Non-dominance and attitudinal prioritisation methods for intuitionistic and interval-valued intuitionistic fuzzy preference relations

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\textbf{A R T I C L E   I N F O}

Keywords:
- Quantifier guided non-dominance degree
- Interval-valued intuitionistic preference relations
- Expected score function
- OWA operator
- Priority vector

\textbf{A B S T R A C T}

A novel intuitionistic fuzzy set (IFS) score function and an intuitionistic fuzzy preference relation (IFPR) quantifier guided non-dominance based prioritisation method are introduced. Based on Yager’s continuous OWA (COWA) operator, the interval-valued intuitionistic fuzzy COWA (IVIF-COWA) operator is defined, and a new attitudinal expected score function for interval-valued intuitionistic fuzzy numbers (IVIFNs) is introduced. The novelty of this attitudinal expected score function is that it allows the comparison of IVIFNs by taking into account of the decision makers’ attitudinal character. Moreover, we show that the new attitudinal expected score function extends: (i) the IFS score function introduced in this paper, which is mathematically equivalent to Chen and Tan’s score function (Chen and Tan, 1994); and (ii) Xu and Chen’s score function for IVIFNs (Xu and Chen, 2007). Using the proposed score functions, a method is developed to construct FPRs from a given IFPR and IVIFPR, respectively. When the hesitancy degree function is null, we prove that the score FPRs coincide with their respective IFPR and IVIFPR. Finally, a ranking sensitivity analysis of the attitudinal expected score function with respect to the attitudinal parameter is provided.

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1. Introduction

Since Atanassov (1986) introduced the concept of intuitionistic fuzzy set (IFS), later generalized by Atanassov and Gargov (1989) with the interval-valued intuitionistic fuzzy set (IVIFS), much work has been done to develop decision making models with them as the information representation format. For example, recent proposals to deal with the use of IFSs in decision making can be found in Li (2011b), Li, Chen, and Huang (2010), Wei (2010), Xu and Hu (2010), Ye (2010a), Yue (2011), while IVIFs have been investigated in Chen, Wang, and Lu (2011), Li (2011a), Park, Park, Kwun, and Tan (2011), Tan (2011), Wang, Li, and Zhang (2012, 2009), Xu (2010), Ye (2010), Zhang, Jiang, Jia, and Luo (2010).

A problem that needs to be addressed in this type of decision making environment is the ranking of intuitionistic fuzzy numbers (IFNs) and interval-valued intuitionistic fuzzy numbers (IVIFNs). Yager (2004) pointed out that this problem has been extensively studied for the case of FNs, and that there is no unique best approach. Recall that FNs are particular cases of IFNs, and these of IVIFNs. Thus, the same conclusion applies to these later type of numbers. It is therefore important to develop a methodology that best captures the decision maker’s preferences regarding the ranking of IFNs and IVIFNs (Yager, 2004).

A widely used approach to rank FNs is to convert them into a representative crisp value, and perform the comparison on them (Yager, 2004). This approach is also the common one used to rank IFNs and IVIFNs. Representative crisp values developed for IFNs and IVIFNs are known with the names of score degree and accuracy degree. Chen and Tan (1994) developed a score function for IFSs based on the membership and non-membership functions, which was later improved by Hong and Choi (2000) with the addition of an accuracy function. Other score and accuracy functions to rank IVIFNs have been proposed by Xu and Chen (2007). Later, Ye (2009) proposed a different accuracy function that he claimed solved some drawbacks associated to the accuracy function developed by Xu and Chen. However, both accuracy functions were proved to be equivalent when ordering IVIFNs (Wang, 2011), and therefore the drawbacks highlighted by Ye (2009) were not properly addressed. Indeed, in some cases, these proposals do not allow the proper discrimination between different IVIFNs. We believe that this is because they are straight forward extensions of their respective proposals for the case of IFNs, and therefore are not rich enough to capture all the information contained in IVIFNs. In this paper,
and to overcome this issue, we implement Yager’s approach to comparing FNs based on the continuous OWA (COWA) operator to the case of IVIFNs. By doing this, we develop a new score function that will ‘increase our capability for modelling users preferences’ (Yager, 2004) in comparing IVIFNs.

For IVIFNs, we note the mathematical equivalence between IFSs and interval-valued FSs (IVFSs) (Cornelis, Atanassov, & Kerre, 2003). Because IVFSs are special cases of interval type-2 FSs, we provide a link between the development of the type reduced set (TRS) of an interval type-2 FS and the score function of an IFS. This will allow us to define the mathematical equivalent to the score function proposed by Chen and Tan (1994). An advantage of the score function proposed in this paper is that it can be used to derive a fuzzy preference relation (FPR) from an intuitionistic FPR (IFPR), which we call the score FPR (SFPR), and that we further exploit to propose a quantifier guided non-dominance based prioritisation method for an IFPR. We also prove that when the hesitancy degree function is null the IFPR coincides with its SFPR.

For IVIFNs, the continuous OWA (COWA) operator introduced by Yager (2004) is extended to define the interval-valued intuitionistic fuzzy COWA (IVIF-COWA) operator, which is used to propose a new IVIFN attitudinal expected score function. The novelty of this attitudinal expected score function is that it allows the comparison of IVIFNs by taking into account the decision makers’ attitudinal character. Moreover, we will show that the new attitudinal expected score function extends: (i) the new IFS score function introduced in this paper; and (ii) Xu and Chen’s score function for IVIFNs. The attitudinal expected score functions are used to derive a FPR given an interval-valued IFS score function, which is mathematically equivalent to the score function proposed by Chen and Tan (1994). An advantage of the expected score function for IVIFNs that implements the decision maker’s requirement when evaluated using an intuitionistic fuzzy number (IFN) (Chen, 2010; Yang & Chiclana, 2011). For interval type-2 FS, Nie and Tan (2003). Because IVFSs are special cases of interval type-2 FSs, and interval-valued FSs (IVFSs) (Cornelis, Atanassov, & Kerre, 2003), we note the mathematical equivalence between IFSs and IVFSs by Yager (2004) is extended to define the interval-valued intuitionistic fuzzy number (IFN) (Chen, 2010; Yang & Chiclana, 2009), and therefore an IFS can be seen as a collection of IFNs. Interval-valued FSs are special cases of type-2 FSSs. Indeed, making the interval type-2 FS over X with lower membership function (LMF) $\mu_A(x)$ and upper membership function (UMF) $\mu_A(x) + \tau_A(x)$ (Greenfield, Chiclana, Coupland, & John, 2008). The Type Reduced Set (TRS) of a type-2 FS plays an important role in the final stage of any type-2 fuzzy decision making problem (Greenfield, Chiclana, Coupland, & John, 2012). The derivation of the TRS of a type-2 FS is a challenging problem and much research has been done in this area recently (Chiclana & Zhou, 2011). For interval type-2 FS, Nie and Tan (2008) propose a computational simple, efficient, approximate method to obtain the TRS for interval type-2 FSs, which involves taking the mean of their LMF and UMF. Experimental evidence (Greenfield & Chiclana, 2011; Greenfield, Chiclana, & John, 2009) strongly suggests that as the domain discretisation is made finer, the Nie-Tan method produces a set that defuzzifies in the same value than that of the TRS. Based on this evidence, we propose a new definition of the score function of an IFS.

Obviously, an IFS becomes a FS when $\mu_A(x) = 1 - \nu_A(x) \forall x \in X$. However, when there exists at least a value $x \in X$ such that $\mu_A(x) < 1 - \nu_A(x)$ an extra parameter has to be taken into account when working with IFSs: the hesitancy degree $\tau_A(x)$ of $x$ to $A$ (Monteiro, Gomez, & Bustince, 2007)

$$\tau_A(x) = 1 - \mu_A(x) - \nu_A(x)$$

(2)

The hesitancy degree $\tau_A(x)$ is an indicator of the hesitation margin of the membership of element $x$ to the IFS $A$. It represents the amount of lacking information in determining the membership of $x$ to $A$ (Burillo & Bustince, 1996; Yang & Chiclana, 2012). If the hesitation degree is zero, the reciprocal relation between membership and non-membership makes the latter one unnecessary in the formulation as it can be derived from the former.

### 2. Non-dominance prioritisation method for IFPRs

Intuitionistic fuzzy sets (IFSs) were introduced by Atanassov (1986).

**Definition 1 (Intuitionistic fuzzy set (IFS)).** An intuitionistic fuzzy set (IFS) $A$ over a universe of discourse $X$ is given by

$$A = \{ (x, \mu_A(x), \nu_A(x)) | x \in X \}$$

where

$$\mu_A : X \rightarrow [0,1], \nu_A : X \rightarrow [0,1]$$

and

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \forall x \in X.$$  

(1)

For each $x$, the numbers $\mu_A(x)$ and $\nu_A(x)$ are the degree of membership and degree of non-membership of $x$ to $A$ respectively.

Note that given an IFS $A$, it is always true that $\mu_A(x) = 1 - \nu_A(x) + \tau_A(x)$. Since IFSs and interval-valued FSs are mathematically equivalent (Deschrijver & Kerre, 2003; Dubois, Gottwald, Hajek, Kaczprzyk, & Prade, 2005), we can call $[\mu_A(x), \mu_A(x) + \tau_A(x)]$ an intuitionistic fuzzy number (IFN) (Chen, 2010; Yang & Chiclana, 2009), and therefore an IFS can be seen as a collection of IFNs. Interval-valued FSs are special cases of type-2 FSSs (Mendel, 2001). Indeed, $[[\mu_A(x), \mu_A(x) + \tau_A(x)] | x \in X]$ is an interval type-2 FS over $X$ with lower membership function (LMF) $\mu_A(x)$ and upper membership function (UMF) $\mu_A(x) + \tau_A(x)$ (Greenfield, Chiclana, Coupland, & John, 2008). The Type Reduced Set (TRS) of a type-2 FS plays an important role in the final stage of any type-2 fuzzy decision making problem (Greenfield, Chiclana, Coupland, & John, 2012). The derivation of the TRS of a type-2 FS is a challenging problem and much research has been done in this area recently (Chiclana & Zhou, 2011). For interval type-2 FS, Nie and Tan (2008) propose a computational simple, efficient, approximate method to obtain the TRS for interval type-2 FSs, which involves taking the mean of their LMF and UMF. Experimental evidence (Greenfield & Chiclana, 2011; Greenfield, Chiclana, & John, 2009) strongly suggests that as the domain discretisation is made finer, the Nie-Tan method produces a set that defuzzifies in the same value than that of the TRS. Based on this evidence, we propose a new definition of the score function of an IFS.
Definition 3 (IFS score function). Given an IFS $A$ over $X$, with IFNs $A(x) = [\mu_A(x), \nu_A(x) + \tau_A(x)]$, $x \in X$, the following score function can be defined

$$S_{SWC}(A): X \to [0,1]$$

$$S_{SWC}(A)(x) = \frac{\mu_A(x) + \tau_A(x)}{2}$$

(5)

Note that this new score function can be re-written as follows:

$$S_{SWC}(A)(x) = \frac{\mu_A(x) - \nu_A(x) + 1}{2}$$

(6)

This means that $S_{SWC}(A)$ and $S_{SF}(A)$ are ordering mathematically equivalent in that they will lead to the same ordering of alternatives when the ordering rule (4) is applied. However, because of its range, we will see that score function $S_{SWC}(A)$ will allow the derivation of a fuzzy preference relation (FPR) from an intuitionistic fuzzy preference relation (IFPR), that we will call the score FPR (SFPR). As mentioned above, when the hesitancy degree is equal to the null function, the IFS becomes a FS, and therefore an IFPR reduces to a FPR, which is equal to the SFPR as we will show in the next subsection.

2.2. Score FPR associated to an IFPR

Given three alternatives $x_i$, $x_j$, $x_k$ such that $x_i$ is preferred to $x_j$ and $x_j$ to $x_k$, the question whether the ‘degree or strength of preference’ of $x_i$ over $x_j$ exceeds, equals, or is less than the ‘degree or strength of preference’ of $x_j$ over $x_k$ cannot be answered by the classical preference modelling (Chiclana, Herrera-Viedma, Alonso, & Herrera, 2009). The introduction of the concept of fuzzy set as an extension of the classical concept of set when applied to a binary relation leads to the concept of a fuzzy relation. The definition of a FPR is the following one (Bezdek, Spillman, & Spillman, 1978; Nurmi, 1981):

Definition 4 (Fuzzy preference relation (FPR)). A fuzzy preference relation (FPR) $P$ on a finite set of alternatives $X = \{x_1, \ldots, x_n\}$ is characterized by a membership function $\mu_\nu: X \times X \to [0,1]$, $\mu_\nu(x_i,x_j) = p_{ij}$, verifying

$$p_{ij} + p_{ji} = 1 \quad \forall i,j \in \{1, \ldots, n\}.$$  

A FPR may be conveniently denoted by the matrix $P = (p_{ij})$. The following interpretation is also usually assumed:

- $p_{ij} = 1$ indicates the maximum degree of preference for $x_i$ over $x_j$.
- $p_{ij} \in [0,1]$ indicates a definite preference for $x_i$ over $x_j$.
- $p_{ij} = 0.5$ indicates indifference between $x_i$ and $x_j$.

We note that a FPR as above is also known as reciprocal ([0,1]-valued) preference relation. Szmidt and Kacprzyk (2002) defined the intuitionistic FPR (IFPR) as a generalisation of the concept of FPR. The adapted definition of an IFPR is the following one:

Definition 5 (Intuitionistic fuzzy preference relation (IFPR)). An intuitionistic fuzzy preference relation (IFPR) $B$ on a finite set of alternatives $X = \{x_1, \ldots, x_n\}$ is characterized by a membership function $\mu_\nu: X \times X \to [0,1]$ and a non-membership function $\nu_\nu: X \times X \to [0,1]$ such that

$$0 \leq \mu_{ij}(x_i, x_j) + \nu_{ij}(x_i, x_j) \leq 1 \quad \forall (x_i, x_j) \in X \times X$$

(7)

The value $\mu_{ij}(x_i, x_j) = \mu_{ij}$ can be interpreted as the certainty degree up to which $x_i$ is preferred to $x_j$, while the value $\nu_{ij}(x_i, x_j) = \nu_{ij}$ represents the certainty degree up to which $x_i$ is non-preferred to $x_j$. Additionally, the following conditions are imposed:

- $\mu_{ii} = \nu_{ii} = 0.5 \quad \forall i \in \{1, \ldots, n\}$.
- $\mu_{ji} = \nu_{ij} \quad \forall i,j \in \{1, \ldots, n\}$.

The IFPR can be conveniently represented by a matrix $B = (b_{ij})$ with $b_{ij} = (\mu_{ij}, \nu_{ij})$. When the hesitancy degree function is the null function we have that $\mu_{ij} + \nu_{ij} = 1$ (\forall i,j), and therefore the IFPR $B = (b_{ij})$ is mathematically equivalent to the FPR $(\mu_{ij})$, i.e. $B = (\mu_{ij})$. In the following, we will provide a method to derive a FPR from an IFPR $B = (b_{ij})$ via the application of the score function $S_{SWC(A)} (5)$, which we call the score FPR (SFPR). We will also prove that this method in consistent with the case when the IFPR reduces to be a FPR, in which case the SFPR coincides with the original FPR.

Theorem 1 (Score FPR (SFPR)). Let $B = (b_{ij})$ be an IFPR. Then $P = (p_{ij})$ where

$$p_{ij} = S_{SWC}(b_{ij})$$

is a FPR. $P$ is called the Score FPR (SFPR) associated to the IFPR $B$.

Proof. We have that

$$p_{ij} + p_{ji} = S_{SWC}(b_{ij}) + S_{SWC}(b_{ji}) = \frac{\mu_{ij} - \nu_{ij} + 1}{2} + \frac{\mu_{ji} - \nu_{ji} + 1}{2}$$

For being $B = (b_{ij})$ an IFPR we have that $\mu_{ij} = \nu_{ji}$ and $\nu_{ij} = \mu_{ji}$, and therefore it is true that

$$p_{ij} + p_{ji} = 1 \quad \forall i,j \in \{1, \ldots, n\} \quad \Box$$

Corollary 1. Let $B = (b_{ij})$ be an IFPR and $P = (p_{ij})$ its associated SFPR. If the hesitancy degree function is the null function then $B = P$.

The above results can be exploited to define concepts for an IFPR via the equivalent known ones in the associated SFPR. In particular, we propose a methodology to derive a priority vector for an IFPR via its corresponding SFPR based on the concept of non-dominance degree introduced by Orlovsky (1978).

2.3. Intuitionistic quantifier guided non-dominance degree

Yager (1996) presented a methodology to formulate linguistic expressions using ordered weighted average (OWA) operators guided by linguistic quantifiers (Zadeh (1983)). Specifically, the linguistic quantifier desired to be implemented is represented mathematically by a basic unit-monotonic (BUM) function $Q$: $[0,1] \to [0,1]$ such that $Q(0) = 0$, $Q(1) = 1$ and $Q(x) \geq Q(y)$ if $x \geq y$, which is used compute the OWA operator weights as follows:

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right), \quad i = 1, \ldots, n$$

Non-decreasing relative linguistic quantifiers have been modelled in the literature with the following BMU function $Q$

$$Q(x) = \begin{cases} 
0 & 0 \leq x < a \\
\frac{x-a}{b-a} & a \leq x \leq b \\
1 & b \leq x \leq 1 
\end{cases}$$

$a,b \in [0,1]$. The election of adequate values for the parameters $a$ and $b$ could lead to the implementation of a suitable linguistic quantifier. The linguistic quantifier ‘most of’ can be implemented by choosing the values $(a,b) = (0.3,0.8)$. Based on this methodology, a quantifier guided non-dominance degree that extends Orlovsky’s non-dominance concept for a FPR was presented by Chiclana, Herrera, and Herrera-Viedma (1998). Here we extend the concept of non-dominance to the case of an IFPR.
Let $X = \{x_1, \ldots, x_n\}$ be a set of alternatives evaluated by a decision maker against a particular criterion using an IFPR $B = (b_j)$, and $Q$ a BUM function. The intuitionistic quantifier guided non-dominance degree associated to the alternative $x_i$, IQGNDD, is defined as follows:

$$IQGNDD_i = \psi_Q\left(1 - p_q^i\right) \tag{8}$$

with $p_q^i = \max(p_{q_1}^i - p_{q_2}^i, 0)$, $p_{q_2} = S_{\text{BIF}}(b_2)$ and $\psi_Q$ is an OWA operator guided by the linguistic quantifier represented by the BUM function $Q$.

**Example 1.** Let

$$B = \begin{pmatrix}
0.50 & 0.05 & 0.02 & 0.01 \\
0.20 & 0.05 & 0.02 & 0.01 \\
0.10 & 0.05 & 0.02 & 0.01 \\
0.10 & 0.05 & 0.02 & 0.01
\end{pmatrix}$$

be an IFPR. The associated SFPR is

$$W = \begin{pmatrix}
0.50 & 0.60 & 0.60 & 0.80 & 0.65 \\
0.40 & 0.50 & 0.70 & 0.65 & 0.35 \\
0.40 & 0.30 & 0.50 & 0.70 & 0.45 \\
0.20 & 0.35 & 0.65 & 0.50 & 0.25 \\
0.35 & 0.65 & 0.55 & 0.75 & 0.50
\end{pmatrix}$$

Using the above linguistic quantifier 'most of', the OWA operator weighting vector is

$$W = (0.0, 0.0, 0.4, 0.5, 0.1)^T$$

The intuitionistic quantifier guided non-dominance degree associated to each one of the alternatives are

$$IQGNDD_1 = 1.00, \quad IQGNDD_2 = 0.87, \quad IQGNDD_3 = 0.82, \quad IQGNDD_4 = 0.57, \quad IQGNDD_5 = 0.97$$

The alternatives can be ranked from best to worst according to the degree up to which an alternative is not dominated by 'most of' the rest of alternatives:

$$x_1 > x_3 > x_2 > x_5 > x_4$$

### 3. Attitudinal based expected score function of an IVIFS

Atanassov and Gargov (1989) introduced the notion of interval-valued intuitionistic fuzzy set (IVIFS), which is characterized by a membership function and a non-membership function that take integer numbers rather than crisp numbers.

**Definition 6** (Interval-valued IFS (IVIFS)). Let $\text{INT}(0,1]$ be the set of all closed subintervals of the unit interval and $X$ be a universe of discourse. An interval-valued IFS (IVIFS) $\tilde{A}$ over $X$ is given as:

$$\tilde{A} = \left\{ (x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x))^T | x \in X \right\} \tag{9}$$

where $\mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) \in \text{INT}(0,1]$, and

$$0 \leq \text{sup}_{A} \mu_{\tilde{A}}(x) + \text{sup}_{A} \nu_{\tilde{A}}(x) \leq 1, \quad \forall x \in X$$

Denoting by $\bar{\mu}_{\tilde{A}}(x), \bar{\nu}_{\tilde{A}}(x)$ and $\tilde{\mu}_{\tilde{A}}(x)$ the lower and upper end points of $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$, respectively, an IVIFS can be represented as

$$\tilde{A} = \left\{ (x, \bar{\mu}_{\tilde{A}}(x), \bar{\nu}_{\tilde{A}}(x), \tilde{\mu}_{\tilde{A}}(x), \tilde{\nu}_{\tilde{A}}(x))^T | x \in X : 0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1, \mu_{\tilde{A}}(x) \wedge \nu_{\tilde{A}}(x) \geq 0 \right\} \tag{10}$$

The hesitancy function of an IVIFS is:

$$\pi_{\tilde{A}}(x) = \left[ 1 - \bar{\mu}_{\tilde{A}}(x) - \bar{\nu}_{\tilde{A}}(x), 1 - \bar{\mu}_{\tilde{A}}(x) - \bar{\nu}_{\tilde{A}}(x) \right] \tag{11}$$

Given $x \in X$,

$$\left[ \bar{\mu}_{\tilde{A}}(x), \bar{\nu}_{\tilde{A}}(x) \right], \left[ \bar{\nu}_{\tilde{A}}(x), \bar{\nu}_{\tilde{A}}(x) \right]$$

will be referred to as an interval-valued intuitionistic fuzzy number (IVIFN). For convenience, an IVIFN will be denoted by $([\mu^-, \mu^+], [\nu^-, \nu^+])$.

Xu and Chen (2007) proposed the following score and accuracy degrees associated to an IVIFN:

**Definition 7** (Xu and Chen (2007) IVIFN score and accuracy degrees). Given an IVIFN $\tilde{A} = ([\mu^-, \mu^+], [\nu^-, \nu^+])$, its score and accuracy degrees can be represented, respectively, by

$$S_{\text{AC}}(\tilde{A}) = \frac{\mu^- + \mu^- + \nu^- + \nu^-}{2} \tag{12}$$

and

$$A_{\text{AC}}(\tilde{A}) = \frac{\mu^- + \mu^- + \nu^- + \nu^-}{2} - 1 \tag{13}$$

Note that $S_{\text{AC}}(\tilde{A}) \in [-1, 1]$, while $H_{\text{AC}}(\tilde{A}) \in [0, 1]$. The score and accuracy values are used by proposing n IVIFNs two level ranking method. In the first level, the score value is used to rank the IVIFNs as per rule (4). The second level is applied when two IVIFNs have same score values, in which case the accuracy values are used to discern which IVIFNs is lower. Two IVIFNs are considered equivalent in term of ordering when both have same score and accuracy values.

Ye (2009) proposed a different expression for the accuracy degree of an IVIFN that has the same range of values, $[-1, 1]$, than the score degree defined above:

**Definition 8** (Ye (2009) Accuracy degree). Given an IVIFN $\tilde{A} = ([\mu^-, \mu^+], [\nu^-, \nu^+])$, its degree of accuracy can be represented by

$$\widetilde{A}_{\text{AV}}(\tilde{A}) = \mu^- + \mu^- + \nu^- + \nu^- - 1 \tag{14}$$

However, both accuracy values are equivalent when ordering IVIFNs as proved by Wang (2011). Moreover, as the following example illustrates, the above score and accuracy values are unable to discriminate between all pairs of IVIFNs in terms of ranking.

**Example 2.** The following two IVIFNs

$$\tilde{A}_1 = ([0.2,0.5], [0.1,0.45]) \quad \text{and} \quad \tilde{A}_2 = ([0.25,0.45], [0.2,0.35])$$

have the same score value $S_{\text{AC}}(\tilde{A}_1) = S_{\text{AC}}(\tilde{A}_2) = 0.075$, and therefore the accuracy value is to be used to rank them. However, in this case we have $H_{\text{AC}}(\tilde{A}_1) = H_{\text{AC}}(\tilde{A}_2) = 0.625$. From (14), we get $A_{\text{AV}}(\tilde{A}_1) = A_{\text{AV}}(\tilde{A}_2) = -0.025$.

Yager (2004) points that the comparison of FNs, which is essential in decision making under uncertainty, has been widely investigated and it is now clear that a unique best approach does not exist. Although we agree with this claim, it is our aim to improve the above proposals to ranking IVIFNs. In the following, and to overcome the highlighted shortcoming of Example 2, we propose a novel score function for IVIFNs that extends the score function (5), which takes into account the decision maker's attitude via the application of the concept of attitudinal character of a BUM and the continuous ordered weighted average (COWA) operator introduced by Yager (2004). We call this the IVIFN attitudinal expected...
score function, and we will prove that a particular case of this new score function is equivalent to Xu and Chan’s score degree (12), in that they both produce the same ranking of IVIFNs.

In the following, we will elaborate the definition of the interval-valued intuitionistic fuzzy COWA (IVIF-COWA) operator, which is fundamental in the definition of the IVIFN attitudinal expected score function:

**Definition 9 (BUM attitudinal character).** The attitudinal character of a BUM function, \( Q \), is

\[
AC(Q) = \int_{0}^{1} Q(y) dy
\]

**Definition 10 (COWA operator).** Let \( INT(\mathbb{R}^+) \) be the set of all closed subintervals of \( \mathbb{R}^+ \). A continuous ordered weighted average (COWA) operator is a mapping \( F_Q : INT(\mathbb{R}^+) \rightarrow \mathbb{R}^+ \) which has an associated BUM function, \( Q \), such that

\[
F_Q([a, b]) = \int_{0}^{1} dQ(y)(b - y(b - a))dy
\]

Denoting \( \lambda = AC(Q) \) we have

\[
F_Q([a, b]) = (1 - \lambda) \cdot a + \lambda \cdot b
\]

where \( \lambda \) is the attitudinal character of the BUM function \( Q \). Thus, \( F_Q([a, b]) \) is the weighted average of the end points of the closed interval with attitudinal character parameter, and it is known as the attitudinal expected value of \([a, b]\).

In the following, we extend the COWA operator to the case in which our argument is an IVIFN and develop the interval-valued intuitionistic fuzzy COWA (IVIF-COWA) operator.

**Definition 11 (IVIF-COWA operator).** Let \( F_Q \) be a COWA operator with associated BUM function \( Q \). An interval-valued intuitionistic fuzzy COWA (IVIF-COWA) operator is a mapping \( F_Q : INT(\mathbb{R}^+) \times INT(\mathbb{R}^+) \rightarrow \mathbb{R}^+ \times \mathbb{R}^+ \) such that

\[
\tilde{F}_Q([a, b], [c, d]) = (F_Q([a, b]), F_Q([c, d]))
\]

Given an IVIFN \( \tilde{A} = ([\mu_1, \mu_2], [\nu_1, \nu_2]) \), then we have

\[
\tilde{F}_Q(\tilde{A}) = (F_Q([\mu_1, \mu_2]), F_Q([\nu_1, \nu_2]))
\]

\[
= ((1 - \lambda) \cdot \mu + \lambda \cdot \mu^+, (1 - \lambda) \cdot \nu + \lambda \cdot \nu^+)
\]

This allows us to propose the following score function of an IVIFN:

**Definition 12 (IVIFN attitudinal expected score function).** The attitudinal expected score degree of an IVIFN \( \tilde{A} = ([\mu, \mu^+], [\nu, \nu^+]) \) is

\[
\tilde{S}_{\text{WC}}(\tilde{A}) = \frac{(1 - \lambda) \cdot (\mu - \nu^+) + \lambda \cdot (\mu^+ - \nu^+)}{2} + \frac{1}{2} = S_{\text{WC}}(\tilde{A})
\]

**Proposition 1.** The IVIFN attitudinal expected score function \( \tilde{S}_{\text{WC}} \) generalises the IFN score function \( S_{\text{WC}} \).

**Proof.** We need to prove that expression (20) when the input value is an IFN yields the same result than the one obtained applying expression (5). An IFN \( \tilde{A} = ([\mu, \nu]) \) is a particular case of IVIFN for which \( \mu^+ = \mu^- = \mu \) and \( \nu^+ = \nu^- = \nu \) and therefore we have

\[
\tilde{S}_{\text{WC}}(\tilde{A}) = \frac{(1 - \lambda) \cdot (\mu - \nu^+) + \lambda \cdot (\mu^+ - \nu^+)}{2} + \frac{1}{2} = S_{\text{WC}}(\tilde{A})
\]

**Proposition 2.** The IVIFN score function \( \tilde{S}_{\text{WC}} \) and the IVIFN attitudinal expected score function \( S_{\text{WC}} \), with attitudinal are related as follows

\[
\tilde{S}_{\text{WC}}(\tilde{A}) = \frac{S_{\text{WC}}(\tilde{A}) + 1}{2}
\]

and therefore are equivalent in the ordering IVIFNs.

**Proof.** When \( \lambda = 0.5 \) the expression of \( \tilde{S}_{\text{WC}} \) reduces to

\[
\tilde{S}_{\text{WC}}(\tilde{A}) = \frac{0.5 \cdot (\mu - \nu^+) + 0.5 \cdot (\mu^+ - \nu^+)}{2} + \frac{1}{2}
\]

\[
= \frac{\mu - \nu^+ + \mu^+ - \nu^+ + 1}{4}
\]

Thus it is true that

\[
\tilde{S}_{\text{WC}}(\tilde{A}) = \frac{S_{\text{WC}}(\tilde{A}) + 1}{2}
\]

and it is obvious that both \( \tilde{S}_{\text{WC}} \) and \( S_{\text{WC}} \) produce the same ranking of IVIFNs.

**Theorem 2.** Let \( \tilde{a}_1 = ([\mu_1, \mu_2], \nu_1, \nu_2) \) and \( \tilde{a}_2 = ([\mu_2, \mu_3], \nu_2, \nu_3) \) be two IVIFNs with attitudinal expected score values \( \tilde{S}_{\text{WC}}(\tilde{a}_1) \) and \( \tilde{S}_{\text{WC}}(\tilde{a}_2) \), respectively. If

\[
\mu_1 - \nu_1 \leq \mu_2 - \nu_2 \text{ and } \mu_2 - \nu_2 \leq \mu_3 - \nu_3 \text{ then } \tilde{S}_{\text{WC}}(\tilde{a}_1) \leq \tilde{S}_{\text{WC}}(\tilde{a}_2)
\]

The attitudinal expected score function can be used to define an ordering relation on the set of IVIFNs. However, this ordering is much dependent on the BUM function used to define the attitudinal expected score function. Indeed, a change on the value of \( \lambda \) could result in a different ordering of two IVIFNs.

**Example 3** (Example 2 continuation). Recall that the two IVIFNs \( \tilde{a}_1 = ([0.2, 0.5], [0.1, 0.45]) \) and \( \tilde{a}_2 = ([0.25, 0.45], [0.2, 0.35]) \) had equal score degrees as per expression (12) and equal accuracy degrees as per expressions (13) and (14). Expression (21) implies that both IVIFNs have the same attitudinal expected score value when \( \lambda = 0.5 \), and therefore in this case we have \( \tilde{a}_1 \sim \tilde{a}_2 \). However, this is not the case for different attitudinal values. Indeed, their attitudinal expected score value as per expression (20) are

\[
\tilde{S}_{\text{WC}}(\tilde{a}_1) = \frac{1.10 - 0.5 \cdot \lambda}{2} \quad \text{and} \quad \tilde{S}_{\text{WC}}(\tilde{a}_2) = \frac{1.05 + 0.5 \cdot \lambda}{2}
\]

respectively. Their ranking depends on the decision maker’s attitudinal character as follows:

1. \( \tilde{a}_1 < \tilde{a}_2 \) if and only if \( \lambda > 0.5 \)
2. \( \tilde{a}_1 \sim \tilde{a}_2 \) if and only if \( \lambda = 0.5 \)
3. \( \tilde{a}_1 > \tilde{a}_2 \) if and only if \( \lambda < 0.5 \)

In the following we will provide a sensitivity analysis of the attitudinal expected score function with respect to the attitudinal character \( \lambda \), i.e., we will provide the conditions under which the ordering of two IVIFNs is not affected by a change in the attitudinal parameter.

Let \( \lambda \) be the attitudinal parameter associated to BUM function \( Q \) under which it has been established that \( \tilde{S}_{\text{WC}}(\tilde{a}_1) \leq \tilde{S}_{\text{WC}}(\tilde{a}_2) \).
Assume that the attitudinal parameter is perturbed by a quantity \( \Delta \alpha \) and denoted by \( \tilde{S}_{\text{IVIFPR}}(\tilde{x}) \) and \( \tilde{S}_{\text{SWC}}(\tilde{x}) \) the new interval-valued expected score degrees. The question to answer is: what are the conditions \( \Delta \alpha \) needs to verify so that \( \tilde{S}_{\text{IVIFPR}}(\tilde{x}) \leq \tilde{S}_{\text{SWC}}(\tilde{x}) \), i.e. the ranking of IVIFNs does not change? The following theorem provides the answer to this question.

**Theorem 3.** Let \( \tilde{x}_1 = \left( \left[ \mu_1^-, \mu_1^+ \right], \left[ v_1^-, v_1^+ \right] \right) \) and \( \tilde{x}_2 = \left( \left[ \mu_2^-, \mu_2^+ \right], \left[ v_2^-, v_2^+ \right] \right) \) be two IVIFNs with expected scores such that \( \tilde{S}_{\text{SWC}}(\tilde{x}_1) \leq \tilde{S}_{\text{SWC}}(\tilde{x}_2) \). Let \( \Delta \alpha \) be a perturbation of the attitudinal character \( \lambda \) with \( 0 \leq \Delta \alpha \leq 1 \). Then we have

\[
\tilde{S}_{\text{IVIFPR}}(\tilde{x}_1) \leq \tilde{S}_{\text{IVIFPR}}(\tilde{x}_2) \iff \Delta \alpha \leq 1 - \lambda,
\]

where \( \lambda = \left( \mu_2^+ - v_1^- \right) - \left( \mu_1^+ - v_2^- \right) \) and \( \tilde{S}_{\text{IVIFPR}}(\tilde{x}_1) \leq \tilde{S}_{\text{IVIFPR}}(\tilde{x}_2) \)

**Proof.** Firstly, we note that \( \Delta \alpha \) is subject to the following constraint:

\[
-\lambda \leq \Delta \alpha \leq 1 - \lambda
\]

We have the following relation between \( \tilde{S}_{\text{SWC}}(\tilde{x}_1) \) and \( \tilde{S}_{\text{SWC}}(\tilde{x}_2) \):

\[
\tilde{S}_{\text{SWC}}(\tilde{x}_1) = \tilde{S}_{\text{SWC}}(\tilde{x}_2) + \frac{\Delta \alpha - \beta_1}{2}
\]

where \( \beta_1 = \left( \mu_2^+ - v_1^- \right) - \left( \mu_1^+ - v_2^- \right) \). The following equivalence holds:

\[
\tilde{S}_{\text{SWC}}(\tilde{x}_1) \leq \tilde{S}_{\text{SWC}}(\tilde{x}_2) \iff \Delta \alpha \leq \beta_1
\]

Three scenarios are possible:

- \( \beta_1 = \beta_2 \). Because \( \tilde{S}_{\text{SWC}}(\tilde{x}_1) \leq \tilde{S}_{\text{SWC}}(\tilde{x}_2) \) then (23) is true for any value of \( \Delta \alpha \), i.e.

\[
-\lambda \leq \Delta \alpha \leq 1 - \lambda
\]

- \( \beta_1 > \beta_2 \). Therefore:

\[
-\lambda \leq \Delta \alpha \leq \frac{\tilde{S}_{\text{SWC}}(\tilde{x}_2) - \tilde{S}_{\text{SWC}}(\tilde{x}_1)}{\beta_2 - \beta_1}
\]

- \( \beta_1 < \beta_2 \). Therefore:

\[
\frac{2 \left( \tilde{S}_{\text{SWC}}(\tilde{x}_2) - \tilde{S}_{\text{SWC}}(\tilde{x}_1) \right)}{\beta_2 - \beta_1} \leq \Delta \alpha \leq 1 - \lambda
\]

4. Prioritisation method for IVIFPR

Xu and Chen (2007) introduced the concept of interval-valued IFPR (IVIFPR) as follows:

**Definition 13.** IVIFPR An interval-valued IFPR (IVIFPR) \( \tilde{R} \) on a finite set of alternatives \( X = \{x_1, x_2, \ldots, x_n\} \) is an IVIFS over the set \( X \times X \), which is represented by a matrix \( \tilde{R} = (\tilde{R}_{ij}) \) where \( \tilde{R}_{ij} = (\mu_{ij}(x_i, x_j), v_{ij}(x_i, x_j)) \). Let \( \sum_{j=1}^{n} \mu_{ij}(x_i, x_j) = \mu_i \) and \( \sum_{j=1}^{n} v_{ij}(x_i, x_j) = v_i \) for each alternative \( x_i \). Let \( \tilde{x} = \left( \mu_i, v_i \right) \) be an IVIFN verifying the following conditions:

- \( \mu_i = \mu_{ij} \) if \( i = j \), \( \mu_i \in [0.5, 0.5] \) \( \forall i \in \{1, \ldots, n\} \)
- \( \mu_i = \mu_{ij} \) if \( i \neq j \), \( \mu_i \in [0.5, 0.5] \) \( \forall i \in \{1, \ldots, n\} \)

Let \( \tilde{R} = (\tilde{R}_{ij}) \) be an IVIFPR associated to the IVIFPR \( \tilde{R} \). The expected preference degree of the alternatives can be used as priority values to produce an ordering of the alternatives, as the following example illustrates.

**Example 4.** Suppose an expert provides the following IVIFPR on a set of four alternatives \( X = \{x_1, x_2, x_3, x_4\} \):

\[
\begin{array}{cccc}
0.5000 & 0.4875 & 0.5250 & 0.6125 \\
0.5125 & 0.5000 & 0.6000 & 0.6000 \\
0.4750 & 0.4000 & 0.5000 & 0.5500 \\
0.3875 & 0.4000 & 0.4500 & 0.5000 \\
\end{array}
\]

Applying expression (24) we obtain

\[
\begin{align*}
\tilde{p} &= 0.53, \\
\tilde{p}_2 &= 0.55, \\
\tilde{p}_3 &= 0.48, \\
\tilde{p}_4 &= 0.43
\end{align*}
\]
which results in the following ordering

\[ x_2 > x_1 > x_3 > x_4 \]

The ordering of alternatives obviously depends on the actual BUM function used to represent the decision maker's attitudinal character.

**Example 5** (Example 4 continuation). Applying expression (24) we obtain

\[ p^*_1 = \frac{42 + 2 \cdot \lambda}{80}, \quad p^*_2 = \frac{45 - 3 \cdot \lambda}{80}, \quad p^*_3 = \frac{38}{80}, \quad p^*_4 = \frac{35 + \lambda}{80} \]

It is easy to see that

\[ p^*_4 < p^*_3 < p^*_2 \quad \forall \lambda \]

and

\[ p^*_1 < p^*_2 \quad \text{if} \quad 0 \leq \lambda < 0.6 \]
\[ p^*_1 = p^*_2 \quad \text{if} \quad \lambda = 0.6 \]
\[ p^*_1 > p^*_2 \quad \text{if} \quad 0.6 < \lambda \leq 1 \]

This means that the following orderings are possible

\[ x_2 > x_1 > x_3 > x_4 \quad \text{if} \quad 0 \leq \lambda < 0.6 \]
\[ x_2 \sim x_1 > x_3 > x_4 \quad \text{if} \quad \lambda = 0.6 \]
\[ x_1 > x_2 > x_3 > x_4 \quad \text{if} \quad 0.6 < \lambda \leq 1 \]

We can see that a value of \( \lambda \) is susceptible to produce a change in the ordering of the alternatives when they are increased or decreased by a value sufficiently large as to make it lower or higher than 0.6. It can be seen that an optimistic decision maker will tend to select alternative \( x_1 \) while a pessimistic one will choose alternative \( x_2 \). Hence, our approach can rank the alternatives according to decision maker's attitudinal character, which might be useful in certain decision making context in which such type of information is provided.

In the following we will provide a sensitivity analysis of the attitudinal expected score function with respect to the attitudinal character \( \lambda \), i.e., we will provide the conditions under which the ordering of the expected preference degree of two alternative is not affected by a change in the attitudinal parameter.

**Theorem 5.** Let \( \tilde{R} = (\tilde{r}_{ij}) \) be an IVIFPR and \( P^i = (p^i_{ij}) \) be its associated ASPRF such that

\[ p^i_j \leq p^i_j \]

Let \( \Delta \lambda \) be perturbation of the attitudinal character \( \lambda \) with \( 0 \leq \lambda + \Delta \lambda \leq 1 \). Then we have

\[ p^{i+\Delta \lambda} \preceq p^{i+\Delta \lambda} \quad \text{implies} \quad \max \left\{ -\Delta, \frac{2n}{2n-1} \frac{p^{i+\Delta \lambda} - p^i}{\lambda - \lambda} \right\} \leq \Delta \lambda \leq 1 - \lambda, \quad \text{if} \quad \delta_1 < \delta_2 \]
\[ -\lambda \leq \Delta \lambda \leq 1 - \lambda, \quad \text{if} \quad \delta_1 = \delta_2 \]
\[ -\lambda \leq \Delta \lambda \leq \min \left\{ 1 - \lambda, \frac{2n}{2n-1} \frac{p^{i+\Delta \lambda} - p^i}{\lambda - \lambda} \right\}, \quad \text{if} \quad \delta_1 > \delta_2 \]

where

\[ \delta_1 = \sum_{k=1}^{n} \left[ (\tilde{p}^+_{jk} - \tilde{v}^+_{jk}) - (\tilde{p}^-_{jk} - \tilde{v}^-_{jk}) \right] \]
\[ \delta_2 = \sum_{k=1}^{n} \left[ (\tilde{p}^+_{jk} - \tilde{v}^+_{jk}) \right] \]

**Proof.** Firstly, we note that \( \Delta \lambda \) is subject to the following constraint:

\[ -\lambda \leq \Delta \lambda \leq 1 - \lambda \]

We have the following relation between \( p^i_1 \) and \( p^{i+\Delta \lambda}_1 \):

\[ p^{i+\Delta \lambda}_1 = p^i_1 + \frac{\Delta \lambda \cdot \delta_1}{2n} \]

where \( \delta_1 = \sum_{k=1}^{n} \left[ (\tilde{p}^+_{jk} - \tilde{v}^+_{jk}) - (\tilde{p}^-_{jk} - \tilde{v}^-_{jk}) \right] \).

The following equivalence holds:

\[ p^{i+\Delta \lambda}_1 \leq p^{i+\Delta \lambda}_1 \iff \Delta \lambda \cdot (\delta_1 - \delta_2) \leq 2n \cdot (p^i_1 - p^i_j) \quad \text{(25)} \]

Three scenarios are possible:

- \( \delta_1 = \delta_2 \). Because \( p^i_1 \leq p^i_j \) then (25) is true for any value of \( \Delta \lambda \), i.e.,

\[ -\lambda \leq \Delta \lambda \leq 1 - \lambda \]

- \( \delta_1 > \delta_2 \). \( \Delta \lambda \leq \frac{2n}{\delta_1 - \delta_2} (p^i_1 - p^i_j) \), and therefore:

\[ -\lambda \leq \Delta \lambda \leq \min \left\{ 1 - \lambda, \frac{2n}{\delta_1 - \delta_2} (p^i_1 - p^i_j) \right\} \]

- \( \delta_1 < \delta_2 \). \( \Delta \lambda \leq \frac{2n}{\delta_1 - \delta_2} (p^i_1 - p^i_j) \), and therefore:

\[ \min \left\{ 1 - \lambda, \frac{2n}{\delta_1 - \delta_2} (p^i_1 - p^i_j) \right\} \leq \Delta \lambda \leq 1 - \lambda \]

5. **Conclusions**

Novel score functions for IFSs and IVIFNs are introduced. Their application to derive score FPRs from IFPRs and IVIFPRs has been exploited to derive prioritisation methods for the latter based on the former. It has been proved that when the hesitancy degree function is null the intuitionistic fuzzy relations coincide with their associated score fuzzy relations, respectively. Based on the continuous ordered weighted arithmetic averaging (COWA), a new methodology to implement the decision maker's attitudinal character in decision making problems with information represented via IVIFNs is presented. The attitudinal expected score function for IVIFNs was proved to extend: (i) the score function for IFSs introduced in this paper, which is mathematically equivalent to Chen and Tan's score function for IFS; and (ii) Xu and Chen's score function for IVIF. Finally, a ranking sensitivity analysis of the attitudinal expected score function with respect to the attitudinal parameter is provided.

**Acknowledgements**

This work was supported by National Natural Science Foundation of China (NSFC) under the Grant No. 71101131.

**References**


