strong solutions of system (1.3) by energy estimates; moreover, an existence result for a class of local weak solutions was also given. In Ref. 33, Tian and Xu obtained the compact and bounded absorbing set and the existence of the global attractor for viscous system (1.3) with the periodic boundary condition in \(H^2\) by uniform prior estimate. By parameterizing \(\tilde{t} = -t\) for system (1.3), it then takes the following form:\(^\text{18}\)
\[
m_t + 2mu_x + m_x u + (mv)_x + nv_x = 0,
\]
\[
n_t + 2nv_x + n_x v + (nu)_x + mu_x = 0.
\]  
(1.4)
In particular, when \(u = 0\) (or \(v = 0\)), the degenerated system (1.4) has the same peakon solitons as the Camassa-Holm equation. Moreover, when \(u = v\), system (1.4) is reduced to the scalar Camassa-Holm equation,
\[
m_t + 4mu_x + 2mu_x = 0.
\]  
(1.5)
System (1.4) can be rewritten as a Hamiltonian system,
\[
\frac{\partial}{\partial t} \begin{pmatrix} m \\ n \end{pmatrix} = - \begin{pmatrix} \partial m + m \partial \sigma + n \partial \rho \\ \partial n + m \partial \sigma + n \partial \rho \end{pmatrix} \begin{pmatrix} \delta H/\delta m = u \\ \delta H/\delta n = v \end{pmatrix}
\]  
(1.6)
with the Hamiltonian \( H = \frac{1}{2} \int (mG \ast m + nG \ast n) dx \), where \( G \ast m = u \), \( G \ast n = v \), and \( G(x) = 1/2e^{-|x|} \). It is easy to verify that system (1.4) has the following conserved quantities:

\[
E_1(u) = \int_R udx, \quad E_2(v) = \int_R vdx, \quad E_3(u) = \int_R mdx, \quad E_4(u) = \int_R ndx, \quad E_5(u, v) = \int_R (u^2 + u_x^2 + v^2 + v_x^2) dx.
\]

In Ref. 18, Fu et al. established the precise blow-up scenarios of strong solutions and several results of blow-up solutions with certain initial profiles in detail. They also obtained the exact blow-up rate for system (1.4). To the best of our knowledge, other mathematical and physical properties of system (1.4) have not been studied.

The purpose of this paper is to construct a continuous semigroup of global conservative solutions and a continuous semigroup of global dissipative solutions to system (1.4). For the conservative equations, by introducing a set of independent and dependent variables and two characteristics, the equations are transformed into a semilinear hyperbolic system, whose solutions are obtained as the fixed points of a contractive transformation. Reverting to the original coordinates, a local, conservative solution of system (1.4) is obtained. Thanks to a uniform bound on the \( H^1 \)-norm, the solutions can be extended forward and backward in time. Our construction yields a continuous semigroup in the \( H^1 \times H^1 \) space. The dissipative case is more delicate because the corresponding ordinary differential equation (O.D.E.) now contains a discontinuous nonlocal source term. Observing that all discontinuities are crossed transversally, the existence and uniqueness of solutions can still be established. Moreover, the hyperbolic system can be established as an O.D.E. in a suitable space, which has locally bounded variations in the directions of two suitable cones, respectively. The well posedness of the Cauchy problem was thus provided in the conservative case. Returning to the original coordinates, a local, dissipative solution of system (1.4) was provided. Thanks to a uniform bound on the \( H^1 \)-norm, these solutions can be extended forward in time for all \( t \geq 0 \). We observe that energy loss can occur only through wave breaking for the dissipative solutions. The uniqueness of dissipative solutions to system (1.4) is a delicate issue. Our approaches are similar to but not the exact as the ones used in Refs. 1 and 2 for the Camassa-Holm equation. We point out that the variables used in Eq. (3.4) in this paper for system (1.4) is different from that in Eq. (2.3) in Refs. 1 and 2 for the Camassa-Holm equation. The approaches adopted in this paper make the computation much more direct and simple. Moreover, in contrast to the single characteristic used for the Camassa-Holm equation in Refs. 1 and 2, a coupled independent characteristics is introduced corresponding to the two components \( u \) and \( v \), respectively, which is critical to handling the two-component equations (1.4). So the analysis in this paper is much more complicated.

The remainder of this paper is organized as follows. Section II is the preliminary. In Sec. III, a global continuous semigroup of weak conservative solutions to system (1.4) will be constructed. In Sec. IV, a global continuous semigroup of weak dissipative solutions to system (1.4) is established.

### II. PRELIMINARY

System (1.4) can be rewritten in the following form:

\[
\begin{align*}
u_t + (u + v)u_x + A + B_x = 0, \\
v_t + (u + v)v_x + C + D_x = 0,
\end{align*}
\]

where \( A = G \ast (uv_v), B = G \ast (u^2 + \frac{1}{2}u_x^2 + u_x v_x + \frac{1}{2}v^2 + \frac{1}{2}v_x^2), C = G \ast (u_x v), D = G \ast (v^2 + \frac{1}{2}v_x^2 + n v_x + \frac{1}{2}u^2 + \frac{1}{2}u_x^2), \) and \( G = \frac{1}{2}e^{-|x|} \).

System (2.1) is also equivalent to the following nonlocal conservation law:

\[
\begin{align*}
u_t + (u + v)u_x + G \ast (uv_v) + \partial_x G \ast (u^2 + \frac{1}{2}u_x^2 + u_x v_x + \frac{1}{2}v^2 - \frac{1}{4}v_x^2) = 0, \quad t > 0, \quad x \in R, \\
v_t + (u + v)v_x + G \ast (u_x v) + \partial_x G \ast (v^2 + \frac{1}{2}v_x^2 + u_x v_x + \frac{1}{2}u^2 - \frac{1}{4}u_x^2) = 0, \quad t > 0, \quad x \in R.
\end{align*}
\]

The purpose of this paper is to construct a continuous semigroup of global conservative solutions and a continuous semigroup of global dissipative solutions to system (1.4). For the conservative equations, by introducing a set of independent and dependent variables and two characteristics, the equations are transformed into a semilinear hyperbolic system, whose solutions are obtained as the fixed points of a contractive transformation. Reverting to the original coordinates, a local, conservative solution of system (1.4) is obtained. Thanks to a uniform bound on the \( H^1 \)-norm, the solutions can be extended forward and backward in time. Our construction yields a continuous semigroup in the \( H^1 \times H^1 \) space. The dissipative case is more delicate because the corresponding ordinary differential equation (O.D.E.) now contains a discontinuous nonlocal source term. Observing that all discontinuities are crossed transversally, the existence and uniqueness of solutions can still be established. Moreover, the hyperbolic system can be established as an O.D.E. in a suitable space, which has locally bounded variations in the directions of two suitable cones, respectively. The well posedness of the Cauchy problem was thus provided in the conservative case. Returning to the original coordinates, a local, dissipative solution of system (1.4) was provided. Thanks to a uniform bound on the \( H^1 \)-norm, these solutions can be extended forward in time for all \( t \geq 0 \). We observe that energy loss can occur only through wave breaking for the dissipative solutions. The uniqueness of dissipative solutions to system (1.4) is a delicate issue. Our approaches are similar to but not the exact as the ones used in Refs. 1 and 2 for the Camassa-Holm equation. We point out that the variables used in Eq. (3.4) in this paper for system (1.4) is different from that in Eq. (2.3) in Refs. 1 and 2 for the Camassa-Holm equation. The approaches adopted in this paper make the computation much more direct and simple. Moreover, in contrast to the single characteristic used for the Camassa-Holm equation in Refs. 1 and 2, a coupled independent characteristics is introduced corresponding to the two components \( u \) and \( v \), respectively, which is critical to handling the two-component equations (1.4). So the analysis in this paper is much more complicated.

The remainder of this paper is organized as follows. Section II is the preliminary. In Sec. III, a global continuous semigroup of weak conservative solutions to system (1.4) will be constructed. In Sec. IV, a global continuous semigroup of weak dissipative solutions to system (1.4) is established.
In the following, the initial condition of system (2.1) is taken as
\[
\begin{align*}
\{ u(0, x) &= \bar{u}(x), \quad x \in R, \\
v(0, x) &= \bar{v}(x), \quad x \in R,
\end{align*}
\]  
(2.2)

where \((\bar{u}, \bar{v}) \in H^1(R) \times H^1(R)\), the space of absolutely continuous functions \((u, v) \in L^2(R) \times L^2(R)\) with derivatives \((u_x, v_x) \in L^2(R) \times L^2(R)\). This space is endowed with the norms,

\[
\|u\|_{H^1} = \left( \int_R \left( u^2 + u_x^2 \right) dx \right)^{1/2}, \quad \|v\|_{H^1} = \left( \int_R \left( v^2 + v_x^2 \right) dx \right)^{1/2}.
\]

Differentiating the first and the second equations in (2.1) with respect to \(x\), respectively, and using the identity \(\partial^2_t G * f = G * f - f\), we have

\[
\begin{align*}
\left\{ \begin{align*}
&u_{xt} + u^2 + u_x v_x + uv_{xx} + A_x + B - \left( u^2 + \frac{u^3}{2} + u_x v_x + \frac{v^2}{2} - \frac{v_x^2}{2} \right) = 0, \\
v_{xt} + v^2 + u_x v_x + uv_{xx} + C_x + D - \left( v^2 + \frac{v^3}{2} + u_x v_x + \frac{u^2}{2} - \frac{u_x^2}{2} \right) = 0.
\end{align*} \right.
\]  
(2.3)

Multiplying the first equation in (2.1) by \(u\) and the second by \(v\), and multiplying the first equation in (2.3) by \(u_x\) and the second by \(v_x\), we obtain the following equalities:

\[
\begin{align*}
\left( \frac{u^2}{2} \right)_t + \left( \frac{u^3}{3} \right)_x + uvu_x + uA + uBx &= 0, \\
\left( \frac{v^2}{2} \right)_t + \left( \frac{v^3}{3} \right)_x + uvv_x + vC + vDx &= 0,
\end{align*}
\]  
(2.4)

\[
\begin{align*}
\left( \frac{u_x^2}{2} \right)_t + \left( \frac{u_x^3}{3} \right)_x + uv_x u_{xx} + u^2 v_x - uA + uB - \frac{v^2 u_x}{2} + \frac{v_x^2}{2} &= 0, \\
\left( \frac{v_x^2}{2} \right)_t + \left( \frac{v_x^3}{3} \right)_x + uv_x v_{xx} + v^2 u_x - vC + v_x D - \frac{u_x^2 v_x}{2} + \frac{u_x^2}{2} &= 0.
\end{align*}
\]  
(2.6)

It follows from (2.4)–(2.7) that the total energy,

\[
E(t) = \int_R \left( u^2 (t, x) + u_x^2 (t, x) + v^2 (t, x) + v_x^2 (t, x) \right) dx,
\]  
(2.8)

is constant for regular solutions in time. Moreover, the following estimates can immediately be obtained:

\[
\begin{align*}
\|A(t)\|_{L^\infty} \cdot \|A_x(t)\|_{L^\infty} &\leq \frac{1}{2} e^{-|x|} \|u_x\|_{L^2} \leq \frac{1}{2} e^{-|x|} \left\| \frac{u_x^2 + v_x^2}{2} \right\|_{L^1} \leq \frac{1}{4} E(0), \\
\|B(t)\|_{L^\infty} \cdot \|B_x(t)\|_{L^\infty} &\leq \frac{1}{2} e^{-|x|} \|uv_x\|_{L^2} \leq \frac{1}{2} e^{-|x|} \left\| \frac{u^2 + u_x^2}{2} \right\|_{L^2} \leq \frac{1}{2} E(0), \\
\|A(t)\|_{L^2} \leq \frac{1}{2} e^{-|x|} \|uv_x\|_{L^2} &\leq \frac{1}{2} e^{-|x|} \left\| \frac{u^2 + v_x^2}{2} \right\|_{L^2} \leq \frac{\sqrt{3}}{4} E(0), \\
\|B(t)\|_{L^2} \cdot \|B_x(t)\|_{L^2} &\leq \frac{1}{2} e^{-|x|} \left\| \frac{u^2 + u_x^2}{2} + u_x v_x + \frac{v^2}{2} - \frac{v_x^2}{2} \right\|_{L^2} \leq \frac{\sqrt{2}}{2} E(0).
\end{align*}
\]  
(2.9)

The estimates for \(C, C_x, D,\) and \(D_x\) are entirely similar. The above estimates are very useful to obtain the global existence of solution to the Cauchy problem (2.1) and (2.2).
III. GLOBAL CONSERVATIVE SOLUTIONS OF SYSTEM (2.1)

A. Equivalent system

Consider the abstract quasilinear evolution equation of the form,

$$\frac{dz}{dt} = -zz_t + f(z), \quad (3.1)$$

where $z = (u, v)$, $z_t = (u_t, v_t)$, and $f(z) = -vu_t - A - Bx_t - uv_x - C - D$.

Definition 3.1: We call $z = (u, v)$ is a solution to the Cauchy problem (2.1) and (2.2) on a time interval $[0, T]$ if $z = (u(t, x), v(t, x))$ is defined on $[0, T] \times \mathbb{R}$, which is Hölder continuous with the following properties: (i) $z(t, \cdot) = (u(t, \cdot), v(t, \cdot)) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ for each fixed $t$. (ii) The map $t \to z(t, \cdot)$ is Lipschitz continuous from $[0, T]$ into $L^2(\mathbb{R})$, satisfying the initial condition (2.2) together with (3.1).

In the following, we will introduce a new set of independent and dependent variables and transform the equations into a semilinear hyperbolic system. First, for given initial data $\tilde{z} = (\tilde{u}, \tilde{v}) \in H^1 \times H^1$, we consider the following initial problem:

$$\begin{align*}
\frac{\partial}{\partial t} q_1(t, \xi) &= u(t, q_1(t, \xi)), & t \in [0, T], \\
\frac{\partial}{\partial t} q_2(t, \xi) &= v(t, q_2(t, \xi)), & t \in [0, T], \\
q_1(0, \xi) &= \tilde{q}_1(\xi), & x \in \mathbb{R}, \\
q_2(0, \xi) &= \tilde{q}_2(\xi), & x \in \mathbb{R},
\end{align*}$$

(3.2)

where $u$ denotes the first component and $v$ the second component of the solution $z$ to system (2.1), and the nondecreasing maps $\xi \mapsto \tilde{q}_1(\xi)$ and $\xi \mapsto \tilde{q}_2(\xi)$ are defined by

$$\int_0^{\tilde{q}_1(\xi)} u_x^2 dx = \xi \quad \text{and} \quad \int_0^{\tilde{q}_2(\xi)} v_x^2 dx = \xi,$$  

(3.3)

respectively.

In the following, we use the notations

$$\begin{align*}
u(t, \xi) &= u(t, q_1(t, \xi)), & v(t, \xi) &= v(t, q_2(t, \xi)), & A(t, \xi) &= A(t, q_1(t, \xi)), \\
B(t, \xi) &= B(t, q_1(t, \xi)), & C(t, \xi) &= C(t, q_2(t, \xi)), & D(t, \xi) &= D(t, q_2(t, \xi)),
\end{align*}$$

and define the variables: $\theta_1 = \theta_1(t, \xi)$, $\theta_2 = \theta_2(t, \xi)$, $w_1 = w_1(t, \xi)$, and $w_2 = w_2(t, \xi)$ as

$$\begin{align*}
\theta_1 &= 2\text{arc sec } u_x, & \theta_2 &= 2\text{arc sec } v_x, & w_1 &= u_x^2 \cdot \frac{\partial q_1}{\partial \xi}, & w_2 &= v_x^2 \cdot \frac{\partial q_2}{\partial \xi},
\end{align*}$$

(3.4)

with $u_x(t, \xi) = u_x(t, q_1(t, \xi))$, $v_x(t, \xi) = v_x(t, q_2(t, \xi))$. Since $\theta_1, \theta_2$ are defined on $[0, \pi) \cup (\pi, 2\pi)$, all subsequent equations involving $\theta_1, \theta_2$ are invariant.

Remark 3.1: (1) The variable transformation used for the Camassa-Holm equation in Eq. (2.3) in Refs. 1 and 2 is $v = 2\text{arctan } u_x$, while the one used here for system (1.4) is $\theta_1 = 2\text{arc sec } u_x$. Although they are similar, the independent and dependent variables introduced here truly make the computation much more direct and simple.

(2) The authors in Refs. 1 and 2 introduce single characteristic to handle the global conservative and dissipative solutions of the Camassa-Holm equation, while in this paper two characteristics are introduced for system (1.4). So the analysis in this paper is much more complicated.

From (3.3)–(3.4), we deduce the following identities,

$$w_1(0, \xi) = w_2(0, \xi) \equiv 1,$$

(3.5)

$$u_x = \sec \frac{\theta_1}{2}, \quad v_x = \sec \frac{\theta_2}{2}, \quad \frac{1}{u_x^2} = \cos^2 \frac{\theta_1}{2}, \quad \frac{1}{v_x^2} = \cos^2 \frac{\theta_2}{2},$$

(3.6)
It follows from (3.7) that

\[
q_1 (t, \xi') - q_1 (t, \xi) = \int_{\xi}^{\xi'} \cos^2 \frac{\theta_1}{2} (t, s) \cdot w_1 (t, s) \, ds,
\]

\[
q_2 (t, \xi') - q_2 (t, \xi) = \int_{\xi}^{\xi'} \cos^2 \frac{\theta_2}{2} (t, s) \cdot w_2 (t, s) \, ds.
\]

Then, from (3.6)–(3.8), we derive the expressions for \( A, A_x, B, B_x, C, C_x, D, \) and \( D_x \) in terms of the new variable \( \xi \), namely

\[
A (\xi) = \frac{1}{2} \int_{-\infty}^{+\infty} \exp \left\{ - \left| \int_{\xi}^{\xi'} \cos^2 \frac{\theta_1 (s)}{2} \cdot w_1 (s) \, ds \right| \right\} \cdot (uv_x) (\xi') \cos^2 \frac{\theta_1 (\xi')}{2} w_1 (\xi') \, d\xi',
\]

\[
A_x (\xi) = \frac{1}{2} \left( \int_{-\infty}^{+\infty} - \int_{-\infty}^{\xi} \right) \exp \left\{ - \left| \int_{\xi}^{\xi'} \cos^2 \frac{\theta_1 (s)}{2} \cdot w_1 (s) \, ds \right| \right\} \cdot (uv_x) (\xi') \cos^2 \frac{\theta_1 (\xi')}{2} w_1 (\xi') \, d\xi',
\]

\[
B (\xi) = \frac{1}{2} \int_{-\infty}^{+\infty} \exp \left\{ - \left| \int_{\xi}^{\xi'} \cos^2 \frac{\theta_1 (s)}{2} \cdot w_1 (s) \, ds \right| \right\}
\cdot \left( v^2 + \frac{1}{2} u_x^2 + u_x v_x + \frac{1}{2} v^2 - \frac{1}{2} v_x^2 \right) (\xi') \cdot \cos^2 \frac{\theta_1 (\xi')}{2} \cdot w_1 (\xi') \, d\xi',
\]

\[
B_x (\xi) = \frac{1}{2} \left( \int_{-\infty}^{+\infty} - \int_{-\infty}^{\xi} \right) \exp \left\{ - \left| \int_{\xi}^{\xi'} \cos^2 \frac{\theta_1 (s)}{2} \cdot w_1 (s) \, ds \right| \right\}
\cdot \left( v^2 + \frac{1}{2} u_x^2 + u_x v_x + \frac{1}{2} v^2 - \frac{1}{2} v_x^2 \right) (\xi') \cdot \cos^2 \frac{\theta_1 (\xi')}{2} \cdot w_1 (\xi') \, d\xi',
\]

\[
C (\xi) = \frac{1}{2} \int_{-\infty}^{+\infty} \exp \left\{ - \left| \int_{\xi}^{\xi'} \cos^2 \frac{\theta_2 (s)}{2} \cdot w_2 (s) \, ds \right| \right\} \cdot (uv_x) (\xi') \cos^2 \frac{\theta_2 (\xi')}{2} w_2 (\xi') \, d\xi',
\]

\[
C_x (\xi) = \frac{1}{2} \left( \int_{-\infty}^{+\infty} - \int_{-\infty}^{\xi} \right) \exp \left\{ - \left| \int_{\xi}^{\xi'} \cos^2 \frac{\theta_2 (s)}{2} \cdot w_2 (s) \, ds \right| \right\} \cdot (uv_x) (\xi') \cos^2 \frac{\theta_2 (\xi')}{2} w_2 (\xi') \, d\xi',
\]

\[
D (\xi) = \frac{1}{2} \int_{-\infty}^{+\infty} \exp \left\{ - \left| \int_{\xi}^{\xi'} \cos^2 \frac{\theta_2 (s)}{2} \cdot w_2 (s) \, ds \right| \right\}
\cdot \left( v^2 + \frac{1}{2} u_x^2 + u_x v_x + \frac{1}{2} v^2 - \frac{1}{2} v_x^2 \right) (\xi') \cdot \cos^2 \frac{\theta_2 (\xi')}{2} \cdot w_2 (\xi') \, d\xi',
\]

\[
D_x (\xi) = \frac{1}{2} \left( \int_{-\infty}^{+\infty} - \int_{-\infty}^{\xi} \right) \exp \left\{ - \left| \int_{\xi}^{\xi'} \cos^2 \frac{\theta_2 (s)}{2} \cdot w_2 (s) \, ds \right| \right\}
\cdot \left( v^2 + \frac{1}{2} u_x^2 + u_x v_x + \frac{1}{2} v^2 - \frac{1}{2} v_x^2 \right) (\xi') \cdot \cos^2 \frac{\theta_2 (\xi')}{2} \cdot w_2 (\xi') \, d\xi',
\]
In view of (3.2), system (2.1) can be rewritten in the new variables \((t, \xi)\) as follows:

\[
\frac{\partial}{\partial t} u(t, \xi) = u_t + uu_x = (-vu_x - A - B_x)(t, \xi),
\]
\[
\frac{\partial}{\partial t} v(t, \xi) = v_t + vv_x = (-uv_x - C - D_x)(t, \xi).
\] (3.10)

From (3.2), (3.4), (3.6), and (2.3), we obtain

\[
\frac{\partial}{\partial t} \theta_1(t, \xi) = \frac{2}{u_x \sqrt{u_x^2 - 1}} (u_{xx} + uu_{xx})
\]
\[= - \csc \frac{\theta_1}{2} + \left(-2vu_{xx} - 2A_x - 2B + 2u^2 + v^2 - v_x^2\right) \cos \frac{\theta_1}{2} \cdot \cot \frac{\theta_1}{2},
\]
\[
\frac{\partial}{\partial t} \theta_2(t, \xi) = \frac{2}{v_x \sqrt{v_x^2 - 1}} (v_{xx} + vv_{xx})
\]
\[= - \csc \frac{\theta_2}{2} + \left(-2uv_{xx} - 2C_x - 2D + 2v^2 + u^2 - u_x^2\right) \cos \frac{\theta_2}{2} \cdot \cot \frac{\theta_2}{2}.
\] (3.11)

Moreover, from (3.2), (3.4), (2.6), and (2.7), we derive

\[
\frac{\partial}{\partial t} w_1(t, \xi) = \left(u_x^2\right)_x + (uu_x^2)_x
\]
\[= (2u^2 - 2vu_{xx} - 2A_x - 2B + v^2 - v_x^2) \frac{u_x}{u_x^2} \cdot w_1
\]
\[= (2u^2 - 2vu_{xx} - 2A_x - 2B + v^2 - v_x^2) \cos \frac{\theta_1}{2} \cdot w_1,
\]
\[
\frac{\partial}{\partial t} w_2(t, \xi) = \left(v_x^2\right)_x + (vv_x^2)_x
\]
\[= (2v^2 - 2uv_{xx} - 2C_x - 2D + u^2 - u_x^2) \frac{v_x}{v_x^2} \cdot w_2
\]
\[= (2v^2 - 2uv_{xx} - 2C_x - 2D + u^2 - u_x^2) \cos \frac{\theta_2}{2} \cdot w_2.
\] (3.12)

In (3.10)–(3.12), the functions \(A, A_x, B, B_x, C, C_x, D, D_x\) are as in (3.9).

**B. Global solutions of the equivalent system**

First, for given initial data \(\bar{z} = (\bar{u}, \bar{v}) \in H^1 \times H^1\), we consider the corresponding Cauchy problem for the variables \((u, v, \theta_1, \theta_2, w_1, w_2)\) in the following semilinear system:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -vu_x - A - B_x, \\
\frac{\partial v}{\partial t} &= -uv_x - C - D_x, \\
\frac{\partial \theta_1}{\partial t} &= (2u^2 - 2vu_{xx} - 2A_x - 2B + v^2 - v_x^2) \cos \frac{\theta_1}{2} \cot \frac{\theta_1}{2} - \csc \frac{\theta_1}{2}, \\
\frac{\partial \theta_2}{\partial t} &= (2v^2 - 2uv_{xx} - 2C_x - 2D + u^2 - u_x^2) \cos \frac{\theta_2}{2} \cot \frac{\theta_2}{2} - \csc \frac{\theta_2}{2}, \\
\frac{\partial w_1}{\partial t} &= (2u^2 - 2vu_{xx} - 2A_x - 2B + v^2 - v_x^2) \cos \frac{\theta_1}{2} \cdot w_1, \\
\frac{\partial w_2}{\partial t} &= (2v^2 - 2uv_{xx} - 2C_x - 2D + u^2 - u_x^2) \cos \frac{\theta_2}{2} \cdot w_2.
\end{align*}
\] (3.13)
with the initial condition

\[
\begin{align*}
\begin{cases}
    u(0, \xi) = \bar{u}(\tilde{q}_1(\xi)), \\
v(0, \xi) = \bar{v}(\tilde{q}_2(\xi)), \\
\theta_1(0, \xi) = 2\arccsc \bar{u}_x(\tilde{q}_1(\xi)), \\
\theta_2(0, \xi) = 2\arccsc \bar{v}_x(\tilde{q}_2(\xi)), \\
w_1(0, \xi) = 1, \\
w_2(0, \xi) = 1.
\end{cases}
\end{align*}
\]

(3.14)

where \(A, A_x, B, B_x, C, C_x, D, \) and \(D_x\) are functions of variables \(u, v, \theta_1, \theta_2, w_1, w_2, \) and \(\xi, \) and are as in (3.9).

System (3.13) can be regarded as an O.D.E. in the Banach space

\[
X = H^1(R) \times H^1(R) \times [L^2(R) \cap L^\infty(R)] \times [L^2(R) \cap L^\infty(R)]
\]

\[
\times [L^2(R) \cap L^\infty(R)] \times [L^2(R) \cap L^\infty(R)],
\]

(3.15)

which is endowed with the norm

\[
\| (u, v, \theta_1, \theta_2, w_1, w_2) \|_X = \|u\|_{H^1} + \|v\|_{H^1} + \|\theta_1\|_{L^2} + \|\theta_2\|_{L^2} + \|w_1\|_{L^\infty} + \|w_2\|_{L^\infty}
\]

The following Lemmata play important role in the proof of the global existence of solution to Cauchy problem (2.1) and (2.2).

**Lemma 3.1:** Let \(z = (\bar{u}, \bar{v}) \in H^s \times H^s, \ s > 3/2, \) then the solution \(z\) to (2.1) satisfies

(i) \(\|u(t, \cdot)\|_{L^\infty}^2 \leq \frac{1}{2} (\|\bar{u}\|_{H^s}^2 + \|\bar{v}\|_{H^s}^2), \ \forall t \in [0, T),\)

(ii) \(\|v(t, \cdot)\|_{L^\infty}^2 \leq \frac{1}{2} (\|\bar{u}\|_{H^s}^2 + \|\bar{v}\|_{H^s}^2), \ \forall t \in [0, T).\)

**Lemma 3.2:** Given \(z = (\bar{u}, \bar{v}) \in H^1 \times H^1.\) Then there exists a unique solution defined on some small time interval \([0, T]\) with \(T > 0\) to the Cauchy problem (3.13) and (3.14).

**Proof:** It is sufficient to show that the operator determined by the right-hand side of (3.13), i.e., the map \((u, v, \theta_1, \theta_2, w_1, w_2)\) to

\[
\begin{align*}
( -vu_x - A - B_x, -uv_x - C - D_x, 2(u^2 - vu_{xx} - A - B + v^2 - v_w^2)cot \frac{\theta_1}{2} - csc \frac{\theta_1}{2}, \\
2\left( v^2 - C_x - D - uv_{xx} + \frac{u^2}{2} - \frac{v^2}{2} \right) cot \frac{\theta_2}{2} - csc \frac{\theta_2}{2}, \\
2\left( u^2 - uv_{xx} - A_x - B + \frac{v^2}{2} - \frac{v_w^2}{2} \right) cot \frac{\theta_1}{2} \cdot w_1, 2\left( v^2 - uv_{xx} - C_x - D + \frac{u^2}{2} - \frac{v^2}{2} \right) cot \frac{\theta_2}{2} \cdot w_2)
\end{align*}
\]

is Lipschitz continuous on every bounded domain \(\Omega \subset X\) of the form

\[
\Omega = \{(u, v, \theta_1, \theta_2, w_1, w_2) : \|u\|_{H^s}, \|v\|_{H^s} \leq \alpha, \|\theta_1\|_{L^2}, \|\theta_2\|_{L^2} \leq \beta, \|w_1\|_{L^\infty}, \|w_2\|_{L^\infty} \leq 2\pi, \|w_1\|_{L^2}, \|w_2\|_{L^2} \leq \gamma, w_1(x) \in [w_1^-, w_1^+], w_2(x) \in [w_2^-, w_2^+] \text{ for a.e. } x \in R \}
\]

(3.16)

for arbitrary constants \(\alpha, \beta, \gamma, w_1^-, w_1^+, w_2^-, w_2^+, \) and \(w_2^+ > 0.\)

It is clear that the maps \(vu_{xx}, uv_x, \left( u^2 - vu_{xx} + v^2 - v_w^2 \right)cot \frac{\theta_1}{2} \cdot csc \frac{\theta_1}{2}, \left( u^2 - uv_{xx} + \frac{v^2}{2} - \frac{v_w^2}{2} \right)cot \frac{\theta_2}{2} \cdot w_1, \left( v^2 - uv_{xx} + \frac{v^2}{2} - \frac{v_w^2}{2} \right)cot \frac{\theta_2}{2} \cdot w_2\) are all Lipschitz continuous as maps from \(\Omega\) into \(L^2(R),\)

and also from \(\Omega\) into \(L^\infty(R).\)
In the following, we will prove the Lipschitz continuity of the maps
\[(u, v, \theta_1, \theta_2, w_1, w_2) \mapsto A, A_x, \quad (u, v, \theta_1, \theta_2, w_1, w_2) \mapsto B, B_x,\]
\[(u, v, \theta_1, \theta_2, w_1, w_2) \mapsto C, C_x, \quad (u, v, \theta_1, \theta_2, w_1, w_2) \mapsto D, D_x, \quad (3.17)\]
as maps from \(\Omega\) into \(H^1 (R)\).

For any \(\xi_1 < \xi_2\), it follows from (3.16) that
\[
\int_{\xi_1}^{\xi_2} \cos^2 \frac{\theta_1 (\xi)}{2} \cdot w_1 (\xi) \, d\xi \geq \left[ \frac{\xi_2 - \xi_1}{2} - \beta^2 \right] w_1^-, \]
\[
\int_{\xi_1}^{\xi_2} \cos^2 \frac{\theta_2 (\xi)}{2} \cdot w_2 (\xi) \, d\xi \geq \left[ \frac{\xi_2 - \xi_1}{2} - \beta^2 \right] w_2^- .
\]

The above are key estimates which guarantee that the exponential terms in (3.9) for \(A, A_x, B, B_x, C, C_x, D, D_x\) decrease quickly as \(|\xi - \xi'| \rightarrow \infty\).

Now we introduce the exponentially decaying function
\[
\Lambda (\xi) = \min \left\{ 1, \exp \left( \frac{\beta}{2} w^- - \frac{|\xi|}{2} w^- \right) \right\}, \quad w^- = \max \{ w_1^-, w_2^- \} \quad (3.18)
\]
with the norm \(\| \Lambda \|_{L^1} = (\int_{|\xi| \leq 2\beta^2} + \int_{|\xi| > 2\beta^2}) \Lambda (\xi) \, d\xi = 4\beta^2 + \frac{4}{w^-} .\)

Next, we will show that \(A, B, B_x, C, D, D_x \in H^1 (R)\), such that
\[
B, \partial_\xi B, B_x, \partial_\xi B_x, D, \partial_\xi D, D_x, \partial_\xi D_x \in L^2 (R) . \quad (3.19)
\]

It follows from (3.9) that
\[
|B (\xi)| \leq \frac{1}{2} \left| \Lambda \ast \left( u^2 + u_x^2 \right) \cdot \cos^2 \frac{\theta_1}{2} \cdot w_1 (\xi) \right| + \frac{1}{2} \left| \Lambda \ast \left( v^2 + v_x^2 \right) \cdot \cos^2 \frac{\theta_2}{2} \cdot w_2 (\xi) \right| \leq \frac{w_1^+}{2} \left| \Lambda \ast \left( u^2 \cos^2 \frac{\theta_1}{2} + 1 \right) (\xi) \right| + \frac{w_2^+}{2} \left| \Lambda \ast \left( v^2 \cos^2 \frac{\theta_2}{2} + 1 \right) (\xi) \right| .
\]

Using the standard properties of convolutions, we have
\[
\| B \|_{L^2} \leq \frac{w_1^+}{2} \cdot (\| \Lambda \|_{L^1} \cdot \| u^2 \|_{L^2} + \| \Lambda \|_{L^1}) + \frac{w_2^+}{2} \cdot (\| \Lambda \|_{L^1} \cdot \| v^2 \|_{L^2} + \| \Lambda \|_{L^1}) < \infty . \quad (3.20)
\]

Furthermore, differentiating the third equation in (3.9), we obtain
\[
\frac{\partial}{\partial \xi} B (\xi) = - \left[ u^2 + \frac{1}{2} u_x^2 + u_x v_x + \frac{1}{2} v^2 - \frac{1}{2} v_x^2 \right] (\xi) \cos^2 \frac{\theta_1 (\xi)}{2} \cdot w_1 (\xi)
+ \frac{1}{2} \int_{-\infty}^{+\infty} \exp \left\{ - \left[ \int_{\xi}^{\xi'} \cos^2 \frac{\theta_1 (s)}{2} \cdot w_1 (s) \, ds \right] \cdot \left[ \cos^2 \frac{\theta_1 (\xi')}{2} \cdot w_1 (\xi') \right] \right\} \cdot \left[ \cos^2 \frac{\theta_1 (\xi)}{2} \right] \cdot \left[ \cos^2 \frac{\theta_1 (\xi')}{2} \right] \cdot w_1 (\xi) \, d\xi' . \quad (3.21)
\]
Therefore,
\[
|\partial_\xi B(\xi)| \leq w_1^+ \left| u^2 (\xi) + 1 \right| + w_2^+ \left| v^2 (\xi) + 1 \right| + \frac{w_4^+}{2} \left| \Lambda \left( u^2 \cos^2 \frac{\theta_1}{2} + 1 \right) \right| + \frac{w_5^+}{2} \left| \Lambda \left( v^2 \cos^2 \frac{\theta_2}{2} + 1 \right) \right|.
\]

\[
\left\| \partial_\xi B \right\|_{L^2} \leq w_1^+ \cdot \left\| u^2 \right\|_{L^2} + \left\| w_1 \right\|_{L^2} + w_2^+ \cdot \left\| v^2 \right\|_{L^2} + \left\| w_2 \right\|_{L^2} + \frac{w_4^+}{2} \left( \| \Lambda \|_{L^1} \cdot \| u^2 \|_{L^2} + \| \Lambda \|_{L^2} \right) + \frac{w_5^+}{2} \left( \| \Lambda \|_{L^1} \cdot \| v^2 \|_{L^2} + \| \Lambda \|_{L^2} \right)
\]

\[
\leq w_1^+ \cdot \| u \|_{L^\infty} \cdot \| u \|_{L^2} + \| w_1 \|_{L^2} + w_2^+ \cdot \| v \|_{L^\infty} \cdot \| v \|_{L^2} + \| w_2 \|_{L^2} + \frac{w_4^+}{2} \left( \| \Lambda \|_{L^1} \cdot \| u^2 \|_{L^2} + \| \Lambda \|_{L^2} \right) + \frac{w_5^+}{2} \left( \| \Lambda \|_{L^1} \cdot \| v^2 \|_{L^2} + \| \Lambda \|_{L^2} \right)
\]

\[
< \infty.
\]

The estimates for \( B_x, \partial_\xi B_x, D, \partial_\xi D, D_x, \) and \( \partial_\xi D_x \) can be obtained similarly. So the conclusion (3.19) holds.

To establish the Lipschitz continuity of the maps in (3.17), it is sufficient to show that their partial derivatives,
\[
\frac{\partial B}{\partial u}, \frac{\partial B}{\partial \theta_1}, \frac{\partial B}{\partial \theta_2}, \frac{\partial B}{\partial w_1}, \frac{\partial B}{\partial w_2}, \frac{\partial B}{\partial \theta_1}, \frac{\partial B}{\partial \theta_2}, \frac{\partial D}{\partial v}, \frac{\partial D}{\partial \theta_1}, \frac{\partial D}{\partial \theta_2}, \frac{\partial D}{\partial w_2}, \frac{\partial D}{\partial \theta_2}, \frac{\partial D_x}{\partial w_2}, \frac{\partial D_x}{\partial \theta_2}, \frac{\partial D_x}{\partial \theta_2}
\]

are uniformly bounded as \((u, v, \theta_1, \theta_2, w_1, w_2)\) range inside the domain \(\Omega\).

For given point \((u, v, \theta_1, \theta_2, w_1, w_2) \in \Omega\), the partial derivative \(\frac{\partial B}{\partial u} : H^1(R) \mapsto L^2(R)\) is the linear operator defined by
\[
\left[ \frac{\partial B}{\partial u} (u, v, \theta_1, w_1, w_2) \cdot \hat{u} \right](\xi) = \int_{-\infty}^{+\infty} \exp \left\{ - \frac{1}{2} \int_{\xi}^{\xi'} \cos^2 \frac{\theta_1 (s)}{2} \cdot w_1 (s) \, ds \right\} \cdot u (\xi') \cos^2 \frac{\theta_1}{2} \cdot w_1 (\xi') \cdot \hat{u} (\xi') \, d\xi'.
\]

Its norm is bounded by
\[
\left\| \frac{\partial B}{\partial u} (u, v, \theta_1, \theta_2, w_1, w_2) \right\|_{L^2} \leq w_1^+ \| \Lambda \|_{L^1} \cdot \| u \|_{L^2}.
\]

(3.24)

Then by (3.21), we know \(\frac{\partial (\partial_\xi B)}{\partial u} : H^1(R) \mapsto L^2(R)\) is a linear operator defined by
\[
\left[ \frac{\partial (\partial_\xi B)}{\partial u} (u, v, \theta_1, \theta_2, w_1, w_2) \cdot \hat{u} \right](\xi)
\]

\[
= -2u (\xi) \cos^2 \frac{\theta_1}{2} \cdot w_1 (\xi) \cdot \hat{u} (\xi) + \int_{-\infty}^{+\infty} \exp \left\{ - \frac{1}{2} \int_{\xi}^{\xi'} \cos^2 \frac{\theta_1 (s)}{2} \cdot w_1 (s) \, ds \right\} \cdot \cos^2 \frac{\theta_1 (\xi)}{2} \cdot w_1 (\xi) \cdot \hat{u} (\xi) \cdot \hat{u} (\xi') \, d\xi'.
\]

Its norm is thus bounded by
\[
\left\| \frac{\partial (\partial_\xi B)}{\partial u} (u, v, \theta_1, \theta_2, w_1, w_2) \right\|_{L^2} \leq 2w_1^+ \| u \|_{L^2} + (w_1^+) \| \Lambda \|_{L^1} \cdot \| u \|_{L^2}.
\]

(3.25)

Therefore, (3.24) and (3.25) yield that \(\frac{\partial B}{\partial u}\) is bounded as a linear operator from \(H^1(R)\) into \(H^1(R)\). Similarly, we can obtain the bounds on the other partial derivatives in (3.23).
Therefore, the uniform Lipschitz continuity of the maps in (3.17) can be established. Moreover, the Lipschitz continuity of the right-hand side of (3.13) on a neighborhood of the initial data in the space $X$ can also be established.

Using the standard theory of O.D.E. in Banach spaces, we can obtain the local existence of a unique solution to the Cauchy problem (3.13) and (3.14) on some small time interval $[0, T]$ with $T > 0$.

The following result of conservative quantity is very critical to obtaining the global existence of solution to the Cauchy problem (3.13) and (3.14).

**Lemma 3.3:** Suppose that initial data $\bar{z} = (\bar{u}, \bar{v}) \in H^1 \times H^1$, then the total energy in terms of the new variables $(u, v, \theta_1, \theta_2, w_1, w_2)$ and $\xi$ remains conservative, namely

$$E(t) = \int_{\mathbb{R}} \left[ (u^2 \cos^2 \frac{\theta_1}{2} + 1) w_1 + \left( v^2 \cos^2 \frac{\theta_2}{2} + 1 \right) w_2 \right](t, \xi) \, d\xi = E(0) = E_0. \tag{3.26}$$

**Proof:** It is sufficient to show that, as long as the solution exists,

$$\frac{d}{dt} \int_{\mathbb{R}} \left[ (u^2 \cos^2 \frac{\theta_1}{2} + 1) w_1 + \left( v^2 \cos^2 \frac{\theta_2}{2} + 1 \right) w_2 \right] \, d\xi = 0,$$

with

$$u_\xi = w_1 \cdot \cos \frac{\theta_1}{2}, \quad v_\xi = w_2 \cdot \cos \frac{\theta_2}{2}. \tag{3.27}$$

It follows from the first equation in (3.13) that

$$u_\xi t = w_1 \left( u^2 - v u_{xx} - A_x - B + \frac{v^2 - v_x^2}{2} \right) \cdot \cos \frac{\theta_1}{2} + \frac{1}{2} w_1. \tag{3.28}$$

On the other hand, from the third and the fifth equations in (3.13), we derive

$$\left( w_1 \cdot \cos \frac{\theta_1}{2} \right)_t = \frac{\partial w_1}{\partial t} \cdot \cos \frac{\theta_1}{2} - \frac{1}{2} w_1 \cdot \sin \frac{\theta_1}{2} \cdot \frac{\partial \theta_1}{\partial t},$$

$$= (2u^2 - 2vu_{xx} - 2A_x - 2B + v^2 - v_x^2) \cdot \cos \frac{\theta_1}{2} \cdot w_1 - \frac{1}{2} w_1 \sin \frac{\theta_1}{2}.$$

$$= w_1 \left( u^2 - v u_{xx} - A_x - B + \frac{v^2 - v_x^2}{2} \right) \cdot \cos \frac{\theta_1}{2} + \frac{1}{2} w_1. \tag{3.29}$$

It follows from (3.28) and (3.29) that, for every $\xi$, $\frac{\partial}{\partial t} \left( u_\xi - w_1 \cdot \cos \frac{\theta_1}{2} \right) = 0$. The case for $v_\xi$ is entirely similar. Since the identities in (3.27) hold at $t = 0$, we infer that (3.27) remains valid for all times $t \geq 0$, as long as the solution is defined.
Next, we deduce from (3.13) that
\[
\frac{d}{dt} \int_R \left( \left( u^2 \cos^2 \frac{\theta_1}{2} + 1 \right) w_1 + \left( v^2 \cos^2 \frac{\theta_2}{2} + 1 \right) w_2 \right) d\xi
\]
\[
= \int_R \left( \left( u^2 \cos^2 \frac{\theta_1}{2} + 1 \right) \frac{\partial w_1}{\partial t} + \left( 2uv \cdot \cos^2 \frac{\theta_1}{2} - u^2 \cos \frac{\theta_1}{2} \sin \frac{\theta_1}{2} \frac{\partial \theta_1}{\partial t} \right) w_1 
+ \left( v^2 \cos^2 \frac{\theta_2}{2} + 1 \right) \frac{\partial w_2}{\partial t} + \left( 2vv \cdot \cos^2 \frac{\theta_2}{2} - v^2 \cos \frac{\theta_2}{2} \sin \frac{\theta_2}{2} \frac{\partial \theta_2}{\partial t} \right) w_2 \right) w_1 d\xi
\]
\[
= \int_R \left( 2 \left( u^2 \cos^2 \frac{\theta_1}{2} + 1 \right) \left( u^2 - uv_{xx} - Ax - B + \frac{1}{2} v^2 - \frac{1}{2} v^2 \right) \cos \frac{\theta_1}{2} \right) w_1 
+ \left( v^2 \cos^2 \frac{\theta_2}{2} + 1 \right) \left( v^2 - uv_{xx} - Cx - D + \frac{1}{2} v^2 - \frac{1}{2} u^2 \right) \cos \frac{\theta_2}{2} w_2
\]
\[
- v^2 \cos \frac{\theta_2}{2} \sin \frac{\theta_2}{2} \left[ 2 \left( u^2 - uv_{xx} - Cx - D + \frac{1}{2} v^2 - \frac{1}{2} u^2 \right) \cos \frac{\theta_2}{2} \right. 
\left. - \left( 2uu + 2uA + 2kB \right) \cos \frac{\theta_1}{2} \right] w_2 d\xi
\]
\[
= \int_R \left[ w_1 \left( 3u^2 - 2uv_{xx} - 2Ax - 2B + v^2 - v^2 \right) \cos \frac{\theta_1}{2} - (2uu + 2uA + 2kB) \cos \frac{\theta_1}{2} \right] d\xi
\]
\[
+ w_2 \left( 3v^2 - 2uv_{xx} - 2Cx - 2D + u^2 - u^2 \right) \cos \frac{\theta_2}{2} - (2uu + 2vC + 2vD) \cos \frac{\theta_2}{2} \right] d\xi
\]
\[
= \int_R \left[ w_1 \left( 3u^2 \cos \frac{\theta_1}{2} - 2B \cos \frac{\theta_2}{2} - 2uB \cos^2 \frac{\theta_2}{2} \right) 
+ w_2 (3v^2 \cos \frac{\theta_2}{2} - 2D \cos \frac{\theta_2}{2} - 2vD \cos^2 \frac{\theta_2}{2}) \right] d\xi.
\]

On the other hand, from (3.7), we derive that
\[
B_\xi = w_1 \cdot B_x \cdot \cos^2 \frac{\theta_1}{2}, \quad D_\xi = w_2 \cdot D_x \cdot \cos^2 \frac{\theta_2}{2}.
\]

Moreover, it follows from (3.27) that
\[
(uB)_\xi = u_\xi B + uB_\xi = w_1 \left( B \cdot \cos \frac{\theta_1}{2} + uB \cdot \cos^2 \frac{\theta_1}{2} \right),
\]
\[
(vD)_\xi = v_\xi D + vD_\xi = w_2 \left( D \cdot \cos \frac{\theta_2}{2} + vD \cdot \cos^2 \frac{\theta_2}{2} \right).
\]

In addition, we obtain that
\[
(u^3)_\xi = 3u^2 u_\xi = 3w_1 u^2 \cdot \cos \frac{\theta_1}{2}, \quad (v^3)_\xi = 3v^2 v_\xi = 3w_2 v^2 \cdot \cos \frac{\theta_2}{2}.
\]

Thus we have
\[
\frac{d}{dt} \int_R \left[ \left( u^2 \cos^2 \frac{\theta_1}{2} + 1 \right) w_1 + \left( v^2 \cos^2 \frac{\theta_2}{2} + 1 \right) w_2 \right] d\xi
\]
\[
= \int_R \partial_\xi (u^3 - 2uB) d\xi + \int_R \partial_\xi (v^3 - 2vD) d\xi = 0.
\]
Now, we have the following result of the global existence and uniqueness of solution to the Cauchy problem (3.13) and (3.14).

**Theorem 3.1:** Suppose that the initial data \( \tilde{z} = (\tilde{u}, \tilde{v}) \in H^1 \times H^1 \), then there exists a unique global solution to the Cauchy problem (3.13) and (3.14).

**Proof:** To ensure that the local solution of (3.13) and (3.14) constructed in Lemma 3.2 can be extended to a global solution, it is sufficient to show that the quantity,

\[
\|u\|_{H^1} + \|v(t)\|_{H^1} + \|\theta(t)\|_{L^2} + \|\dot{\theta}(t)\|_{L^\infty} + \|w(t)\|_{L^\infty} + \|1/w(t)\|_{L^\infty},
\]

(here we define \( |\theta(t)| = \max\{|\theta_1(t)|, |\theta_2(t)|, |w(t)| = \max\{|w_1(t)|, |w_2(t)|) \) remains uniformly bounded on any bounded time interval.

From (3.26) and (3.27), we obtain the bounds on \( \|u(t)\|_{L^\infty} \) and \( \|v(t)\|_{L^\infty} \),

\[
\sup_{\xi \in \mathbb{R}} |u^2(t, \xi)| \leq 2 \int_{\mathbb{R}} |uu_\xi| d\xi \leq 2 \int_{\mathbb{R}} |u| \cdot \left| \cos \frac{\theta_1}{2} \right| w_1 d\xi \leq E_0,
\]

\[
\sup_{\xi \in \mathbb{R}} |v^2(t, \xi)| \leq 2 \int_{\mathbb{R}} |vv_\xi| d\xi \leq 2 \int_{\mathbb{R}} |v| \cdot \left| \cos \frac{\theta_2}{2} \right| w_2 d\xi \leq E_0.
\]

It follows from (3.9) and (3.26) that

\[
\|A(t)\|_{L^\infty}, \|A_t(t)\|_{L^\infty} \leq \left\| \frac{1}{2} e^{-|x|} \right\|_{L^\infty} \cdot \|u(t) v_x(t)\|_{L^1} \leq \left\| \frac{1}{2} e^{-|x|} \right\|_{L^\infty} \cdot \left\| \frac{u^2(t) + v^2(t)}{2} \right\|_{L^1} \leq \frac{1}{2} E_0,
\]

\[
\|B(t)\|_{L^\infty}, \|B_t(t)\|_{L^\infty} \leq \left\| \frac{1}{2} e^{-|x|} \right\|_{L^\infty} \cdot \left\| \left(u^2 + \frac{u_x^2}{2} + u_x v_x + \frac{v_x^2}{2} - \frac{v^2}{2} \right) \right\|_{L^1} \leq \frac{1}{2} E_0,
\]

\[
\|C(t)\|_{L^\infty}, \|C_t(t)\|_{L^\infty} \leq \left\| \frac{1}{2} e^{-|x|} \right\|_{L^\infty} \cdot \left\| \left(v^2 + \frac{v_x^2}{2} + u_x v_x + \frac{u^2}{2} - \frac{u_x^2}{2} \right) \right\|_{L^1} \leq \frac{1}{2} E_0.
\]

We thus recover the estimates (2.9) for the new variables.

Indeed, as long as the solution exists, the fifth equation in (3.13) and the estimates in (3.31) and (3.32) imply that

\[
\left| \frac{\partial w_1}{\partial t} \right| \leq (2E_0 + E_0 + E_0) w_1 = 4E_0 w_1.
\]

Therefore,

\[
e^{-4E_0 t} \leq w_1(t) \leq e^{4E_0 t}.
\]

Similarly, we have \( e^{-4E_0 t} \leq w_2(t) \leq e^{4E_0 t} \).

From the third and fourth equations in (3.13), we obtain that

\[
\|\theta_1(t)\|_{L^\infty}, \|\theta_2(t)\|_{L^\infty} \leq e^{Kt},
\]

for a suitable constant \( K = K(E_0) > 0 \). The estimates of \( \|u(t)\|_{H^1} \) and \( \|v(t)\|_{H^1} \) will thus follow from the bounds on the \( L^1 \) norms of \( B_x, \partial_x B_x, D_x, \) and \( \partial_x D_x \). Denote the right-hand side of (3.33)
as $\lambda$, so that $\lambda^{-1} \leq w_1(t) \leq \lambda$. Then from (3.21) we deduce that

$$\|\partial_\xi B(t)\|_{L^1} \leq \lambda E_0 + \frac{1}{2} \int_R \exp \left\{ \int_{\xi}^{\xi'} \frac{\lambda - \cos^2 \frac{\theta_1(t, s)}{2}}{2} ds \right\} \cdot \left[ u^2 + \frac{1}{2} u_x^2 + u_x v_x + \frac{1}{2} v^2 - \frac{1}{2} v_x^2 \right](t, \xi') \cos^2 \frac{\theta_1(t, \xi)}{2} \lambda d\xi'$$

$$\leq \lambda E_0 + \|\Lambda\|_{L^1} \cdot \lambda E_0,$$

where

$$\Lambda(\xi) = \min \left\{ 1, \exp \left( \frac{18 E_0 \lambda^{-1}}{2} - \frac{|\xi|}{\lambda^{-1}} \right) \right\}.$$

Also we can obtain $\|\Lambda\|_{L^1} = \left( \int_{|\xi| \geq 36 E_0} + \int_{|\xi| \geq 36 E_0} \right) \Lambda(\xi) d\xi = 72 E_0 + \frac{4}{\lambda}$. The estimates of $\|B_\xi\|_{L^1}$, $\|D\|_{L^1}$, and $\|D\|_{L^1}$ can be obtained similarly. This establishes the boundedness of the norm $\|u(t)\|_{H^1}$ and $\|v(t)\|_{H^1}$ for $t$ in bounded intervals. So the local solution of (3.13) and (3.14) can be extended to a global one.

### C. Global conservative solutions of system (2.1)

In this section, we will show that the global solution of the system (3.13) yields a global conservative solution to system (2.1), in the original variables $(t, x)$.

Define

$$q_1(t, \xi) = \tilde{q}_1(\xi) + \int_0^t u(\tau, \xi) d\tau, \quad q_2(t, \xi) = \tilde{q}_2(\xi) + \int_0^t v(\tau, \xi) d\tau. \quad (3.34)$$

Then, for each fixed $\xi$, the functions $t \mapsto q_1(t, \xi)$ and $t \mapsto q_2(t, \xi)$ provide a solution to the Cauchy problem,

$$\begin{align*}
\frac{\partial}{\partial t} q_1(t, \xi) &= u(t, \xi), \\
\frac{\partial}{\partial t} q_2(t, \xi) &= v(t, \xi), \\
q_1(0, \xi) &= \tilde{q}_1(\xi), \\
q_2(0, \xi) &= \tilde{q}_2(\xi).
\end{align*} \quad (3.35)$$

We claim that a solution of system (2.1) can be obtained by setting

$$\begin{align*}
u(t, x) &= u(t, \xi), & \text{if } q_1(t, \xi) = x, \\
v(t, x) &= v(t, \xi), & \text{if } q_2(t, \xi) = x.
\end{align*} \quad (3.36)$$

In order to prove the global existence of solution to the Cauchy problem (2.1) and (2.2), we need the following lemma.

**Lemma 3.4:** Let $(u, v, \theta_1, \theta_2, w_1, w_2)$ be a global solution to the Cauchy problem (3.2) and (3.3). Then we have

$$\frac{\partial}{\partial t} q_1(\xi) = u_\xi(\xi), \quad \frac{\partial}{\partial t} q_2(\xi) = v_\xi(\xi)$$

for a.e. $(t, \xi) \in [0, \infty) \times R$.

**Proof:** First, we establish the identities

$$q_1(\xi) = w_1 \cos^2 \frac{\theta_1}{2}, \quad q_2(\xi) = w_2 \cos^2 \frac{\theta_2}{2}$$

for a.e. $(t, \xi) \in [0, \infty) \times R$. 

(3.37)
In view of (3.27) and (3.13), we obtain that
\[
\frac{\partial}{\partial t} \left( w_1 \cos \frac{\theta_1}{2} \right)(t, \xi) = -w_1 \cdot \cos \frac{\theta_1}{2} \cdot \sin \frac{\theta_1}{2} + \frac{\partial w_1}{\partial t} \cdot \cos^2 \frac{\theta_1}{2}
\]
\[
= -w_1 \cos \frac{\theta_1}{2} \cdot \sin \frac{\theta_1}{2} \left[ 2 \left( u^2 - vu_{xx} - A_\xi - B + \frac{v^2}{2} - \frac{v_x^2}{2} \right) \cos \frac{\theta_1}{2} \cdot \cot \frac{\theta_1}{2} - \csc \frac{\theta_1}{2} \right]
\]
\[
+ 2w_1 \left( u^2 - vu_{xx} - A_\xi - B + \frac{v^2}{2} - \frac{v_x^2}{2} \right) \cdot \cos^2 \frac{\theta_1}{2}
\]
\[
= w_1 \cdot \cos \frac{\theta_1}{2} = u_\xi(t, \xi).
\]

On the other hand, (3.34) implies
\[
\frac{\partial}{\partial t} q_{1\xi}(t, \xi) = u_\xi(t, \xi).
\]

Moreover, in a similar way, we obtain
\[
\frac{\partial}{\partial t} q_{2\xi}(t, \xi) = v_\xi(t, \xi).
\]

This completes the proof of this lemma.

Now, we have the following main results.

**Theorem 3.2:** Let \((u, v, \theta_1, \theta_2, w_1, w_2)\) be a global solution to the Cauchy problem (3.2) and (3.3), then the function \(z = z(t, x)\) defined by (3.34)–(3.36) provides a global solution to the Cauchy problem (2.1) and (2.2) for system (2.1). Moreover, the energy is almost always conserved, namely
\[
\|u\|_{H^1}^2 + \|v\|_{H^1}^2 = \|\bar{u}\|_{H^1}^2 + \|\bar{v}\|_{H^1}^2 \quad \text{for a.e.} \quad t \geq 0.
\]

**Proof:** From (3.31) we derive the uniform bounds \(|u(t, \xi)|, |v(t, \xi)| \leq E_0^{1/2}\). Thus (3.34) implies the estimates
\[
\bar{q}_1(\xi) - E_0^{1/2}t \leq q_1(t, \xi) \leq \bar{q}_1(\xi) + E_0^{1/2}t, \quad t \geq 0.
\]
\[
\bar{q}_2(\xi) - E_0^{1/2}t \leq q_2(t, \xi) \leq \bar{q}_2(\xi) + E_0^{1/2}t, \quad t \geq 0.
\]

This yields that, for each \(t\),
\[
\lim_{\xi \to \pm \infty} \bar{q}_1(t, \xi) = \lim_{\xi \to \pm \infty} \bar{q}_2(t, \xi) = \pm \infty.
\]

Hence the image of the continuous maps \((t, \xi) \mapsto (t, q_1(t, \xi))\) and \((t, \xi) \mapsto (t, q_2(t, \xi))\) covers the entire domain \([0, \infty) \times R\).

In view of (3.37), we can see that the maps \(\xi \mapsto q_1(t, \xi)\) and \(\xi \mapsto q_2(t, \xi)\) are nondecreasing. Therefore the map \((t, x) \mapsto z(t, x)\) in (3.36) is well defined for all \((t, x) \in [0, \infty) \times R\).

Next, for every fixed \(t\), we have
\[
\int_R (u^2 + u_x^2 + v^2 + v_x^2)(t, x)dx
\]
\[
= \int_{[\cos \theta_1 \neq -1]} \left( u^2 \cos^2 \frac{\theta_1}{2} + 1 \right) w_1(t, \xi) d\xi + \int_{[\cos \theta_1 \neq -1]} \left( u^2 \cos^2 \frac{\theta_1}{2} + 1 \right) w_2(t, \xi) d\xi
\]
\[
\leq E_0.
\]

Using a Sobolev inequality in Ref. 2, we obtain the uniform Hölder continuity with exponent 1/2 of \(z\) as functions of \(x\). From the first equation in (3.13) and the uniform bounds on \(\|A\|_{L^\infty}\) and \(\|B_1\|_{L^\infty}\), we can deduce that the map \(t \mapsto u(t, q_1(t))\) is uniformly Lipschitz continuous along every characteristic curve \(t \mapsto q_1(t, \xi)\). Similarly, the map \(t \mapsto v(t, q_2(t))\) is uniformly Lipschitz continuous along every characteristic curve \(t \mapsto q_2(t, \xi)\).
continuous along every characteristic curve $t \mapsto q_1 (t, \xi)$. Therefore, $z = z (t, x)$ is globally Hölder continuous on the entire $t - x$ plane. This completes the proof that $z = (u, v)$ is a global solution of system (2.1) in the sense of Definition 3.1.

**Theorem 3.3:** Suppose that $\tilde{z}$ is a initial data and $\tilde{z}_n = (\tilde{u}_n, \tilde{v}_n)$ is a sequence which converges to $\tilde{z}$. Let $z^n$, $z$ be the corresponding solutions of (3.13) with initial data (3.14). Then, for any $T > 0$, the convergence $z^n (t, \xi) \to z (t, \xi)$ holds uniformly for $(t, \xi) \in [0, T] \times R$.

**Proof:** Let $\tilde{z}_n$ be a sequence of initial data converging to $\tilde{z}$ in $H^1 (R) \times H^1 (R)$. From (3.3) and (3.14), we have that, at time $t = 0$,

$$\sup_{\xi \in R} |q_{1n} (0, \xi) - q_1 (0, \xi)| \to 0, \quad \sup_{\xi \in R} |q_{2n} (0, \xi) - q_2 (0, \xi)| \to 0,$$

$$\sup_{\xi \in R} |z_n (0, \xi) - z (0, \xi)| \to 0.$$

Moreover,

$$\|\theta_{1n} (0, \cdot) - \theta_1 (0, \cdot)\|_{L^2} \to 0, \quad \|\theta_{2n} (0, \cdot) - \theta_2 (0, \cdot)\|_{L^2} \to 0.$$

This implies $z_n (t, \xi) \to z (t, \xi)$ uniformly for $t, \xi$ in bounded sets. The uniformly Hölder continuity of all functions $z$, $z_n$ ensures the convergence $q_{1n} (t, x) \to q_1 (t, x)$, $q_{2n} (t, x) \to q_2 (t, x)$, $z_n (t, x) \to z (t, x)$ uniformly on bounded sets. This completes the proof of this theorem.

**IV. GLOBAL DISSIPATIVE SOLUTIONS OF SYSTEM (2.1)**

In the conservative case, the solutions can be continued after the breaking time, in the sense that the total energy remains constant for almost every time. This section is concerned with global dissipative solutions, where wave breaking might induce a partial or even total loss of energy.

What we prove here is that:

Our constructive procedure (via coordinate transformations) yields a unique semigroup of solutions, defined on the entire space $H^1 (R) \times H^1 (R)$.

All of our solutions satisfy the Oleinik type inequality

$$u_x (t, x), v_x (t, x) \leq C \left(1 + t^{-1}\right) \quad t > 0, \quad (4.1)$$

with a constant $C$ depending only on the norm of the initial data $\|\tilde{z}\|_{H^1}$.

**Definition 4.1:** A solution of the Cauchy problem (2.1) and (2.2) is said to be dissipative if it satisfies the inequality (4.1) for some constant $C$, and moreover its energy $E (t)$ in (2.8) is a nonincreasing function of time $t$.

**A. An equivalent semilinear system**

In order to obtain the global dissipative solutions of system (3.13), it is necessary to modify it suitably. Suppose that, along given characteristic $t \to q_1 (t, \xi)$, the wave breaks at the first time $t = \tau_1 (\xi)$. Recalling our rescaled variable $\theta_1 = 2arctan (u_x)$ implies that $u_x (t, \xi) \to -\infty$, as $t \to \tau_1 (\xi)$. For all $t \geq \tau_1$, we set $\theta_1 (t, \xi) \equiv \pi$. Define $\tau_2 (\xi)$ in the same way, then as $t \to \tau_2 (\xi), v_x (t, \xi) \to -\infty$. Similarly, for all $t \geq \tau_2$, we set $\theta_2 (t, \xi) \equiv \pi$. Moreover, we set $\tau (\xi)$
\[ A(\xi) = \frac{1}{2} \left( \int_{[\xi^{'},\xi],\theta_1(\xi'),\theta_2(\xi)\neq\pi} \cos^2 \frac{\theta_1(s)}{2} \cdot w_1(s) ds \right) \]

\[ \cdot (u_x v_x) - \cos^2 \frac{\theta_1}{2} \cdot w_1(\xi) d\xi', \]

\[ A_x(\xi) = \frac{1}{2} \left( \int_{[\xi^{'},\xi],\theta_1(\xi)^{\prime},\theta_2(\xi)\neq\pi} \cos^2 \frac{\theta_1(s)}{2} \cdot w_1(s) ds \right) \]

\[ \cdot (u_x v_x) - \cos^2 \frac{\theta_1}{2} \cdot w_1(\xi') d\xi', \]

\[ B(\xi) = \frac{1}{2} \left( \int_{[\xi^{'},\xi],\theta_1(\xi),\theta_2(\xi)\neq\pi} \cos^2 \frac{\theta_1(s)}{2} \cdot w_1(s) ds \right) \]

\[ \cdot \left( u^2 + \frac{1}{2} u_x^2 + u_x v_x + \frac{1}{2} u^2 - \frac{1}{2} v_x^2 \right) - \cos^2 \frac{\theta_1}{2} \cdot w_1(\xi) d\xi', \]

\[ C(\xi) = \frac{1}{2} \left( \int_{[\xi^{'},\xi],\theta_1(\xi),\theta_2(\xi)\neq\pi} \cos^2 \frac{\theta_1(s)}{2} \cdot w_1(s) ds \right) \]

\[ \cdot (u v_x) - \cos^2 \frac{\theta_2}{2} \cdot w_2(\xi) d\xi', \]

\[ C_x(\xi) = \frac{1}{2} \left( \int_{[\xi^{'},\xi],\theta_1(\xi),\theta_2(\xi)\neq\pi} \cos^2 \frac{\theta_1(s)}{2} \cdot w_1(s) ds \right) \]

\[ \cdot (u v_x) - \cos^2 \frac{\theta_2}{2} \cdot w_2(\xi) d\xi', \]

\[ D(\xi) = \frac{1}{2} \left( \int_{[\xi^{'},\xi],\theta_1(\xi),\theta_2(\xi)\neq\pi} \cos^2 \frac{\theta_1(s)}{2} \cdot w_1(s) ds \right) \]

\[ \cdot \left( u^2 + \frac{1}{2} u_x^2 + u_x v_x + \frac{1}{2} u^2 - \frac{1}{2} v_x^2 \right) - \cos^2 \frac{\theta_2}{2} \cdot w_2(\xi) d\xi', \]

\[ D_x(\xi) = \frac{1}{2} \left( \int_{[\xi^{'},\xi],\theta_1(\xi),\theta_2(\xi)\neq\pi} \cos^2 \frac{\theta_1(s)}{2} \cdot w_1(s) ds \right) \]

\[ \cdot \left( u^2 + \frac{1}{2} u_x^2 + u_x v_x + \frac{1}{2} u^2 - \frac{1}{2} v_x^2 \right) - \cos^2 \frac{\theta_2}{2} \cdot w_2(\xi) d\xi'. \]
Thus system (3.13) is transformed into the following form:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -vu_x - A - B_x, \\
\frac{\partial v}{\partial t} &= -uv_x - C - D_x, \\
\frac{\partial \theta_1}{\partial t} &= \left\{ \begin{array}{ll}
(2u^2 - 2vu_{xx} - 2A_x - 2B + v^2 - v_x^2) \cos \frac{\theta_1}{2} \cot \frac{\theta_1}{2} - \csc \frac{\theta_1}{2} & \text{if } \theta_1, \theta_2 \neq \pi \\
0 & \text{if } \theta_1 \text{ or } \theta_2 = \pi
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\frac{\partial \theta_2}{\partial t} &= \left\{ \begin{array}{ll}
(2v^2 - 2uv_{xx} - 2C_x - 2D + u^2 - u_x^2) \cos \frac{\theta_2}{2} \cot \frac{\theta_2}{2} - \csc \frac{\theta_2}{2} & \text{if } \theta_1, \theta_2 \neq \pi \\
0 & \text{if } \theta_1 \text{ or } \theta_2 = \pi
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\frac{\partial w_1}{\partial t} &= \left\{ \begin{array}{ll}
(2u^2 - 2vu_{xx} - 2A_x - 2B + v^2 - v_x^2) \cos \frac{\theta_1}{2} \cdot w_1 & \text{if } \theta_1, \theta_2 \neq \pi \\
0 & \text{if } \theta_1 \text{ or } \theta_2 = \pi
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\frac{\partial w_2}{\partial t} &= \left\{ \begin{array}{ll}
(2v^2 - 2uv_{xx} - 2C_x - 2D + u^2 - u_x^2) \cos \frac{\theta_2}{2} \cdot w_2 & \text{if } \theta_1, \theta_2 \neq \pi \\
0 & \text{if } \theta_1 \text{ or } \theta_2 = \pi
\end{array} \right.
\end{align*}
\]

System (4.3) can be regarded as an O.D.E. in a Banach space, where the right-hand side is now discontinuous. The discontinuity occurs precisely when $\theta_1$ or $\theta_2 = \pi$. By the third and the fourth equations in (4.3), it is easy to see that $\theta_1$ and $\theta_2$ approach the values transversally, i.e., with strictly negative derivative $\theta_1$, $\theta_2 = -1$ and $\theta_1$, $\theta_2 = -1$. We note that it is exact the transversality condition that guarantees the well-posedness of system (4.3).

**B. Global solutions of the equivalent semilinear system**

Consider here the Cauchy problem for system (4.3) in more compact form as

\[
\frac{\partial}{\partial t} U(t, \xi) = F(U(t, \xi)) + G(\xi, U(t, \cdot)) \quad \xi \in \mathbb{R},
\]

\[
U(0, \xi) = \bar{U}(\xi),
\]

where $U = (u, v, \theta_1, \theta_2, w_1, w_2) \in \mathbb{R}^6$, with

\[
F(U) = \begin{cases}
(-uv_x, -uv_x, (2u^2 - 2vu_{xx} + v^2 - v_x^2) \cos \frac{\theta_1}{2} \cot \frac{\theta_1}{2} - \csc \frac{\theta_1}{2}, \\
(2v^2 - 2uv_{xx} + u^2 - u_x^2) \cos \frac{\theta_2}{2} \cot \frac{\theta_2}{2} - \csc \frac{\theta_2}{2}, (2u^2 - 2vu_{xx} + v^2 - v_x^2) \cos \frac{\theta_1}{2} \cdot w_1, \\
(2v^2 - 2uv_{xx} + u^2 - u_x^2) \cos \frac{\theta_2}{2} \cdot w_2) & \text{if } \theta_1, \theta_2 \neq \pi,
\end{cases}
\]

\[
(0, 0, 0, 0, 0, 0) & \text{ if } \theta_1 \text{ or } \theta_2 = \pi,
\]

\[
G(\xi, U(\cdot)) = \begin{cases}
(-A - B_x, -C - D_x, (-2A_x - 2B) \cos \frac{\theta_1}{2} \cot \frac{\theta_1}{2}, (-2C_x - 2D) \cos \frac{\theta_2}{2} \cot \frac{\theta_2}{2}, \\
(-2A_x - 2B) \cos \frac{\theta_1}{2} \cdot w_1, (-2C_x - 2D) \cos \frac{\theta_2}{2} \cdot w_2) & \text{if } \theta_1, \theta_2 \neq \pi,
\end{cases}
\]

\[
(-A - B_x, -C - D_x, 0, 0, 0, 0) & \text{ if } \theta_1 \text{ or } \theta_2 = \pi.
\]

The nonlocal operators $A, A_x, B, B_x, C, C_x, D, D_x$ are as in (4.2).
It is easy to see that if \((u, v, \theta_1, \theta_2, w_1, w_2)\) is a solution to Cauchy problem (4.4)–(4.7), then the mapping \((t, \xi) \to (u(t, \xi), v(t, \xi), \theta_1(t, \xi), \theta_2(t, \xi), w_1(t, \xi), w_2(t, \xi))\) provides a solution of system (4.3). Equation (4.4) can be regarded as an O.D.E. in the space \(L^\infty (R; R^6)\). Notice that the vector field \(F : R^6 \to R^6\) in (4.6) is uniformly bounded and Lipschitz continuous as long as \(u, v\) remain in a bounded set. However, the nonlocal operator \(G\) is discontinuous. Indeed, at a time \(t^*\) such that \(\text{meas} (\{\xi; \theta_1(t^*, \xi) = \pi \text{ or } \theta_2(t^*, \xi) = \pi\}) > 0\), the set 
\(\{\xi; t(\xi) > t, \theta_1(t, \xi), \theta_2(t, \xi) \neq \pi\}\) may suddenly shrink. Thus the integral terms \(A, A_1, B, B_1, C, C_1, D, D_1\) in (4.2) are discontinuous.

First, we obtain the local existence and uniqueness of solution to the Cauchy problem (4.4)–(4.7).

**Theorem 4.1:** Given \(\tilde{z} = (\tilde{u}, \tilde{v}) \in H^1 \times H^1\). Then there exists a unique local solution defined on some time interval \([0, T]\) with \(T > 0\) to the Cauchy problem (4.4)–(4.7).

**Proof:** We will first obtain \(a\ p\ r\ i\ o\ r\ i\) estimates on \(F\) and \(G\) in (4.4).

Suppose that \(U = (u, v, \theta_1, \theta_2, w_1, w_2) \in L^\infty (R; R^6)\) satisfies the inequalities

\[
\|u\|_{L^\infty}, \|v\|_{L^\infty} \leq C, \quad \frac{1}{C} \leq w_1(\xi), w_2(\xi) \leq C \text{ for all } \xi \in R, \tag{4.8}
\]

for some constant \(C\). Then there is a constant \(\kappa^*\) depending only on \(C\), such that

\[
\|F(U)\|_{L^\infty} \leq \kappa^*, \quad \|G(U)\|_{L^\infty} \leq \kappa^*.
\]

Moreover, there exists a Lipschitz constant \(\kappa\) such that, if \(\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\theta}_1, \tilde{\theta}_2, \tilde{w}_1, \tilde{w}_2)\) satisfies the same bounds (4.8), then

\[
\|F(U) - F(\tilde{U})\|_{L^\infty} \leq \kappa \|U - \tilde{U}\|_{L^\infty},
\]

\[
\|G(U) - G(\tilde{U})\|_{L^\infty} \leq \kappa \cdot \|U - \tilde{U}\|_{L^\infty} + \text{meas} \left( \{\xi; \theta_1, \theta_2, \tilde{\theta}_1 = \pi, \tilde{\theta}_2 \neq \pi, \tilde{\theta}_1 \neq \pi \} \right) + \text{meas} \left( \{\xi; \theta_1 = \pi, \tilde{\theta}_1 \neq \pi, \tilde{\theta}_2 = \pi \} \right) + \text{meas} \left( \{\xi; \theta_1 = \pi, \tilde{\theta}_1 \neq \pi, \tilde{\theta}_2 \neq \pi \} \right).
\]

Suppose the initial data \(\tilde{z} = (\tilde{u}, \tilde{v}) \in H^1 \times H^1\) be given. Since \(\tilde{u}, \tilde{u}_x, \tilde{v}, \tilde{v}_x \in L^2\), the sets \(\{x \in R; |	ilde{u}_x(x)| \geq \varepsilon\}\) and \(\{x \in R; |	ilde{v}_x(x)| \geq \varepsilon\}\) have finite measure for every \(\varepsilon > 0\). Thus we can find a constant \(C > 0\) such that \(\|u\|_{L^\infty}, \|v\|_{L^\infty} \leq \frac{C}{\varepsilon}\).

Suppose that the inequalities in (4.8) hold. Choose \(\delta > 0\) small enough such that \(\theta_1(\xi) \in (\pi, \pi + \delta]\). This implies that

\[
\frac{\partial \theta_1}{\partial t} = (2u^2 - 2uv_{xx} - 2A_x - 2B + v^2 - v_x^2) \cos \frac{\theta_1}{2} \cot \frac{\theta_1}{2} - \csc \frac{\theta_1}{2} \leq -\frac{1}{2}. \tag{4.9}
\]

Define the sets

\[
\Omega^3 = \{\xi \in R; \tilde{\theta}_1(\xi) \in (\pi, \pi + \delta] \text{ or } \tilde{\theta}_2(\xi) \in (\pi, \pi + \delta]\}.
\]

By reducing the size of \(\delta > 0\), we may assume that

\[
\text{meas} \left( \Omega^3 \right) \leq 1/8\kappa. \tag{4.10}
\]

On a suitable domain \(D \subset C ([0, T], L^\infty \times L^\infty)\), we will obtain the solution \(t \to U(t) = (u(t), v(t), \theta_1(t), \theta_2(t), w_1(t), w_2(t))\) as the unique fixed point of a contractive transformation \(\Pi : D \to D\). Here the Picard operator \(\Pi\) is defined as

\[
(\Pi(U))(t, \xi) = \tilde{U} + \int_0^t [F(U(\tau, \xi)) + G(U(\tau, \cdot))]|d\tau.
\]
Moreover, for given $T > 0$, we assume the domain $D$ as the set of all continuous mappings $t \to U(t)$ from $[0, T]$ into $L^\infty (R, R^6)$ with the following properties,

$$U(0) = \bar{U}, \quad \|U(t) - U(s)\|_{L^\infty} \leq 2k^* |t - s|,$$

$$\theta_1(t, \xi) - \theta_1(s, \xi), \theta_2(t, \xi) - \theta_2(s, \xi) \leq -\frac{t - s}{2}, \quad \xi \in \Omega^0, \quad 0 \leq s < t \leq T.$$

Now we show that $\Pi$ is a strict contraction. Suppose $U, \bar{U} \in D$ and define

$$\eta = \max_{t \in [0, T]} \|U(t) - \bar{U}(t)\|_{L^\infty},$$

and the crossing time

$$\tau(\xi) = \sup \{t \in [0, T]; \theta_1(t, \xi), \theta_2(t, \xi) \neq \pi\}.$$

Similarly, define the function $\bar{\tau}(\xi)$ and replace $\theta_1$ and $\theta_2$ by $\bar{\theta}_1$ and $\bar{\theta}_2$, respectively. Notice that, for each $\xi \in \Omega^\delta$, we can obtain from (4.9) that

$$|\bar{\tau}(\xi) - \tau(\xi)| \leq 2\eta.$$

Choosing $T$ small enough such that for $t \in [0, T]$, we have

$$\|PU(t) - \bar{P}\bar{U}(t)\|_{L^\infty} \leq \int_0^T \|F(U(\tau)) - F(\bar{U}(\tau))\|_{L^\infty} d\tau + \int_0^T \|G(U(\tau)) - G(\bar{U}(\tau))\|_{L^\infty} d\tau$$

$$\leq 2k \cdot \int_0^T \|U(t) - \bar{U}(t)\|_{L^\infty} d\tau + \kappa \int_0^T \max \left\{ \|\xi; \theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2 \neq \pi\| + \max \left\{ \|\xi; \theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2 \neq \pi, \bar{\theta}_2 = \pi\| \right\} d\tau$$

$$\leq 2kT \eta + \kappa \int_{\Omega^0} |\bar{\tau}(\xi) - \tau(\xi)| d\xi \leq 2kT \eta + \kappa \cdot \max \left\{ \Omega^\delta \right\} \cdot 8\eta \leq \eta.$$

This implies that $\Pi$ is a strict contraction. Thus it has a unique fixed point, which yields the desired local solution of the Cauchy problem (4.4)–(4.7).

Next, we will show that the local solutions of the semilinear system (4.3) can be extended globally. The basic ingredient is a global bound on the total energy,

$$E(t) = \int_{[\theta_1(t, \xi), \theta_2(t, \xi) \neq \pi]} \left[ \left( u^2 \cos^2 \frac{\theta_1}{2} + 1 \right) w_1 + \left( v^2 \cos^2 \frac{\theta_2}{2} + 1 \right) w_2 \right] (t, \xi) d\xi. \quad (4.11)$$

We begin by showing that

$$u_\xi = w_1 \cdot \cos \frac{\theta_1}{2}, \quad v_\xi = w_2 \cdot \cos \frac{\theta_2}{2}. \quad (4.12)$$

as long as the local solution of (4.3) exists.

At the initial time $t = 0$, in view of (3.14) and (3.6), and (3.7), we have

$$\frac{\partial u}{\partial \xi} = \bar{u}_\xi \cdot \sec \frac{\bar{\theta}}{2} \cdot \cos^2 \frac{\bar{\theta}}{2} = \cos \frac{\bar{\theta}}{2} = \bar{w}_1 \cdot \cos \frac{\bar{\theta}}{2} (\text{since } \bar{w}_1 = 1).$$
Next, when \( \theta_1, \theta_2 \neq \pi \), from (4.3) we deduce that

\[
\left( w_1 \cdot \cos \frac{\theta_1}{2} \right)_t = \frac{\partial w_1}{\partial t} \cdot \cos \frac{\theta_1}{2} - \frac{1}{2} w_1 \cdot \sin \frac{\theta_1}{2} \cdot \frac{\partial \theta_1}{\partial t}
\]

\[
= \left( 2u^2 - 2uv_{xx} - 2A_x - 2B + v^2 - v_x^2 \right) \cdot \cos \frac{\theta_1}{2} \cdot w_1 - \frac{1}{2} w_1 \sin \frac{\theta_1}{2}
\]

\[
\cdot \left[ \left( 2u^2 - 2uv_{xx} - 2A_x - 2B + v^2 - v_x^2 \right) \cdot \cos \frac{\theta_1}{2} \cdot \cot \frac{\theta_1}{2} - \csc \frac{\theta_1}{2} \right]
\]

\[
= w_1 \left( u^2 - uv_{xx} - A_x + \frac{v^2}{2} - \frac{v_x^2}{2} \right) \cdot \cos^2 \frac{\theta_1}{2} + \frac{1}{2} w_1.
\]

If \( \theta_1 = \pi \), then (4.3) yields

\[
\left( w_1 \cdot \cos \frac{\theta_1}{2} \right)_t = 0.
\]

On the other hand, the first equation in (4.3) implies

\[
u_{\xi t} = \begin{cases} 
w_1 \left( u^2 - uv_{xx} - A_x - B + \frac{v^2}{2} - \frac{v_x^2}{2} \right) \cdot \cos^2 \frac{\theta_1}{2} + \frac{1}{2} w_1 & \text{if } \theta_1(t, \xi), \theta_2(t, \xi) \neq \pi, \\
0 & \text{if } \theta_1(t, \xi) = \pi,
\end{cases}
\]

with \( A_x \) and \( B \) are given in (4.2). Hence the first identity in (4.12) holds at time \( t \geq 0 \). Similarly, we obtain that the second identity in (4.12) holds at time \( t \geq 0 \).

In the following, we prove that the “extended energy,”

\[
\tilde{E}(t) = \int_{\mathbb{R}} \left[ \left( u^2 \cos^2 \frac{\theta_1}{2} + 1 \right) w_1 + \left( v^2 \cos^2 \frac{\theta_2}{2} + 1 \right) w_2 \right](t, \xi) \, d\xi,
\]

remains constant in time. We remark that the energy in (4.13) in general is strictly larger than the energy in (4.11), in the sense that the integration ranges over the entire real line. From (4.3) we obtain that

\[
\frac{d}{dt} \int_{\mathbb{R}} \left[ \left( u^2 \cos^2 \frac{\theta_1}{2} + 1 \right) w_1 + \left( v^2 \cos^2 \frac{\theta_2}{2} + 1 \right) w_2 \right] \, d\xi
\]

\[
= \int_{\{\theta_1(t, \xi) \neq \pi\}} w_1 \left[ (3u^2 - 2uv_{xx} - 2A_x - 2B + v^2 - v_x^2) \cos \frac{\theta_1}{2} - (2uvu_x + 2uA + 2uB_x) \cos \frac{\theta_1}{2} \right] \, d\xi
\]

\[
+ \int_{\{\theta_2(t, \xi) \neq \pi\}} w_2 \left[ (3v^2 - 2uv_{xx} - 2C_x - 2D + u^2 - u_x^2) \cos \frac{\theta_2}{2} - (2uvu_x + 2uC + 2uD_x) \cos \frac{\theta_2}{2} \right] \, d\xi
\]

\[
= \int_{\mathbb{R}} \left[ w_1 \left( 3u^2 \cos \frac{\theta_1}{2} - 2B \cos \frac{\theta_1}{2} - 2uB_x \cos^2 \frac{\theta_1}{2} \right) + w_2 \left( 3v^2 \cos \frac{\theta_2}{2} - 2D \cos \frac{\theta_2}{2} - 2vD_x \cos^2 \frac{\theta_2}{2} \right) \right] \, d\xi.
\]

In the sense that \( \cos \frac{\theta_1}{2} = \cos \frac{\theta_2}{2} = 0 \) whenever \( \theta_1 = \theta_2 = \pi \), we are again integrating over the entire real line \( \mathbb{R} \) on the last identity of (4.14).

Moreover, from (4.2), we obtain that

\[
B_\xi = w_1 \cdot B_x \cdot \cos^2 \frac{\theta_1}{2}, \quad D_\xi = w_2 \cdot D_x \cdot \cos^2 \frac{\theta_2}{2},
\]

(4.15)

On the other hand, when \( \theta_1 = \pi \) and \( \theta_2 = \pi \), a separate computation yields

\[
B_\xi = 0 = w_1 \cdot B_x \cdot \cos^2 \frac{\pi}{2}, \quad D_\xi = 0 = w_2 \cdot D_x \cdot \cos^2 \frac{\pi}{2}.
\]

Thus the identities in (4.15) still hold for the cases \( \theta_1 = \pi \) and \( \theta_2 = \pi \).
From (4.12) and (4.15), we derive

\[(u B)_{\xi} = u_{\xi} B + u B_{\xi} = w_1 \left( B \cdot \cos \frac{\theta_1}{2} + u B_{\xi} \cdot \cos^2 \frac{\theta_1}{2} \right), \]

\[(v D)_{\xi} = v_{\xi} D + v D_{\xi} = w_2 \left( D \cdot \cos \frac{\theta_2}{2} + v D_{\xi} \cdot \cos^2 \frac{\theta_2}{2} \right). \]

In addition,

\[(u^3)_{\xi} = 3 u^2 u_{\xi} = 3 w_1 u^2 \cdot \cos \frac{\theta_1}{2}, \quad (v^3)_{\xi} = 3 v^2 v_{\xi} = 3 w_2 v^2 \cdot \cos \frac{\theta_2}{2}. \]

So,

\[
\frac{d}{dt} \int_R \left[ (u^2 \cos^2 \frac{\theta_1}{2} + 1) w_1 + (v^2 \cos^2 \frac{\theta_2}{2} + 1) w_2 \right] d\xi \\
= \int_R \partial_{\xi} (u^3 - 2uB) d\xi + \int_R \partial_{\xi} (v^3 - 2vD) d\xi = 0.
\]

This proves our claim, namely

\[E(t) = \int_R \left[ (u^2 \cos^2 \frac{\theta_1}{2} + 1) w_1 + (v^2 \cos^2 \frac{\theta_2}{2} + 1) w_2 \right] (t, \xi) d\xi = E(0) = E_0. \quad (4.16)\]

From (4.12) and (4.16), we obtain the bounds

\[
\sup_{\xi \in R} |u^2(t, \xi)| \leq 2 \int_R |u| |u_{\xi}| d\xi \leq 2 \int_R |u| \cdot |\cos \frac{\theta_1}{2}| w_1 d\xi \leq E_0,
\]

\[
\sup_{\xi \in R} |v^2(t, \xi)| \leq 2 \int_R |v| |v_{\xi}| d\xi \leq 2 \int_R |v| \cdot |\cos \frac{\theta_2}{2}| w_2 d\xi \leq E_0. \quad (4.17)
\]

Thus the above provides uniform a priori bound on \(\|u(t)\|_{L^\infty}\) and \(\|v(t)\|_{L^\infty}\), respectively. It follows from (4.16) and (4.2) that

\[
\|A(t)\|_{L^\infty}, \|A_x(t)\|_{L^\infty} \leq \left\| \frac{1}{2} e^{-|\xi|} \right\|_{L^\infty} \cdot \|u(t) v_x(t)\|_{L^1} \leq \frac{1}{2} E_0,
\]

\[
\|B(t)\|_{L^\infty}, \|B_x(t)\|_{L^\infty} \leq \left\| \frac{1}{2} e^{-|\xi|} \right\|_{L^\infty} \cdot \left( \|u^2 + \frac{u_x^2}{2} + u_{v_x} + \frac{v^2}{2} - \frac{v_{\xi}^2}{2} \right) \|_{L^1} \leq \frac{1}{2} E_0,
\]

\[
\|C(t)\|_{L^\infty}, \|C_x(t)\|_{L^\infty} \leq \left\| \frac{1}{2} e^{-|\xi|} \right\|_{L^\infty} \cdot \|v(t) u_x(t)\|_{L^1} \leq \left\| \frac{1}{2} e^{-|\xi|} \right\|_{L^\infty} \cdot \left( \|v^2(t) + \frac{v_{\xi}^2}{2} + u_{v_x} + \frac{u_x^2}{2} - \frac{u_{\xi}^2}{2} \right) \|_{L^1} \leq \frac{1}{2} E_0. \quad (4.18)
\]

Then, from (4.3), (4.17), and (4.18), we deduce that

\[
\left| \frac{\partial w_1}{\partial t} \right| \leq (2E_0 + E_0 + E_0) w_1 = 4E_0 w_1, \quad \left| \frac{\partial w_2}{\partial t} \right| \leq (2E_0 + E_0 + E_0) w_2 = 4E_0 w_2.
\]

Therefore,

\[e^{-4E_0 t} \leq w_1(t) \leq e^{4E_0 t}, \quad e^{-4E_0 t} \leq w_2(t) \leq e^{4E_0 t}. \]

Moreover, the third and fourth equations in (4.3) yield

\[0 \leq \theta_1(t), \quad \theta_2(t) \leq 2\pi. \]
All the above proves a priori bounds (4.8) which we need to construct a local solution with a constant $C$ uniformly valid over any given time interval $[0, T]$. This completes the proof that the local solution can be extended globally for all times $t \geq 0$.

C. Continuous dependence of solutions to system (2.1)

In Sec. III B, by representing the solution of (4.4)–(4.7) as the fixed point of a contraction in a suitable space, we have obtained the local existence theorem. This yields uniqueness and continuous dependence with respect to convergence on the initial data in $L^\infty \times L^\infty$. In the following, we will show the continuous dependence of solutions to system (2.1), with the initial data belongs to $H^1$. This requires further estimates. Suppose that $\bar{z}_n \to \bar{z}$ in $H^1(R) \times H^1(R)$, with $\theta = 2\arcsin z_n$, then we have $\|z_n - \bar{z}\|_{L^\infty} \to 0$, $\|\theta_n - \bar{\theta}\|_{L^2} \to 0$, where $z = (u, v, \theta = (\theta_1, \theta_2)$.

Based on the weaker assumptions (4.1), we obtain the following result on continuous dependence of solutions of (4.3).

**Theorem 4.2:** Let $\bar{z}_n = (\bar{u}_n, \bar{v}_n)$ be a sequence of initial data such that $\|\bar{z}_n - \bar{z}\|_{H^1} \to 0$. Then, for any $T > 0$, the corresponding solutions $z_n(t, \xi) = (u_n(t, \xi), v_n(t, \xi))$ converge to $z(t, \xi) = (\bar{u}(t, \xi), v(t, \xi))$ uniformly for $(t, \xi) \in [0, T] \times R$.

**Proof:** Let $(u, v, \theta_1, \theta_2, w_1, w_2)$ and $(\bar{u}, \bar{v}, \bar{\theta}_1, \bar{\theta}_2, \bar{w}_1, \bar{w}_2)$ be two solutions of (4.3), corresponding to initial data of the form (3.14). Let $E_0$ be an upper bound for the energies of the two solutions. Assuming that at time $t = 0$,

$$
\|z(0) - \bar{z}(0)\|_{L^\infty} \leq \delta_0, \quad \|\theta_1(0, \xi) - \bar{\theta}_1(0, \xi)\|_{L^2}, \quad \|\theta_2(0, \xi) - \bar{\theta}_2(0, \xi)\|_{L^2} \leq \delta_0.
$$

Next, we will establish a priori bound on

$$
\|z(t) - \bar{z}(t)\|_{L^\infty} \leq \delta(t), \quad t \in [0, T].
$$

for $t \in [0, T]$, which depends only on $\delta_0, T,$ and $E_0$.

Define the set

$$
\Lambda = \{\xi \in R; \bar{\theta}_1(T, \xi) = \pi\} \cup \{\xi \in R; \bar{\theta}_2(T, \xi) = \pi\} \cup \{\xi \in R; \bar{\theta}_1(T, \xi) = \pi\} \cup \{\xi \in R; \bar{\theta}_2(T, \xi) = \pi\},
$$

so that $\alpha^* = \text{meas} (\Lambda)$ is a uniformly bounded number.

For each $\xi \in \Lambda$, let $\tau(\xi)$ be the first time at which one of the four solutions reaches the value $\pi$, so that $\tau(\xi) = \inf \{t \in [0, T]; \min \{\bar{\theta}_1(t, \xi), \bar{\theta}_2(t, \xi), \bar{\theta}_1(t, \xi), \bar{\theta}_2(t, \xi)\} = \pi\}$.

We now construct a measure-preserving mapping: $[0, \alpha^*] \to \Lambda$, which is defined as $\alpha \to \xi(\alpha)$ with the additional property,

$$
\alpha \leq \alpha' \quad \text{if and only if} \quad \tau(\xi(\alpha)) \geq \tau(\xi(\alpha')).
$$

Then we define the distance between $(u, v, \theta_1, \theta_2, w_1, w_2)$ and $(\bar{u}, \bar{v}, \bar{\theta}_1, \bar{\theta}_2, \bar{w}_1, \bar{w}_2)$ as

$$
J((u, v, \theta_1, \theta_2, w_1, w_2), (\bar{u}, \bar{v}, \bar{\theta}_1, \bar{\theta}_2, \bar{w}_1, \bar{w}_2)) = \left(\|u - \bar{u}\|_{L^\infty} + \|v - \bar{v}\|_{L^\infty} + \|\theta_1 - \bar{\theta}_1\|_{L^2} + \|\theta_2 - \bar{\theta}_2\|_{L^2} + \|w_1 - \bar{w}_1\|_{L^2} + \|w_2 - \bar{w}_2\|_{L^2}\right)
$$

$$
+ K_0 \int_0^{\alpha^*} e^{K\alpha} \left(|\theta_1(\xi(\alpha)) - \bar{\theta}_1(\xi(\alpha))| + |\theta_2(\xi(\alpha)) - \bar{\theta}_2(\xi(\alpha))|\right) d\alpha.
$$

For convenience, we set

$$
J(t) = J((u, v, \theta_1, \theta_2, w_1, w_2), (\bar{u}, \bar{v}, \bar{\theta}_1, \bar{\theta}_2, \bar{w}_1, \bar{w}_2))(t) = J^*(t) + K_0 J^0(t),
$$

where
where
\[ J^*(t) = \left( \| u - \tilde{u} \|_{L^\infty} + \| v - \tilde{v} \|_{L^\infty} + \| \theta_1 - \tilde{\theta}_1 \|_{L^2} + \| \theta_2 - \tilde{\theta}_2 \|_{L^2} + \| w_1 - \tilde{w}_1 \|_{L^2} + \| w_2 - \tilde{w}_2 \|_{L^2} \right), \]

\[ J^\theta(t) = \int_0^{\alpha} e^{K\alpha} \left( |\theta_1(\xi(\alpha)) - \tilde{\theta}_1(\xi(\alpha))| + |\theta_2(\xi(\alpha)) - \tilde{\theta}_2(\xi(\alpha))| \right) d\alpha. \]

From (4.3) we derive the following estimate:
\[ J^*(t) = \int_0^{\alpha^*} e^{K\alpha} \left( |\theta_1(\xi(\alpha)) - \tilde{\theta}_1(\xi(\alpha))| + |\theta_2(\xi(\alpha)) - \tilde{\theta}_2(\xi(\alpha))| \right) d\alpha. \]

In the following, we will show that the inequality
\[ \frac{d}{dt} J(t) \leq M \cdot J(t), \quad (4.23) \]
holds for suitable constants \( K_0, K, M \) depending only on \( T \) and \( E_0 \). Moreover, this will imply
\[ J(t) \leq e^{MT} J(0), \quad t \in [0, T], \]
thus providing a priori estimate of (4.19).

For each fixed \( t \in [0, T] \), define the sets
\[ \Gamma(t) = \left\{ \xi \in \Lambda : \theta_1(t, \xi), \theta_2(t, \xi), \tilde{\theta}_1(t, \xi) \neq \pi, \tilde{\theta}_2(t, \xi) = \pi \right\} \]
\[ \cup \left\{ \xi \in \Lambda : \theta_1(t, \xi), \theta_2(t, \xi), \tilde{\theta}_1(t, \xi) \neq \pi, \tilde{\theta}_2(t, \xi) = \pi \right\} \]
\[ \cup \left\{ \xi \in \Lambda : \theta_1(t, \xi), \tilde{\theta}_1(t, \xi), \tilde{\theta}_2(t, \xi) \neq \pi, \theta_2(t, \xi) = \pi \right\} \]
\[ \cup \left\{ \xi \in \Lambda : \theta_2(t, \xi), \tilde{\theta}_1(t, \xi), \tilde{\theta}_2(t, \xi) \neq \pi, \theta_1(t, \xi) = \pi \right\}, \]
\[ \Gamma^+(t) = \Gamma^+_1(t) \cup \Gamma^+_2(t) = \left\{ \xi \in \Lambda : \theta_1(t, \xi) = \tilde{\theta}_1(t, \xi) = \pi \right\} \cup \left\{ \xi \in \Lambda : \theta_2(t, \xi) = \tilde{\theta}_2(t, \xi) = \pi \right\}, \]
\[ \Gamma^-(t) = \left\{ \xi \in \Lambda : \theta_1(t, \xi), \theta_2(t, \xi), \tilde{\theta}_1(t, \xi), \tilde{\theta}_2(t, \xi) \neq \pi \right\} \cap \left\{ \xi \in \Lambda : \tau(t) > t \right\}, \]

Then they have the following properties:
\[ \Gamma(t) \cap \Gamma^+(t) = \Gamma(t) \cap \Gamma^-(t) = \Gamma^+(t) \cap \Gamma^-(t) = \Phi, \quad \Gamma(t) \cup \Gamma^+(t) \cup \Gamma^-(t) = \Lambda, \]
for each \( t \in [0, T] \).

Set \( m(t) = \text{meas} \left( \Gamma^{-}(t) \right), \) such that
\[ \Gamma^{-}(t) = \left\{ \xi(\alpha); \alpha \in [0, m(t)] \right\}. \quad (4.24) \]

From (4.3) we derive the following estimate:
\[ \frac{d}{dt} \left( \| u - \tilde{u} \|_{L^\infty} + \| v - \tilde{v} \|_{L^\infty} + \| \theta_1 - \tilde{\theta}_1 \|_{L^2} + \| \theta_2 - \tilde{\theta}_2 \|_{L^2} + \| w_1 - \tilde{w}_1 \|_{L^2} + \| w_2 - \tilde{w}_2 \|_{L^2} \right) \]
\[ \leq K (\| u - \tilde{u} \|_{L^\infty} + \| v - \tilde{v} \|_{L^\infty} + \| \theta_1 - \tilde{\theta}_1 \|_{L^2} + \| \theta_2 - \tilde{\theta}_2 \|_{L^2} + \| w_1 - \tilde{w}_1 \|_{L^2} + \| w_2 - \tilde{w}_2 \|_{L^2} \]
\[ + \text{meas} (\Gamma(t))). \quad (4.25) \]

Moreover, we can deduce from (4.24) that
\[ \frac{d}{dt} \int_0^{\alpha^*} e^{K\alpha} \left( |\theta_1(t, \xi(\alpha)) - \tilde{\theta}_1(t, \xi(\alpha))| + |\theta_2(t, \xi(\alpha)) - \tilde{\theta}_2(t, \xi(\alpha))| \right) d\alpha \]
\[ = \int_{\Gamma(t) \cap \Gamma^+(t) \cap \Gamma^-(t)} e^{K\alpha} \cdot \frac{\partial}{\partial t} \left( |\theta_1(t, \xi(\alpha)) - \tilde{\theta}_1(t, \xi(\alpha))| + |\theta_2(t, \xi(\alpha)) - \tilde{\theta}_2(t, \xi(\alpha))| \right) d\alpha \]
\[ = \int_{\Gamma(t)} e^{K\alpha} \cdot \frac{\partial}{\partial t} \left( |\theta_1(t, \xi(\alpha)) - \tilde{\theta}_1(t, \xi(\alpha))| + |\theta_2(t, \xi(\alpha)) - \tilde{\theta}_2(t, \xi(\alpha))| \right) d\xi \]
\[ + \int_0^{m(t)} e^{K\alpha} \cdot \frac{\partial}{\partial t} \left( |\theta_1(t, \xi(\alpha)) - \tilde{\theta}_1(t, \xi(\alpha))| + |\theta_2(t, \xi(\alpha)) - \tilde{\theta}_2(t, \xi(\alpha))| \right) d\alpha. \quad (4.26) \]
Indeed, the integral over \( \Gamma^+ (t) \) is zero.

Choosing \( \delta > 0 \) sufficiently small (depending only on \( T, E_0 \)), we have

\[
|\theta_1 (t, \xi) - \tilde{\theta}_1 (t, \xi)|, |\theta_2 (t, \xi) - \tilde{\theta}_2 (t, \xi)| \leq \delta \text{ for } \xi \in \Gamma (t).
\]

This implies that

\[
\frac{\partial}{\partial t} |\theta_1 (t, \xi) - \tilde{\theta}_1 (t, \xi)| \leq -\frac{1}{2}, \quad \frac{\partial}{\partial t} |\theta_2 (t, \xi) - \tilde{\theta}_2 (t, \xi)| \leq -\frac{1}{2}.
\]

On the other hand, choosing a constant \( \kappa \) large enough such that

\[
|\theta_1 (t, \xi) - \tilde{\theta}_1 (t, \xi)|, |\theta_2 (t, \xi) - \tilde{\theta}_2 (t, \xi)| \geq \delta.
\]

Then we have

\[
\frac{\partial}{\partial t} |\theta_1 (t, \xi) - \tilde{\theta}_1 (t, \xi)| \leq -\frac{1}{2} + \kappa |\theta_1 (t, \xi) - \tilde{\theta}_1 (t, \xi)|, \quad \frac{\partial}{\partial t} |\theta_2 (t, \xi) - \tilde{\theta}_2 (t, \xi)| \leq -\frac{1}{2} + \kappa |\theta_2 (t, \xi) - \tilde{\theta}_2 (t, \xi)|.
\]

Finally, for \( \xi \in \Gamma^- (t) \), we have the following estimate:

\[
\frac{\partial}{\partial t} |\theta_1 (t, \xi) - \tilde{\theta}_1 (t, \xi)| + \frac{\partial}{\partial t} |\theta_2 (t, \xi) - \tilde{\theta}_2 (t, \xi)| \leq \kappa \cdot \left( \| u - \tilde{u} \|_{L^\infty} + \| u - \tilde{v} \|_{L^\infty} + \| \theta_1 - \tilde{\theta}_1 \|_{L^2} + \| \theta_2 - \tilde{\theta}_2 \|_{L^2} + \| w_1 - \tilde{w}_1 \|_{L^2} + \| w_2 - \tilde{w}_2 \|_{L^2} + \text{meas} (\Gamma (t)) + |\theta_1 (t, \xi) - \tilde{\theta}_1 (t, \xi)| + |\theta_2 (t, \xi) - \tilde{\theta}_2 (t, \xi)| \right).
\]

So,

\[
\int_{m(t)}^{n(t)} e^{K\alpha} \cdot \frac{\partial}{\partial t} |\theta_1 (t, \xi) - \tilde{\theta}_1 (t, \xi)| + |\theta_2 (t, \xi) - \tilde{\theta}_2 (t, \xi)| + \text{meas} (\Gamma (t)) \cdot \int_{0}^{m(t)} e^{K\alpha} d\alpha + \\
\kappa \int_{m(t)}^{n(t)} e^{K\alpha} \left( |\theta_1 (t, \xi) - \tilde{\theta}_1 (t, \xi)| + |\theta_2 (t, \xi) - \tilde{\theta}_2 (t, \xi)| \right) d\alpha \leq \kappa \left( J^* (t) + \text{meas} (\Gamma (t)) \right) \int_{0}^{m(t)} e^{K\alpha} d\alpha + \\
\kappa \int_{\Gamma^- (t)} e^{K\alpha (\xi)} \left( |\theta_1 (t, \xi) - \tilde{\theta}_1 (t, \xi)| + |\theta_2 (t, \xi) - \tilde{\theta}_2 (t, \xi)| \right) d\xi.
\]

Now, we rewrite (4.25) in the form

\[
\frac{d}{dt} J^* (t) \leq \kappa \cdot \left( J^* (t) + \text{meas} (\Gamma (t)) \right).
\]
In the last inequality, we choose the constant \( K = 4\kappa \), such that
\[
\kappa \int_{0}^{m(t)} e^{K(a-m(t))} da \leq \frac{\kappa}{4}.
\]

In view of (4.21) and (4.22), and choosing \( K_0 = 4\kappa \), then we derive from (4.28) and (4.29) that
\[
\frac{d}{dt} \left( J^*(t) + 4\kappa J^\#(t) \right) \leq \kappa \cdot \left( J^*(t) + \text{meas} (\Gamma(t)) \right) + 4\kappa \left( \frac{1}{4} \text{meas} (\Gamma(t)) + \kappa J^\#(t) + \frac{\kappa}{K} e^{K\alpha} J^*(t) \right)
\]
\[
\leq \kappa J^*(t) + 4\kappa^2 J^\#(t) + \kappa e^{4\kappa \alpha} J^*(t).
\]

This proves our claim (4.23), with \( M = \kappa + \kappa e^{4\kappa \alpha} \).

D. Global dissipative solutions of system (2.1)

In this section we will show that the global solution of system (4.3) yields a global dissipative solution to system (2.1), in the original variables \((t, x)\).

Let \((u, v, \theta_1, \theta_2, w_1, w_2)\) be a global solution to (4.3) and
\[
\begin{align*}
q_1(t, \xi) &= \bar{q}_1(\xi) + \int_{0}^{t} u(\tau, \xi) d\tau, \\
q_2(t, \xi) &= \bar{q}_2(\xi) + \int_{0}^{t} v(\tau, \xi) d\tau.
\end{align*}
\]
(4.30)

Then for each fixed \( \xi \), the functions \( t \mapsto q_1(t, \xi) \) and \( t \mapsto q_2(t, \xi) \) provide a solution to the Cauchy problem
\[
\begin{align*}
\frac{\partial}{\partial t} q_1(t, \xi) &= u(t, \xi), \\
\frac{\partial}{\partial t} q_2(t, \xi) &= v(t, \xi), \\
q_1(0, \xi) &= \bar{q}_1(\xi), \\
q_2(0, \xi) &= \bar{q}_2(\xi).
\end{align*}
\]
(4.31)

We claim that a solution of (2.1) can be obtained by setting
\[
\begin{align*}
u(t, x) &= u(t, \xi), & \text{if } q_1(t, \xi) &= x, \\
u(t, x) &= v(t, \xi), & \text{if } q_2(t, \xi) &= x. \\
\end{align*}
\]
(4.32)

Now, we present the main result in this section.

**Theorem 4.3:** Let \((u, v, \theta_1, \theta_2, w_1, w_2)\) be a global solution to the Cauchy problem (4.3) and (3.14). Then the function \( z = z(t, x) \) defined by (4.30)–(4.32) provides a global solution to the initial value problem (2.1) and (2.2) for system (2.1).

**Proof:** Using the uniform bounds \(|u(t, \xi)|, |v(t, \xi)| \leq E_0^{1/2} \), we obtain the estimates from (4.30)
\[
\bar{q}_1(\xi) - E_0^{1/2} t \leq q_1(t, \xi) \leq \bar{q}_1(\xi) + E_0^{1/2} t, \quad t \geq 0,
\]
\[
\bar{q}_2(\xi) - E_0^{1/2} t \leq q_2(t, \xi) \leq \bar{q}_2(\xi) + E_0^{1/2} t, \quad t \geq 0.
\]

The definition of \( \xi \) in (3.3) yields
\[
\lim_{\xi \to \pm \infty} q_1(t, \xi) = \lim_{\xi \to \pm \infty} q_2(t, \xi) = \pm \infty.
\]

Then the image of the continuous maps \((t, \xi) \mapsto (t, q_1(t, \xi)), (t, \xi) \mapsto (t, q_2(t, \xi)) \) cover the entire plane \([0, \infty] \times \mathbb{R}, \) respectively. It is clear that the maps \( \xi \mapsto q_1(t, \xi) \) and \( \xi \mapsto q_2(t, \xi) \) are nondecreasing. This proves that the map \((t, x) \mapsto z(t, x) \) at (4.32) are well defined, for all \( t \geq 0 \) and \( x \in \mathbb{R}. \)
We claim that, for every fixed \( t \), the image of the singular set where \( \theta_1 = \pi \) or \( \theta_2 = \pi \) has measure zero (in the \( x \)-variable). Indeed

\[
\text{meas} \left( (q_1 (t, \xi) : \theta_1 (t, \xi) = \pi) \right) = \int_{[\theta_1 (t, \xi) = \pi]} q_1 (t, \xi) d \xi = \int_{[\theta_1 (t, \xi) = \pi]} w_1 \cos^2 \frac{\theta_1}{2} (t, \xi) d \xi = 0, \\
\text{meas} \left( (q_2 (t, \xi) : \theta_2 (t, \xi) = \pi) \right) = \int_{[\theta_2 (t, \xi) = \pi]} q_2 (t, \xi) d \xi = \int_{[\theta_2 (t, \xi) = \pi]} w_2 \cos^2 \frac{\theta_2}{2} (t, \xi) d \xi = 0.
\]

Moreover, for every fixed \( t \), we have

\[
\int_R \left( u^2 + u_x^2 + v^2 + v_x^2 \right) (t, x) dx
\]

\[
= \int_{[\cos \theta_1 \neq -1]} \left( u^2 \cos^2 \frac{\theta_1}{2} + 1 \right) w_1 (t, \xi) d \xi + \int_{[\cos \theta_2 \neq -1]} \left( u^2 \cos^2 \frac{\theta_2}{2} + 1 \right) w_2 (t, \xi) d \xi \leq E_0.
\]

This implies the uniform Hölder continuity with exponent \( 1/2 \) of \( z \) as a function of \( x \). From the first and second equations in (4.3) and the bounds on \( \| A \|_{L^\infty}, \| B \|_{L^\infty}, \| C \|_{L^\infty} \) and \( \| D \|_{L^\infty} \), we obtain that the map \( t \mapsto z (t, q (t)) \) is uniformly Lipschitz continuous along every characteristic curve \( t \mapsto q (t) \). Hence, \( z = z (t, x) \) is globally Hölder continuous on \( (t, x) \) for all \( t \geq 0 \) and \( x \in R \).

We know that the map \( t \mapsto z (t) \) is Lipschitz continuous with values in \( L^2 (R) \times L^2 (R) \). Since \( L^2 (R) \times L^2 (R) \) is a reflexive space, by the infinite-dimensional version of Rademacher’s theorem, the map \( t \mapsto q (t) \) is differentiable for a.e. \( t \in R \). Notice that the right hand side of (3.1) lies in \( L^2 (R) \times L^2 (R) \), to establish the equality, one may proceed as follows.

For each smooth function with compact support \( \varphi \in C^\infty_c \), at almost every time \( t \geq 0 \), we have

\[
\frac{d}{dt} \int u (t, x) \varphi (x) dx
\]

\[
= \int (-uu_x - vu_x - A - B_x) (t, x) \varphi (x) dx
\]

\[
= \int [u^2 (t, x) \varphi' (x) + (-vu_x - A - B_x) (t, x) \varphi (x) + u (t, x) u_x (t, x) \varphi (x)] dx.
\]

For each \( \xi \in R \), we set

\[
\tau (\xi) = \inf \{ t > 0, \theta_1 (t) = \pi \text{ or } \theta_2 (t) = \pi \}.
\]

Notice that, for almost every time \( t \geq 0 \), we have

\[
\text{meas} (\{ \xi ; \tau (\xi) = t \}) = 0.
\]

Choosing a time \( t \) such that (4.33) holds. Integrating with respect to the variable \( \xi \) and from (3.6), we obtain

\[
\frac{d}{dt} \int u (t, \xi) \varphi (q_1 (t, \xi)) \left( u_1 \cdot \cos^2 \frac{\theta_1}{2} \right) (t, \xi) d \xi
\]

\[
= \int \left\{ u \varphi w_1 \cdot \cos^2 \frac{\theta_1}{2} + u \varphi \overline{q_1} \cdot \cos^2 \frac{\theta_1}{2} + u \varphi w_1 \cdot \cos^2 \frac{\theta_1}{2} - u \varphi w_1 \cdot \theta_1 \left[ \sin \theta_1 \right] \right\} d \xi
\]

\[
= \int_{\theta_1 (t, \xi), \theta_2 (t, \xi) \neq \pi} \left\{ (-vu_x - A - B_x) \varphi w_1 \cdot \cos^2 \frac{\theta_1}{2} + u^2 \varphi' w_1 \cdot \cos^2 \frac{\theta_1}{2} +
\right.
\]

\[
\left. u \varphi \left( 2u^2 - 2vu_x + 2A_x - 2B + v^2 + v_x^2 \right) \cdot \cos \frac{\theta_1}{2} \cdot w_1 \cdot \cos^2 \frac{\theta_1}{2} - u \varphi w_1 \cdot \left[ -\csc \frac{\theta_1}{2} + (2u^2 - 2vu_x - 2A_x - 2B + v^2 + v_x^2) \cos \frac{\theta_1}{2} \cdot \cot \frac{\theta_1}{2} \right] \sin \frac{\theta_1}{2} \right] d \xi
\]

\[
= \int_{\theta_1 (t, \xi), \theta_2 (t, \xi) \neq \pi} \left\{ (-vu_x - A - B_x) \varphi + u^2 \varphi' + uu_x \varphi \right\} w_1 \cdot \cos^2 \frac{\theta_1}{2} d \xi.
\]
Similarly, we obtain that
\[
\frac{d}{dt} \int v(t, x) \varphi(x) dx = \int \left[ v^2(t, x) \varphi'(x) + (-uv_x - C - D_x)(t, x) \varphi(x) + v(t, x) u_x(t, x) \varphi(x) \right] dx.
\]

This completes the proof that \( z = (u, v) \) is a global solution of system (2.1) in the sense of Definition 3.1.

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