Abstract

In this talk we analyze the problem of searching for optimal/equilibrium rules in the case where decision-makers have time-inconsistent preferences, within a stochastic framework. For the so-called sophisticated agents, we derive a modified Hamilton-Jacobi-Bellman equation describing the sub-game perfect equilibria, which extends the equation for a deterministic problem (see [7] for the case of an infinite horizon setting, and [10] for the finite horizon case with free terminal time). This modified HJB equation is used in order to solve (for some utility functions) the classical consumption and portfolio rules model when the instantaneous discount rate of time preference is non-constant.

A relevant result is that for the CRRA (logarithmic and potential) utility functions, the portfolio rule coincides for the pre-commitment, naive and sophisticated solutions. Moreover, it is independent of the discount factor, and thus coincides with the standard solution when the discount factor is an exponential with constant discount rate. This property is no longer satisfied for more general utility functions, such as the CARA (exponential) function. With respect to the consumption rule, it is proved that, in the log-utility case, it coincides for naive and sophisticated agents.

Keywords: Consumption and portfolio rules, Non-constant discounting, Time inconsistency, Naive and sophisticated agents, Bellman equation
1 Introduction

Variable rate of time preferences have received considerable attention in recent years. Virtually every experimental study on time preferences suggests that the standard assumption of time-consistency (related to the assumption of constant discount rate of time preference) is unrealistic (see, for instance, [17], [1] or [9]). In fact, there is substantial evidence that agents are impatient about choices in the short term but are patient when choosing between long-term alternatives. Effects of the so called hyperbolic discount functions, introduced by Phelps and Pollak [13], have been extensively studied in a discrete time context, within the field of behavioural economics. Laibson [8] has made compelling observations about ways in which rates of time preference vary. However, this topic has received less attention in a continuous time setting. The main reason for this may be the complexity involved in the search for solutions in closed form in the non-constant discounting case. In fact, standard optimal control techniques cannot be used in this context, since they give rise to non-consistent policies. The most relevant effect of non-constant discounting is that preferences change with time. In this sense, an agent making a decision in time $t$ has different preferences compared with those at time $t'$. Therefore, we can consider him or her at different times as different agents. An agent making a decision at time $t$ is usually called the $t$-agent. If the horizon planning is a finite interval $[0, T]$, we can understand the dynamic optimization problem with non-constant discounting as a perfect information sequential game with a continuous number of players (the $t$-agents, for $t \in [0, T]$) making their decisions sequentially. A $t$-agent can act in two different ways: naive and sophisticated. Naive agents take decisions without taking into account that their preferences will change in the near future. Then, they will be continuously modifying their calculated choices for the future, and their decisions will be in general non-time consistent. In order to obtain a time consistent strategy, the $t$-agent should be sophisticated, in the sense of taking into account the preferences of all the $t'$-agents, for $t' \in (t, T]$. Therefore, the solution to the problem of the agent with non-constant discounting should be constructed by looking for the subgame perfect equilibria of the associated game with an infinite number of $t$-agents. Historically, in that part of his analysis allowing for time preference, Ramsey [15] assumed an exponential discount factor with constant discount rate, stating: “This is the only assumption we can make, without contradicting our fundamental hypothesis that successive generations are activated by the same system of preferences”. The main property of non-constant discounting is implicit in this statement: it can create a time-consistency problem. In fact, Strotz [16] illustrated how, for a very simple model, preferences are time consistent if, and only if, the discount factor representing time preferences is an exponential with a constant discount rate. In order to avoid such time inconsistency, agents could decide in a sophisticated way, making an analysis of what their actions would be in the future, as a consequence of their changing preferences. For instance, Pollak [14] gave the right solution to the Strotz problem for both naive and sophisticated agents under a logarithmic utility function.
Although the problem was first presented in a continuous time context [16], almost all attention has been given to the discrete time setting introduced by Phelps and Pollak [13]. This is probably a consequence of the non-existence of a well-stated system of equations giving a general method for solving the problem, at least for sophisticated agents. Therefore, each particular problem has been solved individually. This was the case of the Strotz model solved by Pollak in 1968 [14]. Barro [2] studied a modified version of the neoclassical growth model by including a variable rate of time preference.

For the case of naive agents, one should solve a standard optimal control problem for each time $t \in [0, T]$, in order to find the decision rule at time $t$ of a $t$-agent. Unfortunately, this method cannot be used if the agent is sophisticated. Instead, Markov subgame perfect equilibria must be found. This prompts the use of a dynamic programming approach, applying the Bellman optimality principle.

To solve the intra-personal game for sophisticated agents, a continuous-time model of quasi-hyperbolic time preferences was introduced in [6]. In their model, the discount rate declines during the first “period” (instantaneous gratification), and then becomes constant. Grenadier and Wang [5] employed this model to extend the real options framework in order to analyze the investment-timing decisions (in an irreversible investment framework) of entrepreneurs with time-inconsistent preferences.

In a deterministic environment, Karp [7] adapted the approach by Harris and Laibson [6] for the general case of an arbitrary discount rate of time preference, for autonomous infinite time horizon problems. The case of non-autonomous problems in finite horizon (paying special attention to the free terminal time case) was analyzed in [10].

In this paper we extend the results in [7] and [10] to a stochastic environment, in order to analyze how time-inconsistent preferences modify the classical optimal consumption and portfolio rules when the discount rate is constant ([11], [12]). As expected, the rate of time preference plays no role in a pure optimal portfolio management problem. However, if the consumption is introduced in the model, an inter-temporal conflict arises. We show that, within the HARA (hyperbolic absolute risk aversion) functions, if the relative risk aversion is constant (logarithmic and potential utility functions), the optimal portfolio rule does not depend on the rate of time preference, although the consumption rule changes. This nice property is not satisfied for more general utility functions, such as the (constant absolute risk aversion) exponential function.

The paper is organized as follows. In Section 2 we describe the model. The general stochastic dynamic optimization problem with non-constant discount rate of time preference is studied in Section 3, and the dynamic programming (Hamilton-Jacobi-Bellman) equation is derived. In Section 4, this equation is solved for the “optimal” (in fact, equilibrium) consumption and portfolio rules problem for some particular utility functions. The so-called pre-commitment, naive and sophisticated solutions are compared, and the issue of observational equivalence with a problem with constant discount rate is briefly discussed. Finally, Section 5 contains the main conclusions of the paper.
2 The Model

Let \( x = (x^1, \ldots, x^n) \) be the vector of state variables, and \( u = (u^1, \ldots, u^m) \) the vector of control (or decision) variables. In the conventional model, agent preferences at time \( t \) take the form

\[
U_t = E \left[ \int_t^T e^{-\rho(s-t)} L(x(s), u(s), s) \, ds + e^{-\rho(T-t)} F(x(T)) \right],
\]

where the state variables evolve according to the diffusion equations

\[
dx^i(s) = f^i(x(s), u(s), s) \, ds + \sum_{l=1}^L \sigma^i_l(x(s), u(s), s) \, dw^l(s), \quad x^i(t) = x^i_t,
\]

for \( i = 1, \ldots, n \), where \((w^1(s), \ldots, w^L(s))\) is an \( L \)-dimensional Wiener process with independent components \((dw^1(s)dw^l(s) = 0, \text{ for } l \neq l')\). We will denote \( \Sigma = (\sigma^i_l) \), for \( i = 1, \ldots, n \), \( l = 1, \ldots, L \).

In order to maximize \( U_t \), we must solve a stochastic optimal control problem, and since the discount rate is constant, the solution becomes time consistent.

Now, following Karp [7], let us assume that the instantaneous discount rate is non-constant, but a function of time \( r(s) \), for \( s \in [0, T] \). Impatient agents will be characterized by a non-increasing discount rate \( r(s) \). The discount factor at time \( t \) used to evaluate a payoff at time \( t+\tau \), \( \tau \geq 0 \), is \( \theta(\tau) = \exp \left( - \int_0^\tau r(s) \, ds \right) \).

Then, the objective of the agent at time \( t \) (the \( t \)-agent) will be

\[
\max_{\{u(s)\}} E \left[ \int_t^T \theta(s-t) L(x(s), u(s), s) \, ds + \theta(T-t) F(x(T)) \right].
\]

In Problem (2-3), we assume the usual regularity conditions, i.e., functions \( L, F, f^i \) and \( \sigma^i_l \) are continuously differentiable in all their arguments.

In the discrete time case, most papers work with the so-called hyperbolic discounting, first proposed by Phelps and Pollak [13]. The utility function is defined as \( U_t = u_t + \beta (u_{t+1} + \delta^2 u_{t+2} + \delta^3 u_{t+3} + \cdots) \), where \( 0 < \beta \leq 1 \), and \( u_k \) denotes the utility in period \( k \). In fact, Laibson [8] argues that \( \beta \) would be substantially less than one on an annual basis, perhaps between one-half and two-thirds. Harris and Laibson [6] adapted this inter-temporal utility function to the continuous time setting. As a natural extension of the above discount function to the continuous setting, Barro [2] suggested the instantaneous discount rate \( r(\tau) = \rho + be^{-\gamma \tau} \), where \( b \geq 0 \) and \( \gamma > 0 \) (for the general case, he defined the discount factor as \( \theta(\tau) = e^{-[\rho \tau + \gamma(\tau)]} \)). In other applications it is natural to assume that the discount factor is a linear combination of exponentials with constant but different discount rates. In this paper we will not assume any particular discount function.

In particular, in this paper we are interested in the optimal consumption and portfolio rules in continuous time studied by Merton [11], [12]. Let us assume that there are \( m \) risky assets and one risk-free asset. The risk-free asset pays
a constant rate $\mu_0$, while the return of $i$-th risky asset follows a geometric Brownian motion
\[ dP_i = \mu_i P_i ds + \sigma_i P_i dz_i, \quad i = 1, \ldots, m, \]
with $dz_i dz_j = \rho_{ij} dt$ for $i, j = 1, \ldots, m, i \neq j$, and $\mu_i, \sigma_i$ are constants.

If $w_i$ is the share of wealth invested in the $i$-th risky asset, and $c$ denotes the consumption, the consumer’s budget equation is
\[ dW = \left[ \sum_{i=1}^{m} w_i (\mu_i - \mu_0) W + (\mu_0 W - c) \right] ds + \sum_{i=1}^{m} w_i \sigma_i W dz_i. \tag{4} \]
with the initial condition $W_0$. Then the consumer-investor’s problem is
\[ \max \left\{ c, w_j \right\} E \left[ \int_0^T \theta(s-t) u(c_s) ds + \theta(T-t) F(x(T)) \right] \tag{5} \]
s.t. (4) with the initial condition $W(t) = W_t$.

3 Dynamic Programming Equation

For the solution of Problem (2-3) (and, in particular, Problem (4-5)), if the agent is naive, then we can adapt the standard techniques of stochastic optimal control theory as follows. If $V^0 = V^0(x,s)$ is the value function, the $0$-agent will solve the standard Hamilton-Jacobi-Bellman equation
\[ r(s)V^0 - V^0_s = \max_{u} \left\{ L + V^0_x \cdot f + \frac{1}{2} \text{tr} \left( \Sigma \Sigma' V^0_{xx} \right) \right\}, \quad x(0) = x_0, \]
i.e., the naive agent at time 0 solves the problem, assuming that the discount rate of time preference will be $r(s)$, for $s \in [0,T]$. In the equation above we denote $V^0_x = \left( \frac{\partial V^0}{\partial x_1}, \ldots, \frac{\partial V^0}{\partial x_n} \right)$, and $V^0_{xx} = \left( \frac{\partial^2 V^0}{\partial x_i \partial x_j} \right)$, for $i, j = 1, \ldots, n$. The optimal control will be a function $u^0(s)$. In our framework of changing preferences, this solution corresponds to the so-called pre-commitment solution, in the sense that it is optimal as long as the agent can precommit (by signing a contract, for example) his or her future behaviour at time $t = 0$, and it will be denoted by $V^0 = V^P$. If there is no commitment, the $0$-agent will take the action $u^0(0)$, but, in the near future, the $\epsilon$-agent will change his decision rule (time-inconsistency) to the solution of
\[ r(s-\epsilon)V^{\epsilon} - V^{\epsilon}_s = \max_{u} \left\{ L + V^{\epsilon}_x \cdot f + \frac{1}{2} \text{tr} \left( \Sigma \Sigma' V^{\epsilon}_{xx} \right) \right\}, \quad x(\epsilon) = x_\epsilon. \]
Once again the optimal control trajectory $u^\epsilon(s), s \in [\epsilon, T]$ will be changed for $s > \epsilon$ by the following $s$-selves. In general, the solution for the naive agent will be constructed by solving the family of HJB equations
\[ r(s-t)V^t - V^t_s = \max_{u} \left\{ L + V^t_x \cdot f + \frac{1}{2} \text{tr} \left( \Sigma \Sigma' V^t_{xx} \right) \right\}, \quad x(t) = x_t, \]
for \( t \in [0, T] \), and patching together the “optimal” solutions \( u^t(t) \).

If the agent is sophisticated, things become more complicated. The standard HJB equation cannot be used to construct the solution, and a new method is required. In what follows, we will derive a modified HJB equation which will help us to find the solution to Problem (2-3) and then (4-5).

### 3.1 The Modified HJB Equation

We first will derive the dynamic programming equation for a discretized version of Problem (2-3), following a procedure similar to the one used in [7] and [10]. Let us divide the interval \([0, T]\) of Problem (2-3), following a procedure similar to the one used in [7] and [10]. We first will derive the dynamic programming equation for a discretized version.

#### 3.1.1 The Modified HJB Equation

Help us to find the solution to Problem (2-3) and then (4-5).

\[
\max_{\{u_k\}} V_j = E \left[ \sum_{i=0}^{N-j-1} \theta(i\epsilon)L(x_{i+j}, u_{i+j}, \epsilon) + \theta((N-j)\epsilon)F(x(T)) \right], \quad (6)
\]

\[
x_{k+1}^i = x_k^i + f(x_k^i, u_k^i, \epsilon) + \sum_{l=1}^{L} \sigma_l^i(x_k^i, u_k^i, \epsilon)(w_{k+1}^l - w_k^l), \quad (7)
\]

for \( i = 1, \ldots, n \) and \( k = j, \ldots, N - 1 \), with \( x_j \) given.

Let us state the dynamic programming algorithm for the discrete problem (6-7).

In the final period, \( t = N\epsilon = T \), we define \( V_N^* = F(x(T)) = F(x_N) \), as usual. For \( j = N - 1 \), the optimal value for (6) will be given by the solution to the problem

\[
V_{N-1}^* = \max_{\{u_{N-1}\}} E \left[ L(x_{N-1}, u_{N-1}, (N-1)\epsilon) + \theta_1 V_N^* \right],
\]

\[
x_N^* = x_{N-1}^* + f(x_{N-1}, u_{N-1}, (N-1)\epsilon) + \sum_{l=1}^{L} \sigma_l^i(x_{N-1}, u_{N-1}, (N-1)\epsilon)(w_N^l - w_{N-1}^l), \quad (8)
\]

for \( i = 1, \ldots, n \). If

\[
u_{N-1}(x_{N-1}, (N-1)\epsilon) = \arg \max_{\{u_{N-1}\}} E \left[ L(x_{N-1}, u_{N-1}, (N-1)\epsilon) + \theta_1 V_N^* + f(x_{N-1}, u_{N-1}, (N-1)\epsilon) + \sum_{l=1}^{L} \sigma_l^i(x_{N-1}, u_{N-1}, (N-1)\epsilon)(w_N^l - w_{N-1}^l) \right],
\]

let us denote

\[
H_{N-1}(x_{N-1}, (N-1)\epsilon) = L(x_{N-1}, u_{N-1}^*(x_{N-1}, (N-1)\epsilon), (N-1)\epsilon). \quad (9)
\]

6
In general, for $j = 1, \ldots, N - 1$, the optimal value in (6) can be written as

$$V_j^* = \max_{\{u_j\}} E[E[L(x_j, u_j, j\epsilon)] + \sum_{k=1}^{N-j-1} \theta_k H_{j+k}(x_{j+k}, (j + k)\epsilon) + \theta_{N-j} V_{n+1}^*]. \quad (10)$$

Since

$$V_{j+1}^*(x_{(j+1)}, (j+1)\epsilon) = \sum_{i=0}^{N-j-2} \theta_i H_{j+i+1}(x_{j+i+1}, (j+i+1)\epsilon) + \theta_{N-j-1} V_{n+1}^*, \quad (11)$$

then, solving $\theta_{N-j-1} V_{n+1}^*(x_N)$ in (11) and substituting in (10) we obtain:

**Proposition 1** For every initial state $x_0$, the equilibrium value $V^+(x_i)$ of problem (7-6) can be obtained as the solution of the following algorithm, which proceeds backward in time from period $N - 1$ to period 0:

$$V_N^* = F(x(T)), \quad (12)$$

$$\theta_{N-j-1} V_j^*(x_j, j\epsilon) = \max_{\{u_j\}} [\theta_{N-j-1} L(x_j, u_j, j\epsilon) +$$

$$+ \sum_{k=1}^{N-j-1} (\theta_{N-j-1}\theta_k - \theta_{N-j}\theta_{k-1}) H_{j+k}(x_{j+k}, (j + k)\epsilon) +$$

$$+ \theta_{N-j} V_{j+1}^*(x_{j+1}, (j+1)\epsilon)] \quad (13)$$

$$x_{j+1}^i = x_j^i + f^j(x_j, u_j, j\epsilon) + \sum_{l=1}^L \sigma_l^j(x_j, u_j, j\epsilon)(w_{j,l}^l - w_{j,l}^i), \quad (14)$$

for $i = 1, \ldots, n, j = 0, \ldots, N - 1$.

Equations (12-14) are the equilibrium dynamic programming equations in discrete time, and their solution is the Markov Perfect Equilibrium (MPE) solution to problem (7-6).

In order to derive the modified Hamilton-Jacobi-Bellman equation for the problem with non-constant discounting in continuous time for a sophisticated agent, we take the limit $\epsilon \to 0$ of the dynamic programming equations of the discrete stage equilibrium problem described by (12-14).

Let $V^S(x_t, t)$ represent the value function of the sophisticated $t$-agent, with initial condition $x(t) = x_t$. We assume that $V^S(x_t, t)$ is of class $C^1$ in $t$, and of class $C^2$ in $x$. Since $t = j\epsilon$ and $x^i(t + \epsilon) = x^i(t) + f^i(x(t), u(t), t\epsilon) + \sum_{l=1}^L \sigma_l^i(x(t), u(t), t)(w^i(t + \epsilon) - w^i(t))$, then $V^S(x_t, t) = V_j^*(x_j, j\epsilon)$ and

$$V^S(x_{t+\epsilon}, t + \epsilon) = V^S(x_t, t) + \left[V^S_t + V^S_x \cdot f + \frac{1}{2} \text{tr} \left( \sum_{l=1}^L \sigma_l^S \right) \right]_{(x_t, u(t), t)} +$$

$$+ \sum_{i=1}^n \sum_{l=1}^L \frac{\partial V^S}{\partial x^l} \bigg|_{(x_t, t)} (w^l(t + \epsilon) - w^l(t)) + o(\epsilon),$$

where $o(\epsilon)$ denotes a term that is negligible as $\epsilon \to 0$. This equation allows us to derive the modified Hamilton-Jacobi-Bellman equation, which is given by:

$$H(x(t), u(t), t) \cdot f^i(x(t), u(t), t\epsilon) + \frac{1}{2} \text{tr} \left( \sum_{l=1}^L \sigma_l^S \right) w^l(t + \epsilon) - w^l(t) +$$

$$+ \sum_{i=1}^n \frac{\partial V^S}{\partial x^l} \bigg|_{(x_t, t)} (w^l(t + \epsilon) - w^l(t)) + o(\epsilon),$$

where $H(x(t), u(t), t)$ is the Hamiltonian function, $f^i(x(t), u(t), t)$ is the system dynamics, $\sigma_l^S$ are the vector fields associated with the controls, and $V^S(x_t, t)$ is the value function.
where \( \lim_{\epsilon \to 0} \frac{o(\epsilon)}{\epsilon} = 0 \). Since \( \theta_k = \exp \left( - \int_0^{k \epsilon} r(s) \, ds \right) \), then

\[
\theta_{N-j} = \theta_{N-j-1} [1 - r((N-j)\epsilon)\epsilon] + o(\epsilon) = \theta_{N-j-1} [1 - r(T-t)\epsilon] + o(\epsilon) ,
\]

\[
\theta_{k-1} = \theta_k [1 + r(k\epsilon)\epsilon] + o(\epsilon)
\]

and substituting in (13) and simplifying we obtain

\[
0 = \max_{\{u(t)\}} E \left[ \left( L + V_t^S + V_x^S \cdot f + \frac{1}{2} tr \left( \Sigma \Sigma' V_{xx}^S \right) - r(T-t) V^S(x_t, t) - K \right) \epsilon \right.
\]

\[
+ \sum_{i=1}^n \sum_{l=1}^L \sigma_i \frac{\partial V^S}{\partial x^i} (w^j(t + \epsilon) - w^j(t)) + o(\epsilon) \left|_{(x_t, u(t), t)} \right]
\]

(15)

where \( K(x_t, t) \) is given by

\[
K(x_t, t) = E \left[ \sum_{k=1}^{n-1} \theta(k \epsilon) [r(k \epsilon) - r(T-t)] H_{t+k \epsilon}(x(t+k \epsilon), t + k \epsilon) \right] . \quad (16)
\]

Dividing equation (15) by \( \epsilon \), and taking the limit \( \epsilon \to 0 \) (and hence \( n \to \infty \)), since \( T = n \epsilon \) in (15) and (16), we obtain

\[
r(T-t)V^S + K - \frac{\partial V^S}{\partial \theta} = \max_{\{u\}} \left\{ L + V_x^S \cdot f + \frac{1}{2} tr \left( \Sigma \Sigma' V_{xx}^S \right) \right\} ,
\]

where

\[
K(x_t, t) = E \left[ \int_0^{T-t} \theta(s) [r(s) - r(T-t)] H(x(t+s), t+s) \, ds \right] =
\]

\[
= E \left[ \int_t^T \theta(s-t) [r(s) - r(T-t)] H(x_s, s) \, ds \right] ,
\]

and \( H(x_s, s) = L(x_s, u^*(x_s, s), s) \).

Finally, note that for the equilibrium rule \( u^* = u^*(x_s, s), s \in [t, T] \), by solving the stochastic differential equation (2) we can write \( x_{t+s} \) as a function of \( x_t \) and \( s \) \( x_{t+s} = x(x_t, s) \). Therefore, \( H(x_s, s) = H(x_t, s) \).

Hence, we have proved:

**Theorem 1** Let \( V^S(x, t) \) be the value function of the sophisticated t-agent with initial condition \( x(t) = x \). If \( V^S(x, t) \) is continuously differentiable in \( t \) and of class \( C^2 \) in \( x \), it satisfies the dynamic programming equation

\[
r(T-t)V^S + K - V_t^S = \max_{\{u\}} \left\{ L + V_x^S \cdot f + \frac{1}{2} tr \left( \Sigma \Sigma' V_{xx}^S \right) \right\} , \quad (17)
\]

\[
V^S(x, T) = F(x) , \quad (18)
\]
where

\[
K(x, t) = E \left[ \int_t^T \theta(s-t)[r(s-t) - r(T-t)]H(x, s)ds \right].
\] (19)

The solution \( V^S(x, t) \) to equations (17-19) is called the Markov Perfect equilibrium (MPE), and if \( u^* \) is the maximum in Equation (17), then the associated control trajectory \( u^*(x, t), t \in [0, T] \), is the equilibrium rule.

Note that in the case of constant discount rate \( r(t) = \rho \), for every \( t \in [0, T] \), we recover the usual Hamilton-Jacobi-Bellman equation \( (K = 0) \) in this case.

**Remark 1** Assume now that, in addition to the initial state \( x(0) = x_0 \), the final state \( x(T) = x_T \) is given. In this case, the terminal condition \( W(T, x) = F(x) \) makes no sense \( (x(T) \) is fixed). Instead, we have the extra condition \( x(T) = x_T \) in order to integrate the differential equations.

If there is no final function, equations (17-19) can be written as follows:

**Corollary 1** If, in Problem (2-3), there is no final function \( (F(x(T)) = 0) \), then the modified HJB equation can be written as

\[
\bar{K} - V^S_t = \max_{\{u\}} \left\{ L + V^S_x \cdot f + \frac{1}{2} tr(\Sigma \Sigma' V^S_{xx}) \right\},
\]

where

\[
\bar{K}(x, t) = E \left[ \int_t^T \theta(s-t)r(s-t)H(x, s)ds \right]
\]

and \( V^S(x, T) = 0 \).

**Proof:** Along the equilibrium path,

\[
V^S(x, t) = E \left[ \int_t^T \theta(s-t)H(x, s)ds + \theta(T-t)F(x(T)) \right].
\]

Therefore

\[
K(x, t) = E \left[ \int_t^T \theta(s-t)r(s-t)H(x, s)ds - r(T-t) \int_t^T \theta(s-t)H(x, s)ds \right] = \bar{K}(x, t) - r(T-t)V^S(x, t) - \theta(T-t)r(T-t)E[F(x(T))].
\]

Since \( F(x(T)) = 0 \), the result follows by substituting the expression above in (17).

If the discount rate is non-constant but the 0-agent can precommit at time \( t = 0 \) his or her future behaviour, then the corresponding (classical) HJB equation characterizing the precommitment solution becomes

\[
r(t)V^P_t - V^P_t = \max_{\{u\}} \left\{ L + V^P_x \cdot f + \frac{1}{2} tr(\Sigma \Sigma' V^P_{xx}) \right\}.
\] (20)
Note that, when comparing equations (20) and (17), there are two differences. First, the term \( r(t)V(x, t) \) in (20) changes to \( r(T - t)V^S(x, t) \). Second, and more importantly, a new term \( K(x, t) \) appears in equation (17). In fact, the implicit inclusion of the equilibrium rule \( u^*(x, t) \) in the definition of \( K(x, t) \) (via \( H(x, s) \)) makes things much more complicated in general when compared with the problem with constant discounting.

There is another (and more fundamental) difference between stochastic optimal control (SOC) theory and Problem (2-3) with non-constant discounting. In SOC, by using a stochastic a verification theorem, we can verify that the optimal control path obtained from the HJB equation is indeed the optimal control policy for the SOC problem. However, since the problem with non-constant discount rate of time preference is equivalent to a game with a continuous number of agents, each of whom wants to maximize the expected present discounted value of current and future welfare, the notion of optimality is substituted by that of Markov perfect equilibrium. If such equilibria are non unique, the concept of Pareto optimality should be applied here. Such non-uniqueness of candidate equilibria is usual in an infinite time setting, and was addressed in [7] (in a deterministic infinite horizon context), where a Pareto ranking of steady states was established. In the consumption and portfolio rules problem with a finite horizon planning studied in this paper the MPE is unique, and therefore this problem is avoided.

**Remark 2** For the deterministic autonomous infinite horizon problem, Karp [7] noted that the MPE in the problem with non-constant discount rate is produced by solving the necessary condition to an “auxiliary optimal control problem”. In this case, we obtain a similar result by considering the auxiliary problem with discount rate \( r(T - t) \) and Lagrangian (or instantaneous utility function) \( L(x, u, t) - K(x, t) \) where the decision maker treats the function \( K \) as exogenous. Moreover, if sufficiency conditions are satisfied \( L(x, u) - K(x) \) is concave in \( (x, u) \), where \( K(x, t) \) has been calculated previously from the solution to (17-19)), then the solution to the problem with non-constant discounting proves to be equivalent to the pre-commitment solution with discount rate \( r(T-t) \) and instantaneous utility function \( L - K \).

Things are much easier in the (Mayer) problem, where we are just interested in maximizing a final expected utility \( (L = 0 \text{ in (3)}) \). If the discount rate is constant, it is clear that the optimal solution is independent of the discount rate. It is straightforward to see that this property is preserved in the case of a non-constant discount rate of time preference, not only for the pre-commitment and naive solutions (where actually we are solving standard optimal control problems), but also for sophisticated agents. For instance, let \( V \) be the solution when \( r = 0 \). Then \( V \) verifies \( -V_t = \max_{\{u\}} \left\{ V_x \cdot f + \frac{1}{2} \text{tr} (\Sigma \Sigma' V_{xx}) \right\} \). Now, if \( V^S \) is the value function for a sophisticated agent with (arbitrary) non-constant discounting, from (17) we obtain \( (K = 0 \text{ in this case}) \) \( V^S(x, t) = \theta(T - t)V(x, t) \). Therefore, although the value function changes in a factor \( \theta(T - t) \), the optimal/equilibrium control-state pair coincides for both problems. Hence, in a pure
optimal portfolio management problem such as Problem (4-5) where we omit consumption \((u(c) = c = 0 \text{ in the model})\), the introduction of time-inconsistent preferences does not add anything new. However, in problems where the final time \(T\) is a decision variable (\(T\) is not prefixed), the changing preferences of the decision-maker will modify the optimal solution, in general (the “optimal” final time will be different for naive and sophisticated agents, and for different discount rates, see [10]).

3.2 A Simple Example

Before proceeding with the Merton’s model for time-inconsistent investors, let us solve an extension of the stochastic version of the model used by Strotz [16] (and later on solved by Pollak [14]) in order to illustrate the main features of non constant discounting. The model is the following:

\[
\max_c E \left[ \int_t^T \theta(s-t) \ln c \, ds \right], \quad \text{for } t \in [0,T], \quad (21)
\]

\[
dk = (\mu k - c) d\tau + \sigma k dw, \quad k(t) = k_t, \quad k(T) = 0. \quad (22)
\]

If \(\mu = \sigma = 0\) we recover the Strotz’s model, a Ramsey model with no production function in finite horizon, which can be understood as a problem of consumption of a non renewable resource. Let us assume that \(u(c) = \ln c\).

If the discount rate is constant, \(\theta(s-t) = e^{-\rho(s-t)}\), we must solve the HJB equation

\[
\rho V - V_t = \max_c \left\{ \ln c + V_k (\mu k - c) + \frac{1}{2} \sigma^2 k^2 V_{kk} \right\}. \quad (23)
\]

From symmetry considerations (see, for instance, [3]), the value function is necessarily of the form \(V(k,t) = \lambda(t) \ln k + f(t)\), and the optimal consumption becomes

\[
c^* = \frac{r}{1 - e^{-\rho(T-t)} k^*}. \quad (24)
\]

Solution for a naive agent: Naive \(t\)-agents will look for the solution to the HJB equation

\[
r(\tau-t)V^N - V^N_t = \max_c \left\{ \ln c + V^N_k (\mu k - c) + \frac{1}{2} \sigma^2 k^2 V^N_{kk} \right\} \quad (25)
\]

with \(k(t) = k_t\). Once again, since the naive \(t\)-agent is solving a standard stochastic optimal control problem with discount factor \(r(\tau-t)\), for \(\tau \in [t,T]\), the previous symmetry argument applies and \(V^N(k,\tau) = \lambda^N(\tau) \ln k + f^N(\tau)\). Now, \(c(\tau) = k(\tau)/\lambda^N(\tau)\), for some function \(\lambda^N(\tau)\). After several calculations, we find that the feedback control law is given by \(c(\tau) = \frac{r}{T - \theta(s-t)} ds\). Since the \(t\)-agent will not be time consistent for \(\tau > t\), the actual consumption rule is obtained from the equation above for the case \(\tau = t\), and therefore

\[
c^N(t) = \frac{k^*_t}{\int_t^T \theta(s-t) \, ds}. \quad (26)
\]

Solution for a sophisticated agent: We must solve equations (17-19). From the maximization problem, \(c = 1/V^S_k\) and Equation (17) becomes

\[
r(T-t)V^S + K - V^S_t = -\ln V^S_k + \mu k V^S_k - 1 + \frac{1}{2} \sigma^2 k^2 V^S_{kk}. \quad (27)
\]
In order to find the value function, the remarkable fact is that the symmetry analysis developed in [3] can be extended to the problem addressed by the sophisticated agent. Since the value function satisfies the same symmetry as in the previous problems, it will be necessarily of the form \( V^S(k, t) = \lambda^S(t) \ln k + f^S(t) \). By denoting \( \Lambda^S(t) = (\lambda^S(t))^{-1} \), the solution to the stochastic differential equation (22) is

\[
k(s) = k_t e^{(\mu - \frac{1}{2} \sigma^2)(s-t) - \int_t^s \Lambda^S(\tau) d\tau + \sigma w(s) - w(t)}.
\]

Then,

\[
K = E \left[ \int_t^T \theta(s-t) [r(s-t) - r(T-t)] \ln (\Lambda^S(s)k(s)) \, ds \right] =
\]

\[
eq E \left[ \int_t^T \theta(s-t) [r(s-t) - r(T-t)] \left( \ln \Lambda^S(s) + \ln k_t + (\mu - \frac{1}{2} \sigma^2)(s-t) - \int_t^s \Lambda^S(\tau) d\tau + b(w(s) - w(t)) \right) \, ds \right].
\]

Since \( V^S_k = \frac{\lambda^S(t)}{k} \), by substituting in (24) and simplifying we obtain

\[
\left[ \int_t^T \theta(s-t)[r(s-t) - r(T-t)] \, ds - \dot{\lambda}^S(t) + r(T-t)\lambda^S(t) - 1 \right] \ln k_t =
\]

\[
= - \int_t^T \theta(s-t)[r(s-t) - r(T-t)] \left[ -\ln \lambda^S(s) + \left( \mu - \frac{1}{2} \sigma^2 \right)(s-t) - \int_t^s [\lambda^S(\tau)]^{-1} d\tau \right] \, ds - r(T-t)f^S(t) + f^S(t) - \ln \lambda^S(t) + \mu \lambda^S(t) - 1 - \frac{1}{2} \sigma^2 \lambda^S(t).
\]

Since the equation above must be satisfied for every \( k \), then necessarily

\[
\dot{\lambda}^S - r(T-t)\lambda^S + 1 = \int_t^T \theta(s-t)[r(s-t) - r(T-t)] \, ds,
\]

and using that \( \int_t^T \theta(s-t)r(s-t) \, ds = -\theta(s-t)|_t^T = -\theta(T-t) + 1 \) we obtain

\[
\dot{\lambda}^S - r(T-t)\lambda^S = -\theta(T-t) - r(T-t) \int_t^T \theta(s-t) \, ds,
\]

The general solution of this first order linear differential equation is

\[
\lambda^S(t) = \theta(T)\Lambda(T)e^{\int_t^T r(T-s) \, ds} + \int_t^T \theta(s-t) \, ds.
\]

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Finally, the terminal condition $0 = k(T) = \lambda^S(T)c(T)$ implies that $\lambda^S(T) = 0$ ($c(T)$ is necessarily strictly positive). Therefore, $\lambda^S(t) = \int_t^T \theta(s - t) \, ds$ and

$$c^S(t) = \frac{k_t}{\int_t^T \theta(s - t) \, ds},$$

which coincides with the solution obtained for a naive agent. Of course, this is a special feature of the logarithmic utility function. As illustrated in [10] (within a deterministic setting), the coincidence of solutions for naive and sophisticated agents is not preserved for a general isoelastic utility function.

4 Optimal Portfolios for Time-Inconsistent Investors

4.1 General Setting

In this section, we analyze the consequences of introducing a non-constant discount rate in time preference into the classical solution by Merton ([11], [12]) for the optimal consumption and portfolio problem. Let us briefly describe the basic parameters of the problem.

The standard Itô processes model for a financial market consists of $(m+1)$ securities. One of them is risk-free (a bond, for instance), and the price $P_0(t)$ of 1 unit is assumed to evolve according to the ordinary differential equation

$$\frac{dP_0(t)}{P_0(t)} = \mu_0 dt,$$

where $\mu_0 > 0$ and $P_0(0) = p_0 > 0$. There are also $m$ risky assets (stocks, for instance), whose prices $P_i(t)$, $i = 1, \ldots, m$, evolve according to a geometric Brownian motion stochastic process:

$$\frac{dP_i(t)}{P_i(t)} = \mu_i dt + \sum_{k=1}^L \bar{\sigma}_{ik} d\bar{z}_k(t), \quad i = 1, \ldots, m,$$

where $P_i(0) = p_i > 0$, $(\bar{z}_1(t), \ldots, \bar{z}_L(t))$ is a $L$-dimensional standard Brownian motion process, and $\bar{z}_k(t)$ are mutually independent Brownian motions. For the sake of simplicity, we will assume that $\mu_0$ and the drift vector of the risky assets $\mu = (\mu_1, \ldots, \mu_m)$ are constant.

From the diffusion matrix $\Sigma = (\bar{\sigma}_{ik})$, $i = 1, \ldots, m$, $k = 1, \ldots, l$, we can define the variance-covariance matrix $\Sigma = \Sigma' = (\sigma_{ij})$, $i, j = 1, \ldots, m$, whose coefficients are given by $\sigma_{ij} = \sum_{k=1}^l \bar{\sigma}_{ik} \bar{\sigma}_{jk}$. Note that $\Sigma$ is symmetric ($\sigma_{ij} = \sigma_{ji}$). We will assume that $\Sigma$ is positive definite. In particular, this implies that $\sigma_{ii} > 0$ (all $m$ risky assets are indeed risky) and $\Sigma$ is nonsingular ($\det \Sigma > 0$). Elements $\sigma_{ii}$ are usually denoted by $\sigma_i^2$, hence $\sigma_i = (\sigma_{ii})^{1/2}$. 

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By defining \( z_i(t) = \frac{1}{\sigma_i} \sum_{k=1}^{L} \sigma_{ik} \bar{z}_k(t) \), which are correlated standard Brownian motions with \( \text{Cov}(z_i(t), z_j(t)) = \frac{\sigma_i \sigma_j}{\sigma_i^2 \sigma_j^2} \), Equation (25) becomes
\[
dP_i = \mu_i P_i dt + \sigma_i P_i dz_i, \quad i = 1, \ldots, m,
\]
with \( dz_i dz_j = \rho_{ij} dt \) for \( i, j = 1, \ldots, m \), where \( \rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} \). Therefore, the problem for the \( t \)-agent consists in solving (5) subjected to (4).

In terms of the Wiener \( L \)-dimensional process with independent components \( (\bar{z}_1, \ldots, \bar{z}_L) \), the budget equation becomes
\[
dW = \left[ \sum_{i=1}^{m} w_i (\mu_i - \mu_0) W + (\mu_0 W - c) \right] ds + \sum_{i=1}^{m} \sum_{k=1}^{L} w_i \sigma_{ik} W d\bar{z}_k.
\]

Let us briefly recall the solution with a constant discount rate \( \rho \). In this case the agent must solve the HJB equation
\[
\rho V - V_t = \max \left\{ u(c) + \left[ \sum_{j=1}^{m} w_j (\mu_j - \mu_0) W + (\mu_0 W - c) \right] V_W + \right. \]
\[
\left. + \frac{1}{2} \sum_{j=1}^{m} \sum_{k=1}^{m} w_j w_k \sigma_{jk} W^2 V_{WW} \right\}. \tag{28}
\]

By solving the maximization problem in \( w, i = 1, \ldots, m \), we obtain the standard optimal portfolio rule
\[
w = -\frac{V_W}{WW} \Sigma^{-1}(\mu - \mu_0 \cdot 1), \tag{29}
\]
where \( \mu = (\mu_1, \ldots, \mu_m) \) and \( 1 = (1, \ldots, 1) \). As for the optimal consumption, from the maximization problem in \( c \) in Equation (33) we obtain
\[
u'(c) = V_W. \tag{30}
\]

Next, let us assume that the discount rate \( r(t) \) of time preference is non-constant. For the general case, let us describe the so-called pre-commitment solution, and the solution for naive and sophisticated agents.

**Pre-commitment Solution:** If the 0-agent can precommit his future behaviour, he must solve the corresponding HJB equation
\[
r(t)V^P - V^P_t = \max \left\{ u(c) + \left[ \sum_{j=1}^{m} w_j (\mu_j - \mu_0) W + (\mu_0 W - c) \right] V^P_W + \right. \]
\[
\left. + \frac{1}{2} \sum_{j=1}^{m} \sum_{k=1}^{m} w_j w_k \sigma_{jk} W^2 V_{WW}^P \right\}. \tag{31}
\]
Since the right hand term in (31) coincides with that in (28), then the optimal consumption and portfolio rules are given by (30) and (29), respectively, with $V$ replaced by $V^P$.

**Solution for a Naive Agent:** Naive $t$-agents will solve the problem by looking for the solution to the HJB equation

$$r(\tau - t)V^N - V^N_\tau = \max_{\{c,w_i\}} \left\{ u(c) + \left[ \sum_{j=1}^{m} w_j(\mu_j - \mu_0)W + (\mu_0W - c) \right] V^N_W + \right.$$

$$\left. + \frac{1}{2} \sum_{j=1}^{m} \sum_{k=1}^{m} w_jw_k\sigma_{jk}W^2V^N_{WW} \right\} \quad (32)$$

where the value function for the naive $t$-agent is $V^N(W, \tau)$, for $\tau \in [t, T]$. From the maximization problem in (32) we again obtain that the optimal consumption and portfolio rules are given by (30) and (29), with $V$ replaced by $V^N$. In order to construct the actual trajectory, we will follow a procedure similar to the one explained in Section 3.2.

**Solution for a Sophisticated Agent:** From Theorem 1, in order to solve the solution to Problem (5) subject to (27), we analyze the modified HJB equation (17), which for our particular problem becomes

$$r(T - t)V^S + K - V^S_t = \max_{\{c,w_i\}} \left\{ u(c) + \left[ \sum_{j=1}^{m} w_j(\mu_j - \mu_0)W + (\mu_0W - c) \right] V^S_W + \right.$$

$$\left. + \frac{1}{2} \sum_{j=1}^{m} \sum_{k=1}^{m} w_jw_k\sigma_{jk}W^2V^S_{WW} \right\} \quad , (33)$$

where $K$ is given by (19). Once again, the optimal consumption and portfolio rules are given by (30) and (29), with $V$ replaced by $V^S$.

From Equation (29) it becomes clear that if, for every non-constant discount rate of time preference $r(s)$, $\frac{V^S_W}{WV^S_{WW}}$ is constant, then the investment strategy will be independent of $r(s)$ and it will be observationally equivalent to the constant discount rate case (this property will not remain true with respect to the consumption level, in general). From the solution to the problem in [11], [12], natural candidates for this observational equivalence are the CRRA (constant relative risk averse) utility functions, namely the logarithmic and potential functions. In the remaining subsections of the paper, we show how this observational equivalence exists in the investment strategy (not in the consumption rule) for the logarithmic and potential utility functions, but not for more general utility functions, such as the exponential CARA (constant absolute risk averse) utility function.
4.2 Logarithmic Utility Function

First of all, let us analyze the log-utility case, \( u(c) = \ln c \), with final function \( F(W(T)) = a \ln (W(T)) \).

In the case of a constant discount rate \( \rho \), the agent must solve the HJB equation (28). From a symmetry argument [3] it can be proved that \( V(W, t) = \alpha(t) \ln W + \beta(t) \). In fact this symmetry argument can be applied to all the solutions in this section, obtaining in this way unique solutions for each attitude of the \( t \)-agent (pre-commitment, naive or sophisticated). From (30) we obtain

\[
    c = (V_W)^{-1} = \frac{W}{\alpha(t)}
\]

and by substituting in (29) the optimal portfolio rule becomes

\[
    w = \Sigma^{-1}(\mu - \mu_0 \cdot 1).
\]

By substituting in (28), the choice of the value function proves to be consistent and the optimal consumption rule is determined by

\[
    \alpha(t) = \left( a - \frac{1}{\rho} \right) e^{-\rho(T-t)} + \frac{1}{\rho}.
\]

Next, for the general case of non-constant discounting, we solve and compare the solutions for pre-commitment, naive and sophisticated agents.

**Pre-commitment Solution:** We must solve equation (31). Once again, we know that the solution will be of the form \( V^P(W, t) = \alpha^P(t) \ln W + \beta^P(t) \).

Then, the optimal consumption and portfolio rules are given by \( c = \frac{W}{\alpha^P(t)} \) and (35), respectively. By substituting in (31), we obtain that \( \alpha^P(t), \beta^P(t) \) are the solution to the first order linear differential equation system

\[
    \dot{\alpha}^P - r(t)\alpha^P + 1 = 0,
\]

\[
    \dot{\beta}^P - r(t)\beta^P + \left[ \frac{1}{2}(\mu - \mu_0 \cdot 1)\Sigma^{-1}(\mu - \mu_0 \cdot 1) + \mu_0 \right] \alpha^P - \ln \alpha^P - 1 = 0,
\]

with \( \alpha^P(T) = a, \beta^P(T) = 0 \). By solving \( \alpha^P(t) \) we obtain

\[
    \alpha^P(t) = \frac{1}{\theta(t)} \left[ a\theta(T) + \int_t^T \theta(s) \, ds \right]
\]

and therefore \( c^P(t) = \frac{\theta(t)W^*}{a\theta(T) + \int_t^T \theta(s) \, ds} \).

**Solution for a Naive Agent:** Naive \( t \)-agents will solve the problem by looking for the solution of the HJB equation (32). By guessing \( V^N(W, \tau) = \)
\( \alpha^N(\tau) \ln W + \beta^N(\tau) \), we obtain (35) and 
\[ c(\tau) = \frac{W}{\bar{\alpha}^N(\tau)} \] 
where \( \bar{\alpha}^N(\tau) \) is the solution to the first order linear differential equation 
\( \bar{\alpha}^N - r(\tau - t)\bar{\alpha}^N + 1 = 0 \), 
\( \bar{\alpha}^P(T) = a \), which is given by 
\[ \bar{\alpha}^N(\tau) = 1 + \frac{1}{\theta(\tau - t)} \int_t^T \theta(s - t) \, ds \] 
(38).

Since the \( t \)-agent will not be time consistent for \( \tau > t \), the actual consumption rule is obtained for \( \tau = t \), so 
\[ a^N(t) = a\theta(T - t) + \int_t^T \theta(s - t) \, ds \] 
(39) and therefore 
\[ c^N(t) = W \bar{a}^N(\tau) \ln W + \frac{R}{\theta(T - t)} \int_t^T \theta(s - t) \, ds \] 
(40).

**Solution for a Sophisticated Agent:** A sophisticated agent will look for the solution of the modified HJB equation (33), with \( K \) given by (19). We guess 
\[ V_S(W, t) = a^S(t) \ln W + \beta^S(t) \] 
and therefore 
\[ c^S(t) = W \bar{a}^S(\tau) \ln W + \frac{R}{\theta(T - t)} \int_t^T \theta(s - t) \, ds \] 
(40).

### 4.3 Potential Utility Function

Next, let us study the problem with a (isoelastic) potential utility function, 
\[ u(c) = \frac{c^\gamma}{\gamma}, \quad \gamma < 1, \quad \gamma \neq 0, \] 
with final function \( F(W(T)) = a \frac{|W(T)|^\gamma}{\gamma} \).

As above, first we recall the solution with a constant discount rate \( \rho \). From the right hand term in (28) we obtain (30) and (29). As a candidate to the value function we guess 
\[ V(W, t) = \alpha(t) \frac{W(T)^\gamma}{\gamma} \]. Once again, this choice is justified in [3] from a symmetry argument, which is also applied to the pre-commitment, naive and sophisticated solutions. From (30) we obtain 
\[ c = (\alpha(t))^{-\frac{1}{\gamma}} W \] 
(41).
and by substituting in (29) the optimal portfolio rule becomes
\[ w = \frac{1}{1-\gamma} \Sigma^{-1}(\mu - \mu_0 \cdot 1). \] (42)

By substituting in (28), we obtain a Bernoulli equation, whose solution is
\[ \alpha(t) = \left[ \frac{1 - \gamma}{\mu - \delta^p} + \left( \frac{1 - \gamma}{\rho - \delta^p} \right) e^{-\frac{\alpha P}{\gamma}(T-t)} \right]^{1-\gamma}, \] (43)
where
\[ \delta^p = \mu_0 \gamma + \frac{1}{2} \frac{\gamma}{1-\gamma} (\mu - \mu_0 \cdot 1) \Sigma^{-1}(\mu - \mu_0 \cdot 1). \] (44)

Next, we assume that the discount rate of time preference is non-constant.

Pre-commitment Solution: We guess \( V^P(W, t) = \alpha^P(t) \frac{|W(t)|}{\gamma} \) in equation (31). Then the associated equilibrium consumption and portfolio rules are given by \( c = (\alpha^P(t))^{-\frac{\gamma}{1-\gamma}} W \) and (42), respectively. By substituting in (31) we obtain that \( \alpha^P(t) \) is the solution to the Bernoulli equation \( \ddot{\alpha}^P = (\rho - \delta^p)\alpha^P - (1 - \gamma)(\alpha^P)^{-\frac{\gamma}{1-\gamma}}, \) \( \alpha^P(T) = a, \) where \( \delta^p \) is given by (44), whose solution is
\[ \alpha^P(t) = e^{\delta^p(T-t)} \frac{\theta(T)}{\theta(t)} \left[ a + \int_{t}^{T} e^{\frac{1}{\gamma} \frac{\theta(T)-\theta(t)}{\theta(t)} e^{\delta^p(T-s)}} ds \right]^{1-\gamma}. \] (45)

Solution for a Naive Agent: We must solve the HJB equation (32). By guessing \( V^N(W, \tau) = \hat{\alpha}^N(\tau) \frac{|W(t)|}{\gamma} \) and substituting in (32) we obtain (42) and \( c(\tau) = (\hat{\alpha}^N(\tau))^{-\frac{\gamma}{1-\gamma}} W, \) where \( \hat{\alpha}^N(\tau) \) is the solution to \( \hat{\alpha}^N = (r(s-t) - \delta^N)\hat{\alpha}^N - (1 - \gamma)(\hat{\alpha}^N)^{-\frac{\gamma}{1-\gamma}}, \) \( \hat{\alpha}^N(T) = a, \) which is given by
\[ \hat{\alpha}^N(t) = e^{\delta^N(T-t)} \frac{\theta(T-t)}{\theta(t-t)} \left[ a + \int_{t}^{T} e^{\frac{1}{\gamma} \frac{\theta(T-t)}{\theta(t-t)} e^{\delta^N(T-s)}} ds \right]^{1-\gamma}. \]

The actual consumption rule, which is obtained for \( \tau = t, \) is determined by
\[ \hat{\alpha}^N(t) = e^{\delta^N(T-t)} \theta(T-t) \left[ a + \int_{t}^{T} e^{\frac{1}{\gamma} \frac{\theta(T-t)}{\theta(t-t)} e^{\delta^N(T-s)}} ds \right]^{1-\gamma}. \] (46)

Solution for a Sophisticated Agent: Let us look for the solution of the modified HJB equation (33), with \( K \) given by (19). Once again, we guess \( V^S(W, t) = \alpha^S(t) \frac{|W(t)|}{\gamma}. \) As in the previous solutions, by substituting in (33) we obtain (42) and \( c(t) = (\alpha^S(t))^{-\frac{\gamma}{1-\gamma}} W, \) where \( \alpha^S(t) \) is the solution to the integro-differential equation
\[ \dot{\alpha}^S = (r(T-t) - \delta^p)\alpha^S - (1 - \gamma)(\alpha^S)^{-\frac{\gamma}{1-\gamma}} + \frac{1}{\gamma} \int_{t}^{T} \theta(s-t)[r(s-t) - r(T-t)][(\alpha^S)^{-\frac{\gamma}{1-\gamma}} e^{\gamma\int_{s}^{T} \Delta(\tau) d\tau}] ds, \] (47)
where \( \Delta(\tau) = \frac{1}{1-\gamma}(\mu - \mu_0 \cdot 1) \Sigma^{-1}(\mu - \mu_0 \cdot 1) + \mu_0 - (\alpha^S(t))^{-\frac{1}{\gamma}}. \)
4.4 Exponential Utility Function

Finally, let us solve the problem for the constant absolute risk aversion utility function \( u(c) = -\frac{1}{\gamma} e^{-\gamma c}, \gamma > 0 \), with final function \( F(W(T)) = -ae^{-\gamma W} \).

In the constant discount rate case, we guess \( V(W, t) = -ae^{-\gamma(\alpha(t)+\beta(t))W} \) (this choice is justified as the only reasonable possibility in [3] and [4], pp. 193-194, and once again we can replicate the same symmetry argument for the pre-commitment, naive and sophisticated solutions), with \( \alpha(T) = 0, \beta(T) = 1 \). We proceed as before to obtain

\[
c = \alpha(t) + \beta(t)W - \frac{\ln (a\gamma\beta(t))}{\gamma} \tag{48}
\]

and

\[
w = \frac{1}{\gamma\beta(t)W} \Sigma^{-1}(\mu - \mu_0 \cdot 1) . \tag{49}
\]

By substituting in (28), after several calculations we obtain

\[
\beta(t) = \frac{\mu_0}{1 + (\mu_0 - 1)e^{-\mu_0(T-t)}} , \tag{50}
\]

\[
\alpha(t) = -\frac{1}{\gamma} e^{-\int_{\tau}^{T} \beta(s) ds} \int_{t}^{T} [\delta^c(s) - \rho] e^{\int_{s}^{T} \beta(\tau) d\tau} ds , \tag{51}
\]

where \( \delta^c(t) = \beta(t) - \frac{1}{2}(\mu - \mu_0 \cdot 1)\Sigma^{-1}(\mu - \mu_0 \cdot 1) - \beta(t)\ln (a\gamma\beta(t)) \).

If the case of non-constant discounting we obtain the following solutions:

**Pre-commitment Solution:** By guessing \( V^P(W, t) = -ae^{-\gamma(\alpha^P(t)+\beta^P(t))W} \), the associated equilibrium consumption and portfolio rules are given by (48-49), with \( \alpha(t), \beta(t) \) replaced by \( \alpha^P(t), \beta^P(t) \). Moreover, \( \beta^P(t) = \beta(t) \), and

\[
\alpha^P(t) = -\frac{1}{\gamma} e^{-\int_{0}^{T} \beta(s) ds} \int_{t}^{T} [\delta^c(s) - \rho] e^{\int_{s}^{T} \beta(\tau) d\tau} ds . \tag{52}
\]

**Solution for a Naive Agent:** We guess \( V^N(W, \tau) = -ae^{-\gamma(\alpha^N(\tau)+\beta^N(\tau))W} \) in equation (32). As above, the consumption and portfolio rules coincide with those in (48-49), with \( \alpha(t), \beta(t) \) replaced by \( \alpha^N(t), \beta^N(t) \). Once again, \( \beta^N(t) = \beta(t) \). Since \( \alpha^N(\tau) = -\frac{1}{\gamma} e^{-\int_{0}^{T} \beta(s) ds} \int_{\tau}^{T} [\delta^c(s) - \rho] e^{\int_{s}^{T} \beta(\tau) d\tau} ds \), taking \( \tau = t \) we obtain

\[
\alpha^N(t) = -\frac{1}{\gamma} e^{-\int_{0}^{T} \beta(s) ds} \int_{t}^{T} [\delta^c(s) - \rho] e^{\int_{s}^{T} \beta(\tau) d\tau} ds . \tag{53}
\]

**Solution for a Sophisticated Agent:** In order to solve the modified HJB equation (33), with \( K \) given by (19), we guess \( V^S(W, t) = -ae^{-\gamma(\alpha^S(t)+\beta^S(t))W} \).
The consumption and portfolio rules are given by (48-49), with \( \alpha(t), \beta(t) \) replaced by \( \alpha^S(t), \beta^S(t) \). In order to calculate the functions \( \alpha^S(t), \beta^S(t) \), note that equation (27) can be written as
\[
dW(s) = \left[(\mu_0 - \beta^S(s))W + B(s)\right] ds + C(s)d\bar{z},
\]
where \( W(t) = W_t \), and
\[
B(s) = \frac{1}{\gamma \beta^S(s)}(\mu - \mu_0 \cdot 1)\Sigma^{-1}(\mu - \mu_0 \cdot 1) - \alpha^S(s) + \frac{\ln(a \gamma \beta^S(s))}{\gamma}
\]
and \( C(s) = \frac{1}{\gamma \beta^S(s)}(\mu - \mu_0 \cdot 1)\Sigma^{-1} \Sigma \). The solution is
\[
W(s) = e^{\int_0^s (\mu_0 - \beta^S(\tau))d\tau} \left[W(t) + \int_t^s B(s)e^{-\int_0^s (\mu_0 - \beta^S(\tau))d\tau} ds + \int_t^s e^{-\int_0^s (\mu_0 - \beta^S(\tau))d\tau} C(s) d\bar{z} \right].
\]
Let us assume that \( \beta^S(t) = \beta(t) \) (see (50)). This choice will be consistent with equation (33) if \( K \) does not contribute to the calculation of \( \beta^S(t) \). This occurs if, and only if, \( K(W_I, t) = A(t)e^{-\gamma(\alpha^S(t) + \beta^S(t)W_I)} \), for some function \( A(t) \). Let us check it by first noting that if \( \beta^S(t) \) is given by (50), then
\[
e^{\int_0^s (\mu_0 - \beta^S(\tau))d\tau} = \frac{\beta^S(t)}{\beta^S(s)},
\]
hence
\[
W(s) = \frac{1}{\beta^S(s)} \left[\beta^S(t)W(t) + \int_t^s \beta^S(s)B(s) ds + \int_t^s \beta^S(s)C(s) d\bar{z} \right]. \tag{54}
\]
Now, from (19) and (48)
\[
K = E \left[\int_t^T \theta(s - t) [r(s - t) - r(T - t)] \left( \frac{1}{\gamma} e^{-\gamma(\alpha^S(s) + \beta^S(s)W(s) - \ln(a \gamma \beta^S(s))//'\gamma)} \right) ds \right]
\]
and using (54) we obtain
\[
K(W_I, t) = -ae^{-\gamma(\alpha^S(t) + \beta^S(t)W_I)} E \left[\int_t^T \theta(s - t) [r(s - t) - r(T - t)] \beta^S(s) \cdot e^{-\gamma(\alpha^S(s) - \alpha^S(t) + \int_t^s \beta^S(\tau)B(\tau) d\tau + \int_t^s \beta^S(\tau)C(\tau) d\bar{z})} ds \right].
\]
From the expression of \( K \) it follows that our choice of \( \beta^S(t) = \beta(t) \) proves to be consistent with the modified HJB equation. With respect to \( \alpha^S(t) \), it is the solution of a very complicated integro-differential equation.

### 4.5 Comparison between the Different Solutions

First of all, a relevant property of both the logarithmic and potential (CRRA) utility functions is that the portfolio rule is independent of the discount factor, and it is the same for the pre-commitment, naive and sophisticated solutions. This nice property is no longer satisfied for more general HARA utility functions, such as the exponential (CARA) utility function. Although in the exponential case the expression of the portfolio rule is the same for an agent with constant
discount rate and the different solutions with non-constant discounting (they are all given by (49), with \( \beta(t) = \beta^N(t) = \beta^S(t) \)), this property does not imply that the portfolio rule is independent of the discount factor, since the evolution of \( W(t) \) depends on the values of \( \alpha(t) \), \( \alpha^P(t) \), \( \alpha^N(t) \) and \( \alpha^S(t) \), respectively, and all these functions do not coincide, in general.

Concerning the consumption rule, it coincides for naive and sophisticated agents in the logarithmic case. This is a remarkable property, in the sense that naive and sophisticated behaviours are completely different in nature. However, this result is hardly surprising, since it coincides with that obtained by Pollak [14] for the Strotz’s model. In conclusion, naive or sophisticated behaviours have no influence on the solution of the investment-consumption problem with non-constant discount rate of time preference if the utility function is logarithmic. This result is no longer true for the potential and exponential cases.

With respect to the pre-commitment solution, it will be different in general (for the three utility functions) from those of naive and sophisticated agents, unless \( r(t) \) is constant.

Finally, it is easy to prove that, unlike the Ramsey model (see [2]), for the logarithmic utility function the naive (or sophisticated) solution will be in general non-observationally equivalent to the standard solution with some constant discount rate \( \rho \). Note that, for such observational equivalence, it is necessary and sufficient that \( \alpha(t) = \alpha^S(t) \), for some \( \rho \), i.e., from (36) and (40),

\[
\left( a - \frac{1}{\rho} \right) e^{-\rho(T-t)} + \frac{1}{\rho} = a \theta(T-t) + \int_t^T \theta(s-t) \, ds,
\]

i.e.,

\[
a \left[ \theta(T-t) - e^{-\rho(T-t)} \right] = \frac{1}{\rho} \left( 1 - e^{-\rho(T-t)} \right) - \int_0^{T-t} \theta(s) \, ds,
\]

for some \( \rho \). If \( \theta(T-t) \neq e^{-\rho(T-t)} \), by solving \( a \) (which is constant) we obtain

\[
a = \frac{\frac{1}{\rho} \left( 1 - e^{-\rho(T-t)} \right) - \int_t^T \theta(s-t) \, ds}{\theta(T-t) - e^{-\rho(T-t)}}.
\]

By defining \( x(t) = \frac{1}{\rho} \left( 1 - e^{-\rho(T-t)} \right) - \int_t^T \theta(s-t) \, ds \), the above equation becomes \( a = \frac{x(t)}{\dot{x}(t)} \). If \( a = 0 \), then by differentiating the right hand term in (55) with respect to \( t \), we obtain that necessarily \( \theta(T-t) = e^{-\rho(T-t)} \) and the discount factor is constant. If \( a \neq 0 \), the general solution of the equation above is \( x(t) = Ae^{t/a} \), and by identifying \( \dot{x}(t) \) with the denominator in the right hand term in (56) we obtain \( \theta(T-t) = \frac{A}{a} e^{t/a} + e^{-\rho(T-t)} \).

Hence, we have proved that there will be observational equivalence with a standard model with constant discount rate \( \rho \) if, and only if, \( a \neq 0 \) and there exists a constant \( C \) such that the discount factor can be written as

\[
\theta(t) = Ce^{-\frac{1}{a}t} + e^{-\rho t},
\]

i.e., \( \theta(t) \) is a linear combination of two exponentials with constant discount rates \( \rho \) and \( 1/a \).
5 Concluding remarks

In this paper we study the problem of searching for optimal/equilibrium rules in the case where decision-makers have time-inconsistent preferences, within a stochastic framework. For the so-called sophisticated agents, we derive a modified Hamilton-Jacobi-Bellman equation which extends the equation for a deterministic problem (see [7] for the case of an infinite horizon setting, and [10] for the finite horizon case with free terminal time). Although this modified HJB equation seems to be too complicated in general, we illustrate with several examples how it can be managed in order to obtain information about the solution. In particular, this modified HJB equation is used in order to solve (for some utility functions) the classical consumption and portfolio rules model when the instantaneous discount rate of time preference is non-constant.

A relevant result is that for the CRRA (logarithmic and potential) utility functions, the portfolio rule coincides for the pre-commitment, naive and sophisticated solutions. Moreover, it is independent of the discount factor, and thus coincides with the standard solution when the discount factor is an exponential with constant discount rate. This property is no longer satisfied for more general utility functions, such as the CARA (exponential) function.

With respect to the consumption rule, it is proved that, in the log-utility case, it coincides for naive and sophisticated agents. This is a remarkable property which extends to the Merton’s model a similar result already announced in [14] for the Strotz model. This coincidence is no longer satisfied for more general utility functions. In the log-utility case, the observational equivalence problem first studied for the Ramsey model in [2] is analyzed, with a negative answer except for a very particular form of the discount factor.

References


