A Weighted Inner Product Estimator in the Geometric Algebra of Euclidean 3-Space for Source Localization Using an EM Vector-sensor

JIANG Jingfei, ZHANG Jianqiu*

Department of Electronic Engineering, Fudan University, Shanghai 200433, China

Received 21 January 2011; revised 3 March 2011; accepted 28 June 2011

Abstract

In this paper, the source localization by utilizing the measurements of a single electromagnetic (EM) vector-sensor is investigated in the framework of the geometric algebra of Euclidean 3-space. In order to describe the orthogonality among the electric and magnetic measurements, two multivectors of the geometric algebra of Euclidean 3-space (G3) are used to model the outputs of a spatially collocated EM vector-sensor. Two estimators for the wave propagation vector estimation are then formulated by the inner product between a vector and a bivector in the G3. Since the information used by the two estimators is different, a weighted inner product estimator is then proposed to fuse the two estimators together in the sense of the minimum mean square error (MMSE). Analytical results show that the statistical performances of the weighted inner product estimator are always better than its traditional cross product counterpart. The efficacy of the weighted inner product estimator and the correctness of the analytical predictions are demonstrated by simulation results.

Keywords: cross product; electromagnetic; geometric algebra; geometric algebra of Euclidean 3-space; minimum mean square rule; direction finding; vector-sensor

1. Introduction

The cross product algorithm, based on the fact that the instantaneous electric and magnetic vectors of an electromagnetic (EM) plane wave and the direction vector of wave propagation are mutually orthogonal, has been widely applied to such areas as EM source tracking [1-2] and parameter estimations [3-10] in recent years. The reasons for its wide applications are at least threefold. Firstly, since the orthogonal relationships do not depend on the frequency of the EM waves, the cross product algorithm can be equally applied to both the narrowband and wideband sources with very low computational complexity [1]. Secondly, the cross product algorithm is based on the measurements of a single EM vector-sensor, which consists of three orthogonally oriented electric antennas and three orthogonally oriented magnetic antennas. On the one hand, when the six antennas are spatially collocated in a point-like geometry [3-14], there is no need for sensor position calibration and time synchronization since only the instantaneous measurements of an uni-vector-sensor are used. On the other hand, if the six antennas are not spatially collocated but elaborately placed, for example using the ingenious and simple scheme presented in Ref. [15], the mutual coupling among the antennas can be effectively minimized in that the spacing between the antennas can be quite large. Thirdly, the cross product algorithm can be incorporated into the traditional algorithm of direction of arrival (DOA) estimation, which is based on the spatial
phase delay across a vector-sensor array. Creative synergy between these two algorithms has produced several advantages for high resolution vector-sensor array processing, including the improvement of DOA estimation accuracy\cite{14-16}, the DOA estimation without knowing the array geometry\cite{7-8}, self-initializes the multiple signal classification (MUSIC) iteration\cite{9}, array aperture extension\cite{10}, etc. As a result, if the performances of the cross product estimator are improved, better results of the aforementioned applications and algorithms can be achieved.

In this paper, we concentrate on improving the performances of the traditional cross product estimator based on the measurements of a single EM vector-sensor. A weighted inner product estimator, aimed at estimating the propagation direction vector of the incident EM waves, is proposed based on the minimum mean square error (MMSE) fusion rule. To evaluate the performance of the weighted inner product estimator, an analytical comparison to its cross product counterpart is drawn. From the comparison results, we proved that our weighted inner product estimator is always superior to the traditional cross product estimator. However, the price for the performance improvement is that extra computational efforts are required for the MMSE fusion process. In addition, theoretical analyses also reveal that both the cross product and weighted inner product estimators are biased in the presence of mutual coupling. Hence, to make our work meaningful, a simple strategy is proposed to calibrate the undesired coupling before applying our estimators to the sensor outputs.

2. Geometric Algebra of Euclidean 3-Space (G3)

Before starting our discussion, we would like to add a comment on the notation that is used. To facilitate the subsequent discussions, we list here some basic properties of the G3 necessary for our purpose in this paper. For the interested readers, a thorough review about the geometric algebra can be found in Refs. \cite{16-18}, while some properties of the geometric algebra and its applications to physics were discussed in Ref. \cite{19}. In the signal processing field, the application of the geometric algebra to image processing was introduced in Ref. \cite{20}, the techniques of utilizing the geometric algebra to represent power under non-sinusoidal conditions were reported in Ref. \cite{21}, and the Fourier transform in the geometric algebra domain was investigated in Ref. \cite{22}.

2.1. Basics

The geometric algebra of G3, noted as $G_3$, is an 8D algebra system and consists of scalars (0-grade-vectors), vectors (1-grade-vectors), bivectors (2-grade-vectors), and trivectors (3-grade-vectors)\cite{17}. A generic element of the G3 is nominated as a multivector, which is a mathematical object of "mixed" dimensions and can be expressed by

$$A = \alpha_0 + \alpha_1 e_{12} + \alpha_2 e_{23} + \alpha_3 e_{31} + \alpha_4 e_{132} + \alpha_5 e_{231} + \alpha_6 e_{312} + \alpha_7 e_{123}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_7$ are real numbers; $\langle A \rangle_k (k=0, 1, 2, 3)$ is the k-grade-vector part of $A$; $\{1\}, \{e_1, e_2, e_3\}, \{e_{12}, e_{31}, e_{23}\}$, and $\{e_{123}\}$ are respectively the basis elements of the scalar, vector, bivector, and trivector parts of the G3. The multiplication rules of the eight bases are summarized in Table 1. Since the multiplication obeys the left and right distributive rules with respect to addition\cite{18}, the product of any two multivectors in the G3, defined as the geometric product, can be obtained by using the multiplication rules depicted in Table 1.

<table>
<thead>
<tr>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>$e_1$</td>
<td>$e_1$</td>
<td>$e_1$</td>
<td>$e_1$</td>
<td>$e_1$</td>
</tr>
<tr>
<td>$e_2$</td>
<td>$e_2$</td>
<td>$-e_{12}$</td>
<td>$e_{23}$</td>
<td>$-e_{31}$</td>
<td>$e_{13}$</td>
</tr>
<tr>
<td>$e_3$</td>
<td>$-e_{12}$</td>
<td>$e_{23}$</td>
<td>$e_{31}$</td>
<td>$-e_{13}$</td>
<td>$e_{21}$</td>
</tr>
<tr>
<td>$e_{12}$</td>
<td>$e_{12}$</td>
<td>$-e_{23}$</td>
<td>$-e_{31}$</td>
<td>$e_{13}$</td>
<td>$e_{21}$</td>
</tr>
<tr>
<td>$e_{23}$</td>
<td>$e_{23}$</td>
<td>$-e_{31}$</td>
<td>$-e_{13}$</td>
<td>$e_{12}$</td>
<td>$e_{31}$</td>
</tr>
<tr>
<td>$e_{31}$</td>
<td>$e_{31}$</td>
<td>$e_{13}$</td>
<td>$e_{21}$</td>
<td>$-e_{12}$</td>
<td>$-e_{23}$</td>
</tr>
<tr>
<td>$e_{132}$</td>
<td>$e_{132}$</td>
<td>$e_{231}$</td>
<td>$e_{312}$</td>
<td>$-e_{123}$</td>
<td>$-e_{231}$</td>
</tr>
<tr>
<td>$e_{231}$</td>
<td>$e_{231}$</td>
<td>$e_{312}$</td>
<td>$e_{123}$</td>
<td>$-e_{231}$</td>
<td>$-e_{123}$</td>
</tr>
<tr>
<td>$e_{312}$</td>
<td>$e_{312}$</td>
<td>$e_{123}$</td>
<td>$e_{231}$</td>
<td>$-e_{312}$</td>
<td>$-e_{132}$</td>
</tr>
<tr>
<td>$e_{132}$</td>
<td>$e_{132}$</td>
<td>$e_{231}$</td>
<td>$e_{312}$</td>
<td>$-e_{132}$</td>
<td>$-e_{312}$</td>
</tr>
</tbody>
</table>

Table 1 Multiplication rules of geometric algebra of G3

Definition 1 The magnitude of a multivector $A \in G_3$ is a unique scalar calculated by\cite{22}

$$\|A\|^2 = \langle AA^* \rangle_0 = \sum_{k=0}^{3} \langle A \rangle_k \langle A^* \rangle_k$$

(2)

where

$$A^* = a_0 - a_1 e_{12} - a_2 e_{23} - a_3 e_{31} + a_4 e_{132} + a_5 e_{231} + a_6 e_{312} + a_7 e_{123}$$

is the reverse of $A$ and $\|A\|$ means the magnitude of a multivector.

From Table 1 and Definition 1, it can be seen that $e_{12}^2 = -1$ and $e_{132}^2 = -e_{123}$, which are similar to the properties usually attributed to the complex imaginary.
unit $j$ because $j^2 = -1$ and the conjugation of $j$ is $-j$. In fact, it can be proved that $e_{123}$ is mathematically isomorphic to $j$, since $\{1, e_{123}\}$ constructs an algebra system isomorphic to the complex algebra [18].

**Definition 2**  The inner product of two vectors can be extended to two multivectors in the G3. The inner product of a vector $a$ and a $k$-grade-vector $\langle B \rangle_k$ is defined by

$$ a \cdot \langle B \rangle_k = \frac{1}{2} (a \langle B \rangle_k - (-1)^k \langle B \rangle_k a) \tag{4} $$

where $a \langle B \rangle_k$ and $\langle B \rangle_k a$ are the left and right geometric product of $a$ and $\langle B \rangle_k$ (the geometric product is non-commutative); the geometric interpretation of $a \cdot \langle B \rangle_k$ is that it is a $(k-1)$-grade-vector representing the complement (within the subspace defined by $\langle B \rangle_k$) of the orthogonal projection of $a$ onto $\langle B \rangle_k$ [19]. In this way, the inner product by a vector lowers the grade of any simple multivector by one.

**Definition 3**  For a vector $a$ and any multivector $\langle B \rangle_k$ of grade $k$, their outer product is defined by

$$ a \wedge \langle B \rangle_k = \frac{1}{2} (a \langle B \rangle_k + (-1)^k \langle B \rangle_k a) \tag{5} $$

The geometric interpretation of $a \wedge \langle B \rangle_k$ is that it is a $(k+1)$-grade-vector spanned by the $k$-grade-vector $\langle B \rangle_k$ and the vector $a$. As a result, the outer product by a vector raises the grade of any simple multivector by one.

### 2.2. Relations with vector algebra

The vector algebra developed by J. Willard Gibbs in 1884 fits naturally into the G3 [18]. For this purpose, we just give the relations between the outer product and the cross/inner product introduced by Gibbs.

Let $a$ and $b$ be any two vectors in the G3, their cross product $a \times b$ can be expressed by

$$ a \wedge b = e_{123} a \times b \tag{6} $$

or equivalently

$$ a \times b = -a \cdot (be_{123}) \tag{7} $$

The inner product $a \cdot b$ can also be represented by

$$ a \cdot b = -e_{123} (a \wedge (e_{123} b)) \tag{8} $$

### 3. G3 Description of an Uni-vector-sensor

Suppose that a single EM vector-sensor locates at the origin of the coordinated system and there is a narrow-band source, parameterized by $\Theta = [\phi \ \theta \ \gamma \ \eta]$, radiating EM waves to the vector-sensor. When the medium is nonconductive, homogenous and isotropic, the noise free signals that would be received at the sensor are [3-10]

$$ s_{\Theta}(t) = s_{\Theta}(\phi, t) = s_{\Theta}(\theta, t) = s_{\Theta}(\gamma, t) = s_{\Theta}(\eta, t) = \begin{bmatrix} -\sin \phi & \cos \phi & \cos \theta \\ \cos \phi & \sin \phi & \sin \theta \\ 0 & -\sin \theta & \sin \eta \end{bmatrix} \tag{9} $$

where $0 \leq \theta < \pi$ represents the source’s elevation angle measured from the vertical $z$ axis, $0 \leq \phi < 2\pi$ the source’s azimuth angle, $0 \leq \gamma < \pi$ the polarization phase difference angle, $a(\Theta)$ the well-known steering vector, and $p(\Theta)$ the polarization parameter of the source [25]. In Eq. (9), the electric-field vector $[s_E(t) \ s_E(t) \ s_E(t)]^T$ and the magnetic-field vector $[s_M(t) \ s_M(t) \ s_M(t)]^T$ are respectively measured by the electric and magnetic antennas of the vector-sensor. Based on the definitions of elevation angle $\theta$ and azimuth angle $\phi$, the direction vector of the source, which is opposite to the wave propagation vector, is analytically expressed as [3-10]

$$ u = \begin{bmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{bmatrix} \tag{10} $$

where $u_x$, $u_y$, and $u_z$ are the well-known direction cosines of the source along $x$ axis, $y$ axis, and $z$ axis.

For notational simplicity and without loss of generality, the explicit dependence on $\Theta$ will be omitted in the following discussions. $s(t) = [s(t)] e^{i \omega t}$ is the complex envelope of the source’s signal with $\phi$ being the phase angle. Since $e_{123}$ is isomorphic to the complex imaginary unit $j$ [18], we use $e_{123}$ to replace $j$ in Eq. (9) to facilitate our discussions in the framework of the G3.

From the standpoint of G3, the original received data of each vector-sensor are six multivectors (i.e., $S_{E1}, S_{E2}, S_{E3}, S_{M1}, S_{M2}, S_{M3}$) with only scalar and trivector parts. Taking the orthogonality among the electric or magnetic measurements into consideration, we can respectively express them as

$$ S_{E1}(t) = s_{E1}(t)e_1 + s_{E2}(t)e_2 + s_{E3}(t)e_3 \tag{11} $$

$$ S_{M1}(t) = s_{M1}(t)e_1 + s_{M2}(t)e_2 + s_{M3}(t)e_3 \tag{12} $$

Based on the rules in Table 1, it is explicit that $S_{E1}(t)$ and $S_{M1}(t)$ are multivectors consisting the vector and the bivector parts. Compared with Eq. (9), Eqs. (11)-(12) are more coincident with the physical nature of the EM waves, for they can vividly illustrate the orthogonality among the electric and magnetic signals, while the long vector type modeling in Eq. (9) cannot. Similar to Eqs. (11)-(12), the direction vector of the source can be expressed in the following G3 form:
\[ u = e_1 \sin \theta \cos \phi + e_2 \sin \theta \sin \phi + e_3 \cos \theta \] (13)

In applications, the sensor measurements are always corrupted by noise. Then the real data of the array outputs can be modeled as

\[ Y_s(t) = S(t)N_s(t) \] (14)
\[ Y_h(t) = S(t)N_h(t) \] (15)

where \( N_s(t) = n_{s1}(t) e_1 + n_{s2}(t) e_2 + n_s(t) e_3 \) and \( N_h(t) = n_{h1}(t) e_1 + n_{h2}(t) e_2 + n_{h3}(t) e_3 \) are the G3 formulations of the electromagnetic and magnetic measurement noise respectively. In general, the noise components are modeled as complex vectors \(^[3]\). Correspondingly, they can be represented in the G3 framework as

\[ N_E(t) = (n_{s1}(t) + e_{123}n_{e1}(t)) e_1 + (n_{s2}(t) + e_{123}n_{e2}(t)) e_2 + (n_s(t) + e_{123}n_{e3}(t)) e_3 = n_{E1}(t) + e_{123}n_{E1}(t) \] (16)
\[ N_H(t) = (n_{h1}(t) + e_{123}n_{h1}(t)) e_1 + (n_{h2}(t) + e_{123}n_{h2}(t)) e_2 + (n_{h3}(t) + e_{123}n_{h3}(t)) e_3 = n_{H1}(t) + e_{123}n_{H1}(t) \] (17)

where \( n_{E1}, n_{E1}, n_{h1}, \) and \( n_{h1} \) are (real) vectors.

Similar to the complex domain formulation \(^[3]\), some necessary assumptions on the signals and noise components in the G3 framework are listed as follows.

**Assumption 1** Equivalent to the complex formulation \(^[3]\), the source signal sequence \( \{s(1), s(2), \cdots \} \) is a sample from a temporally uncorrelated stationary (complex) random process with zero mean and

\[ E \left[ s(p) s(q) \right] = \sigma^2_s \delta_{p,q}, \quad E \left[ s(p) s(q) \right] = 0 \]

where \( s(t) = s(t)+e_{123}s(t) \) with \( s(t) \) and \( s(t) \) being two real random variables, \( \sigma^2_s \) is an unknown positive real number, \( \delta_{p,q} \) the Kronecker delta, and \( E \{ \ast \} \) the expectation operator.

**Assumption 2** In Eq. (16), the measurement noise of the electric signals is a three-dimensional complex random process with zero mean and

\[ E \left[ N_E(t) N_E(t)^\dagger \right] = 3\sigma^2_E \delta_{p,q}, \quad E \left[ N_E(t) N_E(t)^\dagger \right] = 0 \]

where

\[ E \left[ n_{Em}(t) + e_{123}n_{Em}(t) \right] \left[ n_{Em}(t) + e_{123}n_{Em}(t) \right]^\dagger \] (18)

with \( m=x \) or \( y \) or \( z \).

**Assumption 3** The assumption on the magnetic noise components is similar to that of the electric ones, i.e.,

\[ E \left[ N_H(t) N_H(t)^\dagger \right] = 3\sigma^2_H \delta_{p,q}, \quad E \left[ N_H(t) N_H(t)^\dagger \right] = 0 \]

where

\[ E \left[ (n_{h1}(t) + e_{123}n_{h1}(t)) (n_{h1}(t) + e_{123}n_{h1}(t))^\dagger \right] = \sigma^2_H \]

with \( m=x \) or \( y \) or \( z \).

**Assumption 4** The electric noise is independent of the magnetic noise. The source signals and the measurement noise are uncorrelated.

4. **Weighted Inner Product Estimator**

This section presents a simple algorithm to estimate the direction vector \( u \) of the EM source via the measurements of a uni-vector-sensor. The idea behind this estimator is the fact that the mutually orthogonality among the electric vector, magnetic vector, and the wave propagation vector of an EM wave can be formulated by the inner product between a vector and bivector in the G3, as described below.

4.1. Two independent inner product estimators

Before rendering our discussions, we would like to make the following definitions:

\[ v_1 = -e_1 \sin \phi + e_2 \cos \phi \] (18)
\[ v_2 = -e_1 \cos \phi \cos \theta - e_2 \sin \phi \cos \theta + e_3 \sin \theta \] (19)

According to Eq. (13), Eq. (18) and Eq. (19), it can be checked that \( u \perp v_1 \perp v_2 \) and \( u, v_1, v_2 \) form a right orthogonal triad. By further defining that \( p_1^\dagger \cos \gamma \), \( p_2^\dagger = \sin \gamma \cos \eta \), \( p_3^\dagger = \sin \gamma \sin \eta \), \( s_i(t) = \|s(t)\| \cos \phi \), and \( s_i(t) = \|s(t)\| \sin \phi \), we then have

\[ p(\gamma, \eta) = [p_1, p_2, p_3, e_{123}, p_2] \] (20)
\[ s(t) = s_i(t) + e_{123}s_i(t) \] (21)

Substituting Eqs. (20)-(21) into Eq. (9), Eqs. (18)-(19) imply that Eqs. (11)-(12) can be expressed as

\[ S_E(t) = p_{s1}(t)w_1 - p_{s2}(t)w_2 + p_{s3}(t)w_3 + e_{123} \left( p_{s1}(t)w_1 - p_{s2}(t)w_2 - p_{s3}(t)w_3 \right) = x(t) + e_{123}y(t) \] (22)
\[ e_{123}S_H(t) = uS_E(t) = ux(t) + e_{123}uy(t) \] (23)

where

\[ x(t) = p_{s1}(t)w_1 - p_{s2}(t)w_2 + p_{s3}(t)w_3 \] (24)
\[ y(t) = p_{s1}(t)w_1 - p_{s2}(t)w_2 + p_{s3}(t)w_3 \] (25)

are two vectors. Since \( u \perp v_1 \perp v_2 \), Eqs. (24)-(25) mean that \( x(t) \perp u \) and \( y(t) \perp u \). Consequently, Eq. (23) equals

\[ e_{123}S_H(t) = uS_E(t) = u \times x(t) + e_{123}u \times y(t) \] (26)

Eqs. (24)-(25) also confirm that \( S_E(t) \) and \( e_{123}S_H(t) \) are multivectors with only the vector and bivector parts. Based on such an observation and the inner product definition in Eq. (4), we obtain the following two relations from Eq. (22) and Eq. (26), i.e.,
\[ \langle S_e(t)_1, e_{123}S_H(t)_1 \rangle_2 = x(t)^T(u \wedge x(t)) = -(x(t)^T x(t))u \]  
(27)

\[ \langle e_{123}S_H(t)_1 \rangle_2 \cdot \langle S_e(t)_1 \rangle_2 = e_{123} (u \wedge y(t))^T = -(y(t)^T y(t))u \]  
(28)

Since the inner products \( x(t)^T x(t) \) and \( y(t)^T y(t) \) are scalars, the results in Eqs. (27)-(28) are two vectors parallel to \( u \), which means, after a normalization process, \( u \) can be obtained from Eqs. (27)-(28) by using the noise free sensor data. However, the real sensor data are always corrupted by measurement noise as represented in Eqs. (16)-(17). Using the noise representations in Eqs. (14)-(15), Eqs. (27)-(28) will change to

\[ \langle Y_e(t)_1, e_{123}Y_H(t)_1 \rangle_2 = \left[ \langle S_e(t)_1 + n_{1}(t) \rangle_1 \cdot \langle e_{123}S_H(t)_1 \rangle_2 + e_{123}n_{1}(t) \right] = \left[ -(x(t)^T x(t))u + e_{123} (x(t)^T n_{1}(t)) + e_{123} (n_{1}(t)^T n_{1}(t)) + e_{123} (u \wedge x(t)) \right] \]  
(29)

\[ \langle e_{123}Y_H(t)_1 \rangle_2 \cdot \langle Y_e(t)_1 \rangle_2 = \left[ \langle e_{123}S_H(t)_1 - n_{1}(t) \rangle_1 \cdot \langle S_e(t)_1 \rangle_2 + e_{123}n_{1}(t) \right] = \left[ -(y(t)^T y(t))u + e_{123} (y(t)^T n_{1}(t)) + e_{123} (n_{1}(t)^T n_{1}(t)) + e_{123} (u \wedge y(t)) \right] \]  
(30)

where the last three items on the right sides are the noise perturbation components for the inner product results. If an average process is used to reduce the perturbation, we can get two estimators of the direction vector \( u \) as

\[
\begin{align*}
\hat{u}_x &= \frac{1}{K} \sum_{k=1}^K \langle Y_e(t)_1, e_{123}Y_H(t)_1 \rangle_2 \\
\hat{u}_y &= \frac{1}{\| w_y \|_2} w_y \\
\hat{u}_y &= \frac{1}{K} \sum_{k=1}^K \langle e_{123}Y_H(t)_1, \langle Y_e(t)_1 \rangle_2 \rangle \\
\end{align*}
\]

(31)

(32)

where \( \| \cdot \|_2 \) is the \( l_2 \)-norm of a vector.

**Theorem 1** Under Assumption 1 to Assumption 4, it is almost sure that \( \hat{u}_x \rightarrow_{\text{a.s.}} u \) and \( \hat{u}_y \rightarrow_{\text{a.s.}} u \).

**Proof** See Appendix A.

### 4.2. An optimal weighted inner product estimator

As shown in Eqs. (29)-(30), the noise components in the two estimators \( \hat{u}_x \) and \( \hat{u}_y \) are independent of each other. Motivated by such an observation, the idea of fusing \( \hat{u}_x \) and \( \hat{u}_y \) into a better estimator comes naturally. In this subsection, we introduce a weighted inner product estimator to fulfill this idea.

Rewrite \( \hat{u}_x \) and \( \hat{u}_y \) as \( \hat{u}_x = \sum_{i=1}^3 \tilde{u}_{ix} e_i \) and \( \hat{u}_y = \sum_{i=1}^3 \tilde{u}_{iy} e_i \). Since \( \tilde{u}_x \rightarrow_{\text{a.s.}} u \) and \( \tilde{u}_y \rightarrow_{\text{a.s.}} u \), it is easy to check that

\[ \tilde{u}_x = \sum_{i=1}^3 \tilde{u}_{ix} e_i = \sum_{i=1}^3 \left[ \omega_i \tilde{u}_{ix} + (1 - \omega_i) \tilde{u}_{ix} \right] e_i \rightarrow_{\text{a.s.}} u \]  
(33)

where \( 0 \leq \omega_i \leq 1 \). Let the variances of \( \tilde{u}_x \), \( \tilde{u}_y \), and \( \tilde{u}_ix \) be \( \sigma^2_x \), \( \sigma^2_y \), and \( \sigma^2_{ix} \) respectively. From Eq. (33), one has

\[ \sigma^2_{ix} = \omega_i^2 \sigma^2_x + (1 - \omega_i)^2 \sigma^2_y \]  
(34)

The method we applied to obtain \( \omega_i \) is to minimize Eq. (34), which is known as the MMSE fusion rule for a multi-sensor system [24]. Thus, one has

\[ \omega_i = \frac{1}{\sigma^2_x + \frac{1}{\sigma^2_y}} \]  
(35)

and the minimum variance of \( \tilde{u}_ix \) is

\[ \sigma^2_{ix} = \frac{1}{\sigma^2_x + \frac{1}{\sigma^2_y}} \]  
(36)

According to Eq. (36), it can be easily checked that \( \sigma^2_{ix} < \sigma^2_x \) and \( \sigma^2_{ix} < \sigma^2_y \). As a result, if \( \sigma^2_x \) and \( \sigma^2_y \) can be obtained in advance, the optimal weight factor \( \omega_i \) can be calculated to yield the MMSE estimation of \( u \).

In applications, the on-line optimum estimation of \( u \) requires the on-line estimation of \( \sigma^2_x \) and \( \sigma^2_y \) respectively. Let the \( m \)-th estimation of \( u \) using Eqs. (29)-(30) be \( \hat{u}_x(m) = \sum_{i=1}^3 \tilde{u}_{ix}(m) e_i \) and \( \hat{u}_y(m) = \sum_{i=1}^3 \tilde{u}_{iy}(m) e_i \).

When the data length for the on-line estimation of \( \sigma^2_x \) and \( \sigma^2_y \) is \( M \), their variances can be estimated as

\[ \hat{\sigma}^2_x = \frac{1}{M} \sum_{m=1}^M \left[ \hat{u}_x(m) - \frac{1}{M} \sum_{q=m-M+1}^m \hat{u}_x(q) \right]^2 \]  
(37)

\[ \hat{\sigma}^2_y = \frac{1}{M} \sum_{m=1}^M \left[ \hat{u}_y(m) - \frac{1}{M} \sum_{q=m-M+1}^m \hat{u}_y(q) \right]^2 \]  
(38)

Equations (37)-(38) can be considered as two moving window estimators with a window length \( M \). These windows keep moving forward with the new data entering. In each time, what are seen from the windows are the most recent \( M \) estimates given by \( \hat{u}_x \) and \( \hat{u}_y \), and what are produced by the moving window estimators are the current variances for all the components of \( \hat{u}_x \) and \( \hat{u}_y \).
In summary, our weighted inner product estimator can be implemented by the steps as follows.

**Step 1** Select the number of snapshots $K$.

**Step 2** Choose the window length $M$ for the variance estimations.

**Step 3** Use the most recent $K$ snapshots of sensor data to get the newest estimations of $\hat{u}_s$ and $\hat{u}_p$ by Eqs. (31)-(32).

**Step 4** Estimate $\sigma^2_{u_s}$ and $\sigma^2_{u_p}$ by Eqs. (37)-(38).

**Step 5** Obtain $\sigma_i (i=1,2,3)$ by substituting the estimated $\sigma^2_{u_s}$ and $\sigma^2_{u_p}$ in Eq. (35).

**Step 6** Calculate $\hat{u}_e$ by Eq. (33) and normalize it to a unit vector.

**Step 7** Repeat Step 3-Step 6 to get the next estimate of $\hat{u}_e$.

5. A Comparison with Cross Product Estimator

Nehorai and Paldi proposed a cross product estimator\(^{(3)}\) by averaging the Poynting vectors. Since the cross product estimator and our weighted inner product estimator have the same objective—estimating the propagation vector of the EM waves, we will draw a comparison of them in the aspects of performance and computational issues.

5.1. Performance comparison

Based on Eq. (16), Eq. (17), Eq. (24), and Eq. (25), using complex imaginary part $j$ to replace $e_{123}$ and $a_s e_1 + a_p e_1 + a_s e_2$ to represent a three-dimensional vector $[a_s, a_p, a_s]$, the complex vector of the sensor data can be written as

\[
y^{(i)}(t) = x(t) + jy(t) + n_{ei}(t) + jn_{ei}(t)
\]

(39)  \[
y^{(i)}(t) = x(t) \times y(t) + n_{ei}(t) + jn_{ei}(t)
\]

(40) where the superscript "$i$" means the $i$th antenna of an EM vector-sensor. The cross product estimator\(^{(3)}\) can be equivalently represented in the G3 framework as

\[
\hat{w}_e = \frac{1}{K} \sum_{i=1}^{K} \text{Re}\left\{y^{(i)}(t) \times \left(y^{(i)}(t)^\ast\right)\right\}
\]

(41)

where $\text{Re}\{\cdot\}$ is the operator to extract the real part of a complex vector and $\cdot^\ast$ denotes the conjugation of a complex vector.

In Eq. (41), $\text{Re}\{y^{(i)}(t) \times (y^{(i)}(t))^\ast\}$ can be represented as

\[
\text{Re}\left\{y^{(i)}(t) \times (y^{(i)}(t))^\ast\right\} = \langle x(t) \times x(t) + y(t) \times y(t) + (x(t) \times n_{ei}(t) + y(t) \times n_{ei}(t)) + \right.
\]

(42) Using the relationships between the cross product estimator and our weighted inner product estimator, $\hat{u}_s$ satisfies

\[
\hat{u}_s = \frac{\hat{w}_e + \hat{w}_p}{\| \hat{w}_e + \hat{w}_p \|}.
\]

(44) Let the variance of $\hat{u}_s$ and $\hat{u}_p$ be $\sigma^2_{u_s}$ and $\sigma^2_{u_p}$ respectively. The following theorem holds.

**Theorem 2** When the number of snapshot $K$ is sufficiently large, the variances $\sigma^2_{u_s}$ and $\sigma^2_{u_p}$ satisfy

\[
\sigma^2_{u_s} \leq \sigma^2_{u_p}
\]

(45) and the equality holds if and only if

\[
\frac{\sigma^2_{u_s}}{\sigma^2_{u_p}} = \frac{\| w_x \|}{\| w_y \|}.
\]

(46) **Proof** See Appendix B.

Theorem 2 implies that if Eq. (46) is satisfied, the cross product estimator has a statistical performance equivalent to our weighted inner product one. If the requirement in Eq. (46) is not met, our weighted inner product estimator is always superior over the cross product counterpart.

In applications, $\sigma^2_{u_s}$ and $\sigma^2_{u_p}$ are affected by several factors. To begin with, the requirements of the noise statistic in Assumption 2 and Assumption 3 are difficult to satisfy due to the time-varying surrounding environment\(^{[23]}\). Besides, the analysis in the next section will show that, in the presence of the mutual coupling, $\sigma^2_{u_s}$ and $\sigma^2_{u_p}$ are not only dependent on the complex noise, but also rely on the coupling matrix of the antennas. As the coupling matrix is unknown, its influences on $\sigma^2_{u_s}$ and $\sigma^2_{u_p}$ are also unknown. Finally, since the cross product estimator is the final step of such algorithms as estimate signal parameters via rotation invariance techniques (ESPRIT)\(^{[6,8]}\) and self-initiate MUSIC\(^{[9]}\), the six-dimensional EM vector for $\hat{u}_s$ and $\hat{u}_p$ is the results of some manipulations of the original sensor data. $\sigma^2_{u_s}$ and $\sigma^2_{u_p}$ are therefore related to the accuracy of the aforementioned algorithms. Consequently, it is difficult to estimate whether
\[ \sigma_{ij}^2 \text{ and } \sigma_{ij}^2 \text{ satisfy Eq. (46) or not. Seen in this light, our weighted inner product estimator is meaningful, for it always yields the optimal estimation of the wave propagation vector.} \]

As a side note, Eqs. (39)-(40) also indicate that the weighted inner product estimator can be equivalently implemented in the complex domain. The main manipulations are concluded as: 1) separate the array outputs into real and imaginary parts; 2) compute ̂\( \hat{\mathbf{u}}_x \) and ̂\( \hat{\mathbf{u}}_y \) from the cross product of the real and imaginary parts respectively; and 3) fuse ̂\( \hat{\mathbf{u}}_x \) and ̂\( \hat{\mathbf{u}}_y \) together using Eq. (33).

### 5.2. Computational issues

According to Eqs. (39)-(40), the computational complexity of ̂\( \hat{\mathbf{u}}_x \) (or ̂\( \hat{\mathbf{u}}_y \)) is almost 1/4 of the traditional cross product estimator ̂\( \hat{c} \). The reason is that ̂\( \hat{\mathbf{u}}_x \) (or ̂\( \hat{\mathbf{u}}_y \)) is equivalent to the cross product of real (or imaginary) part of the sensor outputs, while ̂\( \hat{\mathbf{u}}_x \) is cross product of the complex sensor outputs before extracting the real part from the result. However, the variance estimation and fusion process of the weighted inner product estimator require extra computational efforts, which are clearly the costs for the performance improvement.

### 6. Further Discussions

As is well-known, deviations from the true array manifold, typically resulting from mutual coupling effects, can seriously degrade the performance of many high resolution direction finding algorithms. In this section, we will investigate the performances of our estimators in the presence of mutual coupling, and provide a simple approach to compensate the undesired effects.

Considering the coupling among the six antennas of the vector-sensor, the received array data in Eqs. (39)-(40) are rewritten as

\[
\begin{bmatrix}
Y^{(i)}_{1Ei}(θ,t) \\
Y^{(i)}_{2Ei}(θ,t)
\end{bmatrix} = \mathbf{C}a(θ)s(t) + \mathbf{n}_{Ei}(t) = \begin{bmatrix}
Y^{(i)}_{1Ei}(θ,t) \\
Y^{(i)}_{2Ei}(θ,t)
\end{bmatrix} + \begin{bmatrix}
\Delta Y_E(t) \\
\Delta Y_H(t)
\end{bmatrix}
\]

where the 6×6 matrix \( \mathbf{C} \) describes the coupling effects among the antennas, and

\[
\begin{bmatrix}
\Delta Y_E(t) \\
\Delta Y_H(t)
\end{bmatrix} = (\mathbf{C} - \mathbf{I})a(θ)s(t)
\]

symbolizes the model error, which is dependent on the matrix \( \mathbf{C} \) and the source signal \( s(t) \). As array perturbations considered herein are due to unknown mutual coupling, the matrix \( \mathbf{C} \) can be modeled without direction of arrival (DOA) dependence \[^{[29]}\].

Similar to Eqs. (11)-(15), the array output in the G3 formulation is now

\[
\begin{align*}
Y_{Ei}(t) &= Y_E(t) + \Delta Y_{Ei}(t) + e_{123}Y_{Ei}(t) \\
e_{123}Y_{1Ei}(t) &= e_{123}Y_{1Ei}(t) + e_{123}Y_{2Ei}(t) - \Delta Y_{1Ei}(t)
\end{align*}
\]

where \( \Delta Y_E = \Delta Y_{Ei} + j\Delta Y_{Hi} \) and \( \Delta Y_H = \Delta Y_{Hi} + j\Delta Y_{He} \).

Following Eqs. (49)-(50), the inner product estimator in Eq. (29) can be redefined as

\[
\begin{align*}
\langle Y_{Ei}(t) \rangle_1 \langle e_{123}Y_{1Ei}(t) \rangle_2 &= \langle Y_E(t) \rangle_1 (e_{123}Y_{1Ei}(t)) + \Delta Y_{Ei} (e_{123}Y_{2Ei}(t)) + \Delta Y_{Ei} (e_{123}Y_{2Ei}(t)) \\
&= \langle Y_{Ei}(t) \rangle_1 (e_{123}Y_{1Ei}(t)) + \Delta Y_{Ei} (e_{123}Y_{2Ei}(t)) + \Delta Y_{Ei} (e_{123}Y_{2Ei}(t)) \\
&= \langle Y_{Ei}(t) \rangle_1 (e_{123}Y_{1Ei}(t)) + \Delta Y_{Ei} (e_{123}Y_{2Ei}(t)) + \Delta Y_{Ei} (e_{123}Y_{2Ei}(t))
\end{align*}
\]

Since \( \Delta Y_{Ei} \) and \( \Delta Y_{Hi} \) are dependent on the source signal \( s(t) \), it can be checked that the mathematical expectations of last three items in Eq. (51) are not zeros. Following the procedure in the proof of Theorem 1, it is explicit that ̂\( \hat{\mathbf{u}}_x \) is biased in the presence of mutual coupling. In fact, one can correspondingly verify that all the four estimators, i.e., the traditional cross product estimator ( ̂\( \hat{c} \)), the estimator-x ( ̂\( \hat{u}_x \)), the estimator-y ( ̂\( \hat{u}_y \)), and the weighted inner product estimator ( ̂\( \hat{u}_x \)), are biased. As a result, an array calibration process is therefore very necessary. For the special problem investigated in this paper, a simple method is designed as follows.

According to Eq. (47), one can find that

\[
E\{Y_{Ei}(θ,t) s^*(t)\} = E\{\mathbf{C}a(θ)s(t)s^*(t)\} + E\{n_{Ei}(t)s^*(t)\}
\]

Based on Assumption 4, one can rewrite Eq. (52) as

\[
E\{Y_{Ei}(θ,t) s^*(t)\} = C a(θ) \sigma_s^2
\]

Owing to the finite number of snapshots, Eq. (53) means that

\[
C a(θ) = \frac{1}{Q \sigma_s^2} \sum_{q=1}^{Q} Y_{Ei}(θ,q) s^*(q)
\]

where \( Q \) is the number of available snapshots.

Given the data collected from \( n \) calibration sources of known steering vector \( \{a(θ_i)\}_{i=1, \ldots, n} \) and transmitting signal sequence \( \{s(t)\}_{t=1, \ldots, n} \), we have

\[
\mathbf{C} = \mathbf{D}
\]

where \( \mathbf{A} = \{a(θ_1) a(θ_2) \cdots a(θ_n)\} \) and \( \mathbf{D} = \{d(θ_1) d(θ_2) \cdots d(θ_n)\} \) with

\[
d(θ_i) = \frac{1}{Q \sigma_s^2} \sum_{q=1}^{Q} Y_{Ei}(θ,q) s^*(q)
\]

Eq. (55) implies that the least square estimate of \( \mathbf{C} \) is given by

\[
\hat{\mathbf{C}} = \mathbf{A}^H (\mathbf{A A}^H)^{-1} \mathbf{D}
\]
Hence, to estimate the mutual coupling matrix \( C \), we need to know the direction and signal sequence of the \( n \) calibration sources. For more elaborate algorithms to estimate \( C \) without knowing the signal sequence, one can refer to See’s method \([27]\). Using the estimated \( \hat{C} \), the array output can be calibrated as

\[
\hat{y}_{\text{Est}}(\Theta, t) = \hat{C}^{-1} y_{\text{Est}}(\Theta, t)
\]

before implementing the weighted inner product estimator.

7. Simulation Results

In this subsection, simulation examples are presented to verify the performance of the weighted inner product estimator and the analytical predictions in Section 3. A single-source single-vector-sensor scenario with the source parameters \( \Theta = [22.92^\circ, 11.46^\circ, 28.65^\circ, 23.57^\circ] \) is considered.

In the first simulation, three cases of Theorem 2, i.e.,

\[
1 < \frac{\sigma_i^2}{\sigma_j^2} \neq \| w_i \|_2 / \| w_j \|_2,
\]

\[
1 = \frac{\sigma_i^2}{\sigma_j^2} = \| w_i \|_2 / \| w_j \|_2,
\]

\[
1 \gg \frac{\sigma_i^2}{\sigma_j^2} \neq \| w_i \|_2 / \| w_j \|_2,
\]

are taken into consideration. Each case is implemented by adjusting the noise and source signal powers. The estimation mean square errors (MSEs) of \( \hat{x}_u \) and \( \hat{y}_u \) in Eq. (31) and Eq. (32) are respectively set as shown in Fig. 1(a). Namely, the first 150 MSEs of \( \hat{x}_u \) in Fig. 1(a) are designed much larger than those of \( \hat{y}_u \) and reverse them for the last 200 estimated MSEs, while the MSEs of the two estimators between the 151st and 350th data are almost set the same. To validate the effectiveness of the weighted inner product estimator, a window of length \( M = 50 \) is chosen. Following the steps in Section 4.2, the MSEs of the weighted inner product and cross product estimators are shown in Fig. 1(b). In the middle of Fig. 1(b), it can be seen that the MSEs of the weighted inner product and cross product estimators are almost the same, which verifies the declaration of Theorem 2 when the requirement in Eq. (46) is met. Moreover, when the requirement in Eq. (46) is not met, the estimate MSEs of our weighted inner product estimator for the first 150 and last 200 data shown in Fig. 1(b) are smaller than those of the cross product one. Such results validate that the estimate errors of the weighted inner product estimator are reduced by the given weighted process, as predicted by Theorem 2.

In the second simulation, the statistical performances of the weighted inner product estimator are tested by Monte Carlo runs. The root mean square (RMS) errors are calculated by

\[
\text{RMS}(\hat{u}) = \sqrt{\| \hat{u} - u \|^2},
\]

where \( \hat{u} \) is the estimate results respectively obtained by the estimator-x, estimator-y, weighted inner product or cross product estimators while \( u \) is the true one. The same source parameters as the first simulation are used, the snapshots for \( \hat{u}_x \) and \( \hat{u}_y \) are \( K = 1000 \) and the window length is \( M = 50 \). For each of the nine signal-to-noise ratios (SNRs) ranging from \(-20\) to \(-20\) dB, 300 estimates of the estimator-x, estimator-y, weighted inner product and cross product estimators are used to obtain their respective average RMS estimate errors. Figure 2(a) depicts their results when the requirement in Eq. (46) is satisfied, i.e., \( 1 = \sigma_i^2 / \sigma_j^2 = \| w_i \|_2 / \| w_j \|_2 \). It can be seen from Fig. 2(a) that the RMS errors of the weighted inner product and the cross product estimators are almost the same, which confirms the statement of Theorem 2 again. Slight difference among the RMS errors is due to the fact that the finite data length of \( K \) is used. Figure 2(b) illustrates the RMS error analyses of a scenario where \( 1 \gg \sigma_i^2 / \sigma_j^2 \neq \| w_i \|_2 / \| w_j \|_2 \). From it, one can find that the RMS errors of the cross product estimator are larger than those of the estimator-x and smaller than those of the estimator-y, while the RMS errors of the weighted inner product estimator are the smallest. These results confirm that the weighted inner product estimator is the best.
The performance degradation due to the mutual coupling is studied in the third experiment. The configuration of the vector-sensor and the source are identical to that of Fig. 2(b), but the array data are perturbed by a coupling matrix given by

\[
C = \begin{bmatrix}
C_1 & C_2 \\
C_2 & C_1
\end{bmatrix}
\]

where \(C_1\) and \(C_2\) are symmetrical Toeplitz matrices with the first rows given by \(c_1=[1, 0.2+0.1j, 0.2+0.1j]\) and \(c_2=[0.15+0.15j, 0.1+0.05j, 0.1+0.05j]\) respectively. The simulation results with un-calibrated sensor data are displayed in Fig. 3(a), whereas those with calibrated sensor data using Eq. (58) are depicted in Fig. 3(b). Notice that, as expected in Section 6, the mutual coupling perturbation deteriorates the performances of all estimators. The RMS errors for the un-calibrated sensor data are almost a constant with increasing SNRs. By contrast, after the calibration process, the RMS errors of all the estimators decrease to zero with the increasing SNRs. The above results validate the correctness of the analytical predictions and the compensation approach of Section 6.

8. Conclusions

The estimation of the propagation direction vector of an EM wave has been investigated in the framework of the G3 by utilizing the measurement data from a single EM vector-sensor. In particular, two estimators for the direction vector are first reported by the inner product between a vector and a bivector in the G3. Since the noise components for the two estimators are uncorrelated, a weighted inner product estimator, in the sense of MMSE, is proposed by fusing the two estimators together. We have analytically proved that the weighted inner product estimator is always statistically optimal. The performances of the weighted inner product estimator are always superior over its cross product counterpart.

References


Biographies:

JIANG Jingfei received B.S. and M.S. degrees in the Electronic Engineering Department from Fudan University, in 2008 and 2011 respectively. His research interests include array signal processing and geometric algebra with applications in signal processing and image processing. E-mail: 082021030@fudan.edu.cn

ZHANG Jianqiu received B.S. degree from the Electronic Engineering Department from East of China Institute of Engineering, Nanjing, China, in 1982, M.S. and Ph.D. degrees from the Department of Electrical Engineering, Harbin Institute of Technology (HIT), Harbin, China, in 1992 and 1996, respectively. He is currently a professor with the Department of Electronic Engineering, Fudan University, Shanghai, China. From 1999 to 2002, he was a Senior Research Fellow at the School of Engineering, University of Greenwich, Medway Campus, Chatham Maritime, U.K. In 1998, he was a visiting research scientist at the Institute of Intelligent Power Electronics, Helsinki University of Technology, Espoo, Finland. He was an associate professor from 1995 to 1997 and a lecturer from 1989 to 1994 with the Department of Electrical Engineering, HIT. From 1982 to 1987, he was an assistant electronic engineer at 544th factory, Hunan, China. His main research interests are signal processing and its application for advanced sensors, intelligent instrumentation systems and control, and communications. E-mail: jqzhang@fudan.ac.cn

Appendix A: Proof for Theorem 1

Let \( z(t) = 5(Y_1(t))_t \cdot (e_123 Y_3(t))_t \), we have

\[
E[z(t)] = E(a_x)u + e_123E(B_x) + e_123E(C_x) + E(d_x) \tag{A1}
\]

where \( a_x(t) = -x(t) \cdot x(t), B_x(t) = x(t) \wedge n_{1x}(t), C_x(t) = n_{2x}(t) \wedge n_{1x}(t), \) and \( d_x(t) = n_{3x}(t) \cdot (u \wedge x(t)) \). Let \( x(t) = x_1 e_1 + x_2 e_2 + x_3 e_3 \) and \( n_{1x}(t) = n_{1x1} e_1 + n_{1x2} e_2 + n_{1x3} e_3 \), then

\[
B_x(t) = x(t) \wedge n_{1x}(t) = (x_1 n_{1x2} - x_2 n_{1x1}) e_1 + (x_2 n_{1x3} - x_3 n_{1x2}) e_2 + (x_3 n_{1x1} - x_1 n_{1x3}) e_3 \tag{A2}
\]

Since the sources and the noise are uncorrelated (see Assumption 4) and \( x(t) \) is a linear combination of the real and imaginary parts of the source signal, we have \( E(B_x(t)) = 0 \). Similarly, under Assumption 4, we also have \( E(C_x(t)) = 0 \). Using the rule that \( a \cdot (b \wedge c) = (a \cdot b) c - (a \cdot c) b \) and the well-known \((a \cdot c)b = a \wedge (b \cdot c), d_x(t)\) can be rewritten as

\[
d_x(t) = -n_{3x}(t) \times (u \times x(t)) \tag{A3}
\]
Similar to $x(t)$, $u^*x(t)$ is also a linear combination of the real and imaginary parts of the source signal. Like Eq. (A2), we also have $E(d_k(t))=0$. With the above results, it is clear that

$$E(z(t)) = E(a_k)u = -[p^2_x E(s^2_x(t)) + p^2_y E(s^2_y(t)) + p^2_z E(s^2_z(t))] u$$

(A4)

Let $\sigma^2_z = E(s^2_z(t))$ and $\sigma^2_w = E(s^2_w(t))$, then $\sigma^2_z = (p^2_x + p^2_y + p^2_z) \sigma^2_w + p^2_w \sigma^2_y$ is a finite constant. As a result, the expectation of $z(t)$ is a constant vector with the same direction as $u$. Since $z(t)$ is an independent identical distribution (IID) random vector with finite constant expectation, by the Kolmogorov strong law of large numbers $^{[27]}$, we have

$$w_t = \frac{1}{K} \sum_{i=1}^{K} z(t) \xrightarrow{a.s.} -p^2_z u$$

(A5)

Since $u$ is a unit vector, we have

$$\hat{u}_t = \frac{w_t}{\|w_t\|_2} \xrightarrow{a.s.} -p^2_z u$$

(A6)

With similar process, one can also obtain that

$$\hat{u}_t = \frac{w_t}{\|w_t\|_2} \xrightarrow{a.s.} -p^2_x u$$

(A7)

where $p^2_z = (p^2_x + p^2_y + p^2_z) \sigma^2_w + p^2_w \sigma^2_y$.

Appendix B: Proof for Theorem 2

As proved in Appendix A, when $K$ is sufficiently large we have

$$E(w_x) = -p^2_x u, \quad E(w_y) = -p^2_y u$$

(B1)

and

$$\|w_x\|_2 = p^2_x, \quad \|w_y\|_2 = p^2_y$$

(B2)

Similar to the facts shown in Appendix A, we can get

$$E(w_z) = -p^2_z u$$

(B3)

and

$$\|w_z\|_2 = p^2_x + p^2_y$$

(B4)

Rewrite $w_z$ as $w_z = \sum_{i=1}^{3} w_{z_{ei}}$, and let the variance of $w_{zi}$ be $\sigma^2_{w_{zi}}$. Since $w_z = \frac{1}{K} \sum_{i=1}^{K} z(t)$ and $z(t)$ is a IID random vector (see Appendix A) with a constant expectation $-p^2_z u$ and a constant variance matrix generated by the random vector $e_{123} \mathbf{B} + e_{123} \mathbf{C} + d_k$, by the Lindberg-Levy central limit theorem $^{[25]}$, we have

$$w_{zi} \sim N\left(-p^2_z u, \sigma^2_{w_{zi}}\right)$$

(B5)

where $N(\mu, \sigma^2)$ denotes a Gauss random process with a mean equivalent to $\mu$ and a variance given by $\sigma^2$. As a result, we have

$$u_{zi} = \frac{w_{zi}}{\|w_{zi}\|_2} \sim N\left(u, \sigma^2_{u_{zi}}\right)$$

(B6)

where $\sigma^2_{u_{zi}} = \sigma^2_{w_{zi}}/p^4_z$.

Similarly, we also have

$$u_{zi} = \frac{w_{zi}}{\|w_{zi}\|_2} \sim N\left(u, \sigma^2_{u_{zi}}\right)$$

(B7)

$$u_{zi} = \frac{w_{zi}}{\|w_{zi}\|_2} \sim N\left(u, \sigma^2_{u_{zi}}\right)$$

(B8)

where $\sigma^2_{u_{zi}} = \sigma^2_{u_{zi}}/p^4_z$ and $\sigma^2_{u_{zi}} = (\sigma^2_{w_{zi}} + \sigma^2_{w_{yi}})/(p^2_z + p^2_z)$. As shown in Eq. (36), the optimal variance of the weighted inner product estimator is given by

$$\sigma^2_{u_{zi}} = \frac{1}{1 + \frac{1}{\sigma^2_{u_{zi}}}} = \frac{\sigma^2_{w_{zi}}/\sigma^2_{w_{yi}}}{p^2_z + \sigma^2_{w_{yi}}/\sigma^2_{w_{zi}}}$$

(B9)

Thus

$$\sigma^2_{u_{zi}} - \sigma^2_{u_{zi}} = \frac{(\sigma^2_{w_{zi}}/\sigma^2_{w_{yi}} - \sigma^2_{u_{zi}})^2}{(p^2_z + p^2_z) (p^2_z + \sigma^2_{w_{yi}} + p^2_z)} \geq 0$$

(B10)

in which the equality holds if and only if $p^2_z \sigma^2_{w_{zi}} - p^2_z \sigma^2_{w_{yi}} = 0$, i.e.,

$$\sigma^2_{u_{zi}} = \frac{\|w_{zi}\|_2}{\|w_{zi}\|_2}$$

(B11)