Signless Laplacian eigenvalues and circumference of graphs

JianFeng Wang\textsuperscript{a,}\textsuperscript{*}, Francesco Belardo\textsuperscript{b}
\textsuperscript{a} Department of Mathematics, QingHai Normal University, XiNing, QingHai 810008, PR China
\textsuperscript{b} Department of Mathematics and Computer Science, University of Messina, 98166 Sant’Agata, Messina, Italy

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\textbf{A B S T R A C T}

In this paper, we investigate the relation between the Q-spectrum and the structure of G in terms of the circumference of G. Exploiting this relation, we give a novel necessary condition for a graph to be Hamiltonian by means of its Q-spectrum. We also determine the graphs with exactly one or two Q-eigenvalues greater than or equal to 2 and obtain all minimal forbidden subgraphs and maximal graphs, as induced subgraphs, with respect to the latter property.

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1. Introduction

All graphs considered here are finite, undirected, simple (namely, loops and multiple edges are not allowed) and connected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where its order and size are $|V(G)| = n(G) = n$ and $|E(G)| = m(G) = m$, respectively. For a graph $G$, let $M = M(G)$ be a corresponding graph matrix defined in a prescribed way. The $M$-polynomial of $G$ is defined as $\det(A - M)$, where $I$ is the identity matrix. The $M$-eigenvalues of $G$ are the roots of its $M$-polynomial. The $M$-spectral radius of $G$ is the largest $M$-eigenvalue of $G$. The $M$-spectrum, denoted by $\text{Spec}_M(G)$, of $G$ is a multiset consisting of the $M$-eigenvalues together with their multiplicities. In the literature, there are several graph matrices including the adjacency matrix $A(G)$, the Laplacian matrix $L(G)$, the signless Laplacian matrix $Q(G)$ and so on. Here we will mainly consider the signless Laplacian matrix $Q$.

The signless Laplacian matrix, as it is named in [15], has recently attracted the interest of many researchers. This is due to the seminal paper [3] and the subsequent papers [4–6] which standardized the so-called Q-theory of graph spectra. Recall that the signless Laplacian matrix $Q(G) = D(G) + A(G)$, where $D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$ is the diagonal matrix of vertex degrees. The $Q$-polynomial of $G$, namely $\det(kI - Q(G))$, is denoted by $\varphi(G, \kappa) = \varphi(G)$ (we omit the variable if clear from the context). The $Q$-spectrum of a graph $G$ is here denoted by $\text{Spec}_Q(G) = \{\kappa_1(G), \kappa_2(G), \ldots, \kappa_n(G)\}$, with $\kappa_1(G) \geq \kappa_2(G) \geq \cdots \geq \kappa_n(G)$.

We now introduce some notations and terminology. $G \cup H$ stands for the disjoint union of graphs $G$ and $H$, and $kG$ stands for the disjoint union of $k$ copies of $G$. The matching number $\mu(G)$ of $G$ is the maximum number of pairwise independent edges in $G$. The length of the longest cycle in $G$, i.e. the circumference, is denoted by $c(G)$. Let $m_{\mu I}$ denote the number of the $M$-eigenvalues with multiplicities which belong to the real interval $I$. Let $P_n, K_{1,n-1}$ and $K_n$ be respectively the path, the star and the complete graph on $n$ vertices. A $n$-cycle $C_n$ is the cycle of order $n$, and the 3-cycle is usually called a triangle. Another recurrent graph is $T_{i,j,k}$, obtained from three vertex disjoint paths $P_i, P_j$ and $P_k$ by identifying one of their end vertices.
A connected graph is said to be 2-connected if it has no cut-vertices. A block in a graph $G$ is a maximal subgraph that is 2-connected; clearly, a connected graph is said to be itself a block if it is 2-connected. Some other recurrent graphs are depicted later in the paper.

The organization of this paper is as follows. In Section 2, we study the relation between the $Q$-spectrum of a graph and its circumference, and as a consequence we give a necessary condition for a graph to be Hamiltonian in terms of the $Q$-eigenvalues. In Section 3 we investigate the graphs with exactly one or two $Q$-eigenvalues greater than or equal to 2 and obtain all minimal forbidden subgraphs and maximal graphs, as induced subgraphs, with respect to the latter property.

2. The $Q$-eigenvalues and the circumference

In this section, exploiting the relation between the $Q$-spectrum and the circumference of a graph, we will give a new necessary condition for a graph to be Hamiltonian in terms of its $Q$-eigenvalues.

It is well-known that the problem of determining whether a graph is Hamiltonian is one of the most difficult classical problems in graph theory. Although there are large numbers of conditions to judge the Hamiltonicity of a graph (see [10], for example), only several ones are established by spectral means. A first theorem in this direction was due to Mohar [18], while other spectral conditions for Hamiltonian cycles or Hamiltonian paths have been given in [2,9,21,16,17,25].

For a graph $G$, we say that $H$ is a subgraph of $G$, denoted by $H \subseteq G$, which means that $H$ is obtained from $G$ by deleting some edges or vertices; in particular, if $H$ is obtained from $G$ after the deletion of some vertices, then $H$ is an induced subgraph of $G$. Conversely, $H \not\subseteq G$ means that $H$ is not a subgraph of $G$. By $G[V']$, where $V' \subseteq V(G)$, we denote the subgraph of $G$ induced by the vertices of $V'$. In [22] we studied a sort of interlacing property for the $Q$-eigenvalues of a graph and its subgraphs, and, among others, we obtained the following result:

**Theorem 2.1.** Let $G$ be a simple graph and $H \subseteq G$. Then, for $i = 1, 2, \ldots, n(H)$, we have

$$\kappa_i(H) \leq \kappa_i(G).$$

**Lemma 2.2 ([11]).** Let $G$ be a graph, then its $L$-eigenvalues and $Q$-eigenvalues are the same if and only if $G$ is bipartite.

Guo and Tan [12] proved Lemma 2.3 with respect to the Laplacian matrix. Since a tree is bipartite, then, in view of Lemma 2.2, it holds for the $Q$-matrix too.

**Lemma 2.3.** Let $T$ be a tree with order $n$. If $n > 2\mu(T)$, then $\kappa_{\mu(T)} > 2$.

In view of the above lemma, we immediately deduce the following result.

**Lemma 2.4.** Let $G$ be a connected graph with order $n$.

(i) If $n > 2\mu(G)$, then $\kappa_{\mu(G)} > 2$;

(ii) If $n = 2\mu(G)$, then $\kappa_{\mu(G)-1} > 2$.

**Proof.** Since $G$ is connected, there must exist a spanning tree such that $\mu(T) = \mu(G)$. Assume that $n > 2\mu(G)$, then $n(T) = n > 2\mu(G) = 2\mu(T)$. By Lemma 2.3 and Theorem 2.1 we deduce that $\kappa_{\mu(G)}(G) \geq \kappa_{\mu(T)}(T) > 2$; so (i) is proved.

Next we prove (ii). If $G$ is a block, then $G$ has no cut-vertices and so $G - v$ is connected for any $v \in V(G)$. If $G$ is not a block, then $G$ has one block with exactly one cut-vertex in $G$ (see [1], Exercise 3.2.4) and thus $G - v$ is also connected for some vertex $v \in V(G)$. In view of $n = 2\mu(G)$, we have $n(G - v) = n - 1 > 2(\mu(G) - 1) = 2\mu(G - v)$ and, by (i),

$$\kappa_{\mu(G-1)}(G - v) = \kappa_{\mu(G)-1}(G - v) > 2.$$ 

From Theorem 2.1 it follows that $\kappa_{\mu(G)-1}(G - v) > 2$.

This completes the proof. □

The following corollary stems from Lemma 2.4.

**Corollary 2.5.** Let $G$ be a connected graph with order $n \geq 7$ and $\mu(G) \geq 3$. Then $\kappa_3(G) > 2$.

Recall that $c(G)$ denotes the circumference of $G$. We now give the main result of this section.

**Theorem 2.6.** Let $G$ be a connected graph and $\kappa_i(G)$ be its $i$-th largest $Q$-eigenvalue ($i \geq 3$).

(i) If $\kappa_i(G) = 2$, then $c(G) \leq 2i$ with equality if $G = C_{4t}$ and $i = 2t$ ($t \geq 2$);

(ii) If $\kappa_i(G) < 2$, then $c(G) \leq 2i - 1$ with equality if $G = C_{4t+1}$ and $i = 2t + 2$ ($t \geq 1$).

(ii) Let $i$ be odd. If $\kappa_i(G) \leq 2$, then $c(G) \leq 2i - 2$ with equality if $G = C_{4t}$ and $i = 2t + 1$, where $t \geq 1$. 
Proof. Recall that $\text{Spec}_Q(C_n) = \{2 + 2\cos \frac{j \pi}{n}| j = 1, 2, \ldots, n\}$.

For (I), let $i = 2a$ $(a \geq 2)$.

We first show (i). By way of contradiction we set $c(G) \geq 2i + 1$, i.e., $c(G) \geq 4a + 1$. If $c(G) = 4a + 1$, then $G$ contains $C_{4a+1}$ as its subgraph. By Theorem 2.1 we have $\kappa_i(G) \geq \kappa_i(C_{4a+1}) = \kappa_2(C_{4a+1}) = 2 + 2\cos \frac{2\pi}{4a+1}a > 2$, a contradiction. If $c(G) \geq 4a + 2$, then $\mu(G) \geq 2a + 1 = i + 1$. Since $G$ contains a $(i + 1)$-matching then $n(G) \geq 2i + 2$, if so, by Lemma 2.4, we get $\kappa_i(G) > 2$, a contradiction. Therefore, $c(G) \leq 2i$. If $G = C_4$ and $i = 2t$ $(t \geq 1)$, then $\kappa_{2t}(C_4) = 2 + 2\cos \frac{2\pi}{4t}t = 2$. Clearly, $c(C_4) = 4t = 2i$.

For (ii), by way of contradiction, we set $c(G) \geq 2i$, i.e., $c(G) \geq 4a$. If $c(G) = 4a$, then $G$ contains $C_{4a}$ as its subgraph. By Theorem 2.1 we have $\kappa_i(G) \geq \kappa_i(C_{4a}) = \kappa_2(C_{4a}) = 2 + 2\cos \frac{2\pi}{4a}a = 2$, a contradiction. If $c(G) \geq 4a + 1$, then $\mu(G) \geq 2a + 1 = i$. If $\mu(G) = i$, then $n(G) \geq c(G) \geq 4a + 1$. Since $G$ contains a $(i + 1)$-matching then $n(G) \geq 2i + 2$, and by Lemma 2.4 $\kappa_i(G) > 2$, a contradiction. Therefore, $c(G) \leq 2i - 1$. If $G = C_{4t + 3}$ and $i = 2t + 2$, then $\kappa_{2t+2}(C_{4t+3}) = 2 + 2\cos \frac{2\pi}{4t+3}2t < 2$. Clearly, $c(C_{4t+3}) = 4t + 3 = 2i - 1$.

For (II), let $i = 2a + 1$ $(a \geq 1)$. Assume that $c(G) \geq 2i - 1 = 4a + 1$. If $c(G) = 4a + 1$, then $G$ contains $C_{4a+1}$ as its subgraph. By Theorem 2.1 it follows that $\kappa_i(G) \geq \kappa_i(C_{4a+1}) = \kappa_2(C_{4a+1}) = 2 + 2\cos \frac{2\pi}{4a+1}a = 2$, a contradiction. If $c(G) = 4a + 2$, we can similarly obtain $\kappa_i(G) \geq \kappa_i(C_{4a+2}) = \kappa_2(C_{4a+2}) = 2 + 2\cos \frac{2\pi}{4a+2}a > 2$, a contradiction. If $c(G) \geq 4a + 3$, then $\mu(G) \geq 2a + 1 = i$. If $\mu(G) = 2a + 1 = i$, then $n(G) \geq 4a + 3 > 4a + 2 = 2\mu(G) = 2i$ and so $\kappa_i(G) > 2$ by Lemma 2.4(i), a contradiction. If $\mu(G) \geq 2a + 2 = i + 1$, since $G$ contains a $(i + 1)$-matching then $n(G) \geq 2i + 2$, and, in view of Lemma 2.4, $\kappa_i(G) > 2$, a contradiction. Thus, $c(G) \leq 2i - 2$.

Finally, if $G = C_6$ and $i = 2t + 1$, then $\kappa_{2t+1}(C_6) = 2 + 2\cos \frac{2\pi}{6}2t = 2$. Clearly, $c(C_6) = 4t = 2i - 2$.

This completes the proof. $\square$

Note that $G$ is Hamiltonian if $c(G)$ equals the order of $G$. Then from Theorem 2.6 we have the following corollary:

Corollary 2.7. Let $G$ be a connected graph with order $n$ and $\kappa_i(G)$ be its $i$-th largest $Q$-eigenvalue $(i \geq 3)$.

(I) Let $i$ be even.

(i) If $\kappa_i(G) = 2$ and $i < \frac{n}{2}$, then $G$ is not Hamiltonian;

(ii) If $\kappa_i(G) < 2$ and $i < \frac{n+1}{2}$, then $G$ is not Hamiltonian.

(II) Let $i$ be odd. If $\kappa_i(G) \leq 2$ and $i < \frac{n+2}{2}$, then $G$ is not Hamiltonian.

In view of the above corollary, we can give the following necessary condition for a graph to be Hamiltonian.

Theorem 2.8. Let $G$ be a Hamiltonian graph of order $n > 3$ and let $\kappa_i(G)$ be its $i$-th largest $Q$-eigenvalue. Then

(i) if $n$ is even, then $\kappa_i \geq 2$ and $\kappa_{\frac{n+1}{2}} > 2$;

(ii) if $n$ is odd, then $\kappa_{\frac{n+1}{2}} > 2$.

Remark 2.9. In the above theorem a simple condition for checking whether the graph could be Hamiltonian is provided. We observe that even if all $Q$-eigenvalues are greater than 2, then the graph needs not to be Hamiltonian. Indeed, consider a graph obtained of two complete graphs on 5 vertices by identifying a vertex, such a graph has 3 as smallest $Q$-eigenvalue but it is not Hamiltonian, due to the presence of a cut-vertex.

3. Graphs with at most two $Q$-eigenvalues exceeding 2

Another application of Theorem 2.6 is to characterize all connected graphs with a given number of $Q$-eigenvalues at least 2. Recall that the complete graph $K_n$ maximizes the spectral radius of both the Laplacian and signless Laplacian matrices over the class of graphs on $n$ vertices. In particular, we have that the $L$-spectral radius of $K_n$ equals $n$, while its $Q$-spectral radius equals $2n - 2$.

In the literature, there are several papers which investigated step by step the graphs with the property $m_q(2, n] \leq 3$. Such interest comes from the connection between photoelectron spectra of saturated hydrocarbons (alkanes) and the $L$-eigenvalues of the underlying molecular graphs [13,14]. Petrović et al. [20] determined all connected bipartite graphs with $m_l(2, 3, n] = 2$. Fan and Li [8] and Zhang [24] extended their results to all connected graphs with $m_l(2, 2, n] = 2$. Petrović et al. [19] and Zhang [23] determined all connected graphs with $m_q(2, 2, n] = 3$. Hence, inspired by the above cited papers, we consider their signless Laplacian variant, namely we investigate the graphs with the property $m_q(2, 2n - 2] \leq 2$.

3.1. Graphs with at most one $Q$-eigenvalue exceeding 2

Here we first identify the connected graphs with either $k_1 \leq 2$ or $m_q(2, 2n - 2] = 1$.

Lemma 3.1. Let $G$ be a connected graph. Then $\kappa_1(G) \leq 2$ if and only if $G \in \{P_1, P_2\}$. 


Proof. If $G$ has at least 2 edges and $G$ is connected, then $G$ contains $P_2$ as its subgraph and consequently so $\kappa_1(G) \geq \kappa_1(P_2) = 3$. Hence $\kappa_1(G) \leq 2$ has either no edges or just one edge and thus $G \in \{P_1, P_2\}$. \hfill $\square$

In the sequel we assume that $G$ has at least 3 vertices. Fan and Li [8] determined the following graphs with $m_k(2,n) = 1$:

**Lemma 3.2.** Let $G$ be a connected graph with order $n$. Then $G$ has exactly one $L$-eigenvalue exceeding 2 if and only if $G \in \{P_4, C_4, K_{1,n-1} \mid n \geq 3\}$.

Note that all graphs mentioned in Lemma 3.2 are bipartite. Hence each of them has the property $m_k(2,2n-2) = 1$, as well. In order to determine all connected graphs with $m_k(2,2n-2) = 1$, it is sufficient to restrict ourselves to non-bipartite graphs. In the following lemmas we will consider some special graphs as $G_3(4; 1, 0, 1)$ (a triangle with the addition of a single pendant vertex), $G_3(5; 2, 0, 1)$ (a triangle with two additional pendant vertices at one vertex), $G_3(5; 1, 1, 1)$ (a triangle in which two vertices each have a pendant vertex) and $G_4(5)$ (a triangle with a hanging path of length 2); all of them are special cases of the graphs depicted in Fig. 6. Also let $K_4'$ be the graph obtained from $K_4$ by deleting one edge.

**Lemma 3.3.** Let $G$ be a connected non-bipartite graph with order $n$. Then $\kappa_1(G) > 2 \geq \kappa_2(G)$ if and only if $G \in \{C_5, K_4, K_4', G_3(4; 1, 0, 1)\}$.

**Proof.** The sufficiency can be directly verified. We next show the necessity.

Let $G$ have $\kappa_2(G) \leq 2$ and be a non-bipartite graph, then $G$ contains a cycle of odd order. From $\text{Spec}_Q(C_n) = \{2 + 2 \cos \frac{2\pi j}{n} \mid j = 1, 2, \ldots, n\}$, we have $\kappa_2(C_{2i+1}) > 2$, for $i \geq 2$. By Theorem 2.1, $G$ cannot contain any odd cycle of order greater than 3, otherwise $\kappa_2(G) \geq \kappa_2(C_{2i+1}) > 2$. Hence $c(G) \leq 4$ and $G$ must contain at least a triangle. Assume first that $3 \leq n(G) \leq 4$. Since both $K_4$ and $P_4$ verify $m_k(2, 2n-2) = 1$ then any graph on either 3 or 4 vertices verifies $m_k(2, 2n-2) = 1$. In particular, if we look to non-bipartite graphs, then $G \in \{C_5, K_4, K_4', G_3(4; 1, 0, 1)\}$. Assume now that $n(G) \geq 5$. If so $G$ contains $G \in \{G_3(5; 2, 0, 1), G_3(5; 1, 1, 1), G_4(5)\}$ as its subgraph. The reader can verify that $\kappa_2(G) > 2$ (cf. also Theorem 3.10) which together with Theorem 2.1 yields $\kappa_2(G) \geq \kappa_2(G') > 2$, a contradiction. \hfill $\square$

The following result can be derived by combining Lemmas 3.2 and 3.3:

**Theorem 3.4.** Let $G$ be a connected graph with order $n$. Then $G$ satisfies $m_k(2, 2n-2) = 1$ if and only if $G \in \mathcal{G} = \{C_5, K_4, K_4', P_4, C_4, G_3(4; 1, 0, 1), K_{1,n-1} \mid n \geq 3\}$.

We now look to the (connected) graphs whose second largest $Q$-eigenvalue is equal to 2. A direct calculation shows that the graphs in Fig. 1 are graphs for which $\kappa_2 > 2$.

The reader can directly check the below corollary.

**Corollary 3.5.** Let $G$ be a connected graph with order $n$. Then $\kappa_2(G) = 2$ if and only if $G \in \{K_4, K_4', C_4, G_3(4; 1, 0, 1), P_4\}$.

In view of the above results we obtain the following theorem:

**Theorem 3.6.** Let $G$ be a graph with $\kappa_2(G) > 2$, then $G$ contains $T_{1,1,2}$ or $P_5$.

3.2. Graphs with exactly two $Q$-eigenvalues exceeding 2

In this section, we identify the graphs with $m_k(2, 2n-2) = 2$. Let $G$ be a graph; then we say that:

- $G$ satisfies Property A if $k_3(G) \leq 2 < \kappa_2(G) < k_1(G)$;
- $G$ satisfies Property B if $\kappa_2(G) \leq 2$.

Observe that all graphs with Property B not considered in Lemma 3.1 and Theorem 3.4 will verify Property A as well. Suppose that $G$ is a connected graph with Property B. Then, such a property is hereditary because, according to Theorem 2.1, any subgraph $H \subseteq G$ has Property B as well. Therefore, there are minimal graphs against Property B; any such a graph will be denoted as minimal forbidden subgraph. It is not difficult to check that the graphs $F$ and $C_5$, depicted in Fig. 2, are some minimal graphs against Property B.

The following result is a direct consequence of Theorem 2.6(II).

**Corollary 3.7.** Let $G$ be a connected graph with Property B. Then $c(G) \leq 4$. 

![Fig. 1. Graphs for which $k_2 > 2$.](image-url)
We observe that all graphs with Property A and of order at least 7 have been identified (as a special case) by Fan et al. in [7]. Here we give a different proof through Corollary 3.7, and we will determine all minimal forbidden subgraphs with respect to Property A. Furthermore, we characterize the graphs with Property A and of order at most six, which have not been (directly) considered in [7].

3.2.1. Bipartite graphs with Property A

The graphs considered in this section are depicted in Fig. 3. Set $B_4 = B_4(p, q)$ ($p \geq 1, q \geq 0$), $B_5 = B_5(p, q, m)$ ($p \geq q \geq 0, m \geq 1$), $\mathcal{F} = \{B_i|1 \leq i \leq 5\}$, $\mathcal{S} = \{P_3 = B_5(0, 0, 1), P_4 = B_4(1, 1) = B_5(1, 0, 1) = B_5(0, 1, 1), K_{1,p+1} = B_4(p, 0), C_4 = B_5(0, 0, 2)|p \geq 1\}$ and $\mathcal{R} = \mathcal{F} \setminus \mathcal{S}$.

Petrović et al. [20] determined all connected bipartite graphs with exactly two $L$-eigenvalues greater than two and all corresponding minimal forbidden subgraphs. Of course, in view of Lemma 2.2, the results given in [20] hold for the $Q$-spectrum as well.

**Theorem 3.8.** Let $G$ be a connected bipartite graph with order $n$. Then $G$ has Property A if and only if $G \in \mathcal{R}$.

The above theorem can be expressed in terms of forbidden subgraphs. It is obtained from Theorem 2 in [20] by using the well-known relation between the $Q$-polynomial of a graph and the $A$-polynomial of the corresponding line graph (cf. [3] for more details). Some additional graphs, considered in the theorem below, are depicted in Fig. 4.

**Theorem 3.9.** Let $G$ be a connected bipartite graph with $\kappa_3(G) > 2$. Then $G$ contains as subgraph one of the graphs in $\mathcal{F}$, where

$\mathcal{F} = \{P_7, C_6, T_{1,1,4}, T_{1,2,3}, T_{2,2,2}, D_1, D_2, D_3, P_5 \cup P_3, T_{1,1,2} \cup P_3, 3P_3\}$.

3.2.2. Non-bipartite graphs with Property A and of order $\leq 6$

Fan et al. [7] pointed out that graphs, with order at most six and Property A, can be derived via computer check. However, we prefer to determine these graphs through a theoretical proof. The graphs used in this subsection are depicted in Figs. 5 and 6.

Let $\mathcal{H} = \{G_1(5), G_2(5), G_3(5;r, s, t), G_4(5), G_1(6), G_2(6), G_3(6;r, s, t), G_4(6)\}$ and $\mathcal{R} = \{H_i|1 \leq i \leq 16\}$.
Proof. The sufficiency follows from the direct calcula- tions. Next we show the necessity. Let $G$ have Property A. Since Spec($K_4$) = \{6, 2, 2, 2\}, then any graph on 4 or less vertices does not verify Property A. Hence, $5 \leq n(G) \leq 6$. Since $G$ is non-bipartite, from Corollary 3.7 we conclude that $G$ must contain a cycle of order 3. We distinguish the following cases according to the circumference.

Case 1. $c(G) = 3$. Let $k$ denote the number of such cycles. Clearly, $k \leq 2$ (since $n(G) \leq 6$). Suppose that $k = 1$. If $n(G) = 5$, then $G \in \{G_3(5; 2, 0, 1), G_3(5; 1, 1, 1), G_4(5)\}$. If $n(G) = 6$, since $F$ is a minimal forbidden graph, then $G \in \{G_3(6; 3, 0, 1), G_3(6; 2, 1, 1), G_4(6), H_{14}, H_{15}, H_{16}\}$. Now let $k = 2$. Note that these two triangles cannot share any edge, otherwise $c(G) = 4$. If $n(G) = 5$, then $G = H_{10}$. If $n(G) = 6$, then $G \in \{H_{11}, H_{12}, H_{13}\}$.

Case 2. $c(G) = 4$. The following subcases are taken into account:

Subcase 2.1. $K_4 \subseteq G$. If $n(G) = 5$, then $G = G_1(5)$. If $n(G) = 6$, by noticing that $F$ is a minimal forbidden subgraph, we get that $G \in \{G_1(6), H_1, H_2\}$.

Subcase 2.2. $K_4 \not\subseteq G$ and $K_4^- \subseteq G$. Label the vertices of $K_4^-$ in the graph $G_2(n)$ depicted in Fig. 6. Then any vertex of $V(G) \setminus V(K_4^-)$ cannot be adjacent to $u_1$ and $u_4$ simultaneously, otherwise the forbidden subgraph $C_3$ appears. If $n(G) = 5$, then $G \subseteq G_2(5), G_3(5; 1, 0, 2), G_3(5; 0, 0, 3)$. If $n(G) = 6$, from $F$ being a minimal forbidden subgraph, we have $G \in \{G_2(6), G_3(6; 2, 0, 2), G_3(6; 1, 1, 2), G_4(6; 0, 0, 4), H_{3i}[1 \leq i \leq 8]\}$. Since $n(G) = 6$, then $G \subseteq G_4(n)$. Recall that $G$ contains a triangle. Then $C_4$ and this triangle have no common edge. Since $n(G) \leq 6$, then $G = H_3$.

We proceed to investigate the graphs in Fig. 5. Consider $H_1$, for example. Clearly, $n(H_1) = 6$ and $\mu(H_1) = 3$. Take $G$ such that $H_1$ is an induced subgraph of $G$, hence $H_1 \subseteq G$ and $n(G) = n(H_1) + 1 = 7$. From $\mu(G) \geq 3$, in view of Corollary 2.5, we get $k_3(G) > 2$. Consequently, $H_1$ cannot be an induced subgraph of any graph with Property A. From this point of view, $H_1$ is a maximal graph with respect to Property A. In the sequel we say that a graph is a maximal forbidden (induced) subgraph if: (a) it has Property A, (b) if it appears as an induced subgraph of $G$, then $G$ does not have Property A.

In view of the above fact, we deduce the following corollary of Theorems 3.8 and 3.10.

**Corollary 3.11.** The graphs $B_1, B_2$ and $B_3$ in Fig. 3 and the graphs $H_i$ in Fig. 5, with the exception of $H_7$ (that is an induced subgraph of $G_3(n; r, s, t)$) and of $H_{10}$ (that is an induced subgraph of $H_{11}$), are maximal forbidden subgraphs.

**3.2.3. Non-bipartite graphs with Property A and of order $\geq 7$**

Since the graphs with Property A have been already identified by Fan et al. [7], the sufficiency of Lemmas 3.12 and 3.13 can be found in their paper. Here, we give a different proof and we identify all the maximal and minimal forbidden subgraphs with respect to Property A.

Set $\mathcal{G}_2 = \{G_1(n), G_2(n), G_3(n, r, s, t), G_4(n) \mid n \geq 7, r \geq s \geq 0, t \geq 1\}$, whose graphs are depicted in Fig. 6.

**Lemma 3.12.** Let $G$ be a connected non-bipartite graph with $c(G) = 3$ and of order $n \geq 7$. Then $G$ has Property A if and only if $G \in \{G_3(n; r, s, 1), G_4(n) \mid r \geq s \geq 0\}$.
Proof. We now show the necessity. Since \( c(G) = 3 \), then \( G \) only contains triangles among all cycles. We first prove that \( G \) contains a single triangle. Indeed, assume for a contradiction that \( G \) contains (at least) two triangles \( \triangle_1 \) and \( \triangle_2 \). Clearly, these two triangles have no common edge, otherwise, \( c(G) = 4 \). Thus, \( \triangle_1 \) and \( \triangle_2 \) have at most one common vertex. If they have one common vertex, then \( H_{11}, H_{12} \subset G \) contradicting that \( H_{11} \) and \( H_{12} \) are maximal forbidden graphs (Corollary 3.11). If \( \triangle_1 \) and \( \triangle_2 \) have no common vertices, then either \( H_{13} \subset G \) or \( H_{15} \subset G \), which contradicts the fact that both \( H_{13} \) and \( H_{15} \) are maximal forbidden subgraphs (Corollary 3.11). Hence, \( G \) merely contains one triangle \( \triangle = \{v_1, v_2, v_3\} \).

Since \( F \) is a minimal forbidden graph, then at most two vertices in \( \triangle \) are adjacent to the vertices in \( V(G) \setminus \{v_1, v_2, v_3\} \).

If two vertices of \( \triangle \) are adjacent to vertices from \( V(G) \setminus \{v_1, v_2, v_3\} \), then \( H_{14} \) is a maximal forbidden subgraph (Corollary 3.11), then \( G = G_3(n; r, s, 1) \).

Assume next that just one vertex (say \( v_1 \)) of \( \triangle \) is adjacent to vertices from \( V(G) \setminus \{v_1, v_2, v_3\} \). If \( G' = G[V(G) \setminus \{v_1, v_2, v_3\}] \) consists of isolated vertices, then \( G' = G_3(n; r, 0, 1) \). If \( G' \) has at least one edge, then we show that \( G' \) is the star \( K_{1,n-4} \). Since \( G \) has only one triangle, then \( G' \) must be a forest. If one component of \( G' \) has diameter at least three, then \( H_{15} \subset G' \), a contradiction. So, each component of \( G' \) has diameter equal to one or two. If the diameter is equal to one, then \( G' = aK_1 \cup bP_2 \) \((a \geq 0, b \geq 1, a + 2b \geq 4, \text{ since } n(G) \geq 7)\) which implies \( H_{16} \subset G \), a contradiction. Hence, each component of \( G' \) has diameter two, and, consequently, \( G' \) is connected (otherwise, \( H_{16} \subset G \)). Therefore, \( G' = K_{1,n-4} \), as above claimed. Since \( H_{12} \) is a maximal forbidden subgraph, then \( v_1 \) can be adjacent just to the vertex of degree \( n - 4 \) in \( K_{1,4} \), which implies that \( G = G_4(n) \).

This completes the proof. \( \square \)

Lemma 3.13. Let \( G \) be a non-bipartite connected graph with \( c(G) = 4 \) and of order \( n \geq 7 \). Then \( G \) has Property A if and only if \( G \in \{G_1(n), G_2(n), G_3(n; r, s, t), |r \geq s \geq 0, t \geq 2\} \).

Proof. For the necessity, since \( c(G) = 4 \), we distinguish the cases according to whether one among \( K_4, K_4^- \) and \( C_4 \) appears as induced subgraph.

Case 1. \( K_4 \subset G \). Since \( F \) and \( C_3 \) are minimal forbidden subgraphs, then only one vertex in \( K_4 \) is adjacent to some vertex in \( V(G) \setminus V(K_4) \). From \( H_{11} \) and \( H_2 \) being maximal forbidden subgraphs (Corollary 3.11), we deduce that \( G = G_3(n) \).

Case 2. \( K_4^- \not\subset G \) and \( C_4 \subset G \). Label the vertices of \( K_4^- \) as in Fig. 6. Let \( \overline{N(u)} \) denote the set of vertices adjacent to \( u \) and \( N(u) = N(u) \cup \{u\} \). Note that \( 2 \leq |N(u_1) \cap N(u_2)| \leq n - 2 \).

If \( |N(u_1) \cap N(u_2)| > 2 \), then \( G[N(u_1) \cap N(u_2)] \) is the empty graph, namely \( tK_1 \), with \( t = n - |N(u_1) \cap N(u_2)| \), otherwise \( K_4 \) (or \( C_4 \)) appears as subgraph. If \( |N(u_1) \cap N(u_2)| = n - 2 \), then \( G = G_3(n; 0, 0, n - 2) \); if \( |N(u_1) \cap N(u_2)| < n - 2 \), then any vertex in \( N(u_1) \cap N(u_2) \) cannot be adjacent to some vertex in \( V(G) \setminus (\overline{N(u_1)} \cup \overline{N(u_2)}) \). For \( n(G) \geq 7 \), the other minimal forbidden subgraph \( F \) will appear. Hence, only \( u_1 \) and \( u_2 \) can be adjacent to vertices in \( V(G) \setminus (\overline{N(u_1)} \cup \overline{N(u_2)}) \). Finally, since \( H_3 \) and \( H_5 \) are maximal forbidden subgraphs (Corollary 3.11), then \( G = G_3(n; r, s, t) \) with \( r \geq s \geq 0, t \geq 3 \).

Now let \( |N(u_1) \cap N(u_2)| = 2 \). If \( G[V(G) \setminus V(K_4^-)] \) has at least one edge, from \( \mu(K_4^-) = 2 \), we get \( \mu(G) \geq 3 \) which implies \( \kappa_3(G) > 2 \) (Corollary 2.5), a contradiction. Consequently, \( G[V(G) \setminus V(K_4^-)] \) is an empty graph. In view of \( H_8 \) as maximal forbidden subgraph, we get \( G \in \{G_1(n), G_3(n; r, s, 2)\} \).

Case 3. \( K_4^- \not\subset G \) and \( K_4 \subset G \). Recall that \( G \) contains at least one triangle \( \Delta \) and \( n(G) \geq 7 \). If \( C_4 \) and \( \Delta \) have either a common vertex or none, then \( G \) will satisfy \( n(G) \geq 7 \) and \( \mu(G) \geq 3 \), and thus \( \kappa_3(G) > 2 \) (Corollary 2.5), a contradiction. If \( C_4 \) and \( \Delta \) have three vertices in common then \( K_4^- \) appears. Hence, \( C_4 \) and \( \Delta \) share exactly two vertices. If so, the minimal forbidden subgraph \( C_5 \) appears. Thereby, in this case no graph has Property A.

This completes the proof. \( \square \)

The following results follow from Theorems 3.8–3.10, Lemmas 3.12 and 3.13.

Theorem 3.14. Let \( G \) be a connected graph with order \( n \). Then \( G \) has Property A if and only if \( G \in \mathcal{B} \cup \mathcal{A} \cup \mathcal{F}_1 \cup \mathcal{F}_2 \), where \( \mathcal{B}, \mathcal{A}, \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are defined in Sections 3.2.1–3.2.3.

Theorem 3.15. There exist exactly 13 minimal forbidden subgraphs with \( \kappa_3 > 2 \) which are depicted in Figs. 2 and 4.

Theorem 3.16. There exist exactly 17 maximal forbidden subgraphs with Property A that are stated in Corollary 3.11.

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References