Missile target accuracy estimation with importance splitting

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A B S T R A C T

Missile safety takes a more and more important place in the design and the evaluation of a missile. For that purpose, missile performance collateral damages can be characterised with rare quantiles around the missile target. Usual methods like Monte Carlo simulations are unfortunately not efficient to estimate rare quantiles. Consequently, we propose to apply an advanced method of rare event estimation called importance splitting to decrease the quantile relative error. We show the performance of this algorithm on a realistic missile case.

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1. Introduction

Performances of a missile are often characterised with a CEP (Circular Equal Probable). It is defined as a circle, centered on the target, whose boundary includes 50% of the missile impact population within it. The radius of the circle reflects the missile ability to reach its target. Nevertheles, safety and reliability of a missile are also a very important domain of research and analysis. In this article, we propose to evaluate the possible collateral damage zone of a missile, that is to determine CEP that includes 99.99% and more of the missile impact population within it. A usual method to estimate a CEP is Monte Carlo simulations but they cannot be applied in this case with a sufficient reliability. In this article, we propose an alternative algorithm called importance splitting. Instead of estimating the CEP through a very tough simulation, one should consider the estimation of several conditional probabilities that are easier to evaluate by simulation. We firstly describe in this article the application case that will be considered, then present the principle of importance splitting and finally show different results on rare CEP estimation.

2. Missile simulation

2.1. Missile system presentation

We consider in this article an air launched missile that is 6.32 m long and weighs 900 kg. It is a supersonic stand-off missile powered by a liquid-fuel ramjet. It flies at Mach 2 to Mach 3, with a range between 80 km and 300 km depending on flight profile. This missile is modelled with a continuous black-box computer code $\phi$ with 51 independent inputs $X$ and one output $D = \phi(X)$ where $D$ is the target distance. The main inputs of this black-box simulation code are the following:

- Launch aircraft inertial measurement unit in position and speed (10 inputs). These parameters describe the performances of aircraft accelerometers and gyroscopes to estimate aircraft speed and position.
- Missile inertial measurement unit in position and speed (10 inputs). These parameters describe the performances of missile accelerometers and gyroscopes to estimate missile speed and position.
- Meteorological conditions (4 inputs). Wind is notably taken into account since it can deviate the missile trajectory.
- Ramjet combustion process (8 inputs). It influences the performances of the missile propulsion and notably the missile highspeed.
- Missile design and weight (6 inputs). The weight and the size of the different parts of the missile are also slightly random. Deviation of one to two kilograms is possible.
- Internal hydraulic cylinders efficiency (6 inputs). It analyses the performances of the missile maneuvers.
- Aerodynamic and flight mechanic parameters (7 inputs). It describes the forces that act on the missile aircraft in flight and how it influences the speed and the position of the missile.

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estimating rare defined as quantiles of target distance and is consequently a random variable with PDF. The usual way to estimate a quantile \( q_\alpha \) is to consider Monte Carlo techniques. In this article, we focus on the \( \alpha \)-quantile \( q_\alpha \) estimation of the target distance \( D \) with PDF \( g \). It is well known that a quantile \( q_\alpha \) is obtained with the following integral:

\[
\alpha = \int_{-\infty}^{q_\alpha} g(y) \, dy
\]

(1)

The usual way to estimate a quantile \( q_\alpha \) is to consider Monte Carlo techniques. In that case, the cumulative distribution function (CDF) \( G(d) \) of \( D \) is defined in the following way:

\[
G(d) = \int_{-\infty}^{d} g(y) \, dy = \int_{-\infty}^{d} f_\phi(x) \, dx
\]

(2)

In practice, one generates a set of \( N \) independent and identically distributed samples of \( X \) (\( X_1, \ldots, X_N \)) and applies the function \( \phi \) on these samples to determine a set of samples (\( d_1, \ldots, d_N \)) that follows the PDF \( g \) of \( D \). The Monte Carlo estimator of the CDF \( \hat{G}^{MC} \) of \( D \) is given by:

\[
\hat{G}^{MC}(d) = \frac{1}{N} \sum_{i=1}^{N} 1_{d \geq d_i}
\]

(3)

The Monte Carlo estimator of the \( \alpha \)-quantile of the variable \( Y \) is then obtained with:

\[
q^{\alpha, N}_{\hat{G}^{MC}} = \inf \{d, \hat{G}^{MC}(d) > \alpha \}
\]

(4)

3. Quantile estimation with Monte Carlo simulations

3.1. Principle

The missile simulation code is defined with \( \phi \) a scalar continuous function and \( \phi : \mathbb{R}^{51} \rightarrow \mathbb{R} \). In this article, we focus on the \( \alpha \)-quantile \( q_\alpha \) estimation of the target distance \( D \) with PDF \( g \). It is well known that a quantile \( q_\alpha \) is obtained with the following integral:

\[
\alpha = \int_{-\infty}^{q_\alpha} g(y) \, dy
\]

(1)

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\]

(2)

In practice, one generates a set of \( N \) independent and identically distributed samples of \( X \) (\( X_1, \ldots, X_N \)) and applies the function \( \phi \) on these samples to determine a set of samples (\( d_1, \ldots, d_N \)) that follows the PDF \( g \) of \( D \). The Monte Carlo estimator of the CDF \( \hat{G}^{MC} \) of \( D \) is given by:

\[
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\]

(3)

The Monte Carlo estimator of the \( \alpha \)-quantile of the variable \( Y \) is then obtained with:

\[
q^{\alpha, N}_{\hat{G}^{MC}} = \inf \{d, \hat{G}^{MC}(d) > \alpha \}
\]

(4)

3.2. Application on missile CEP estimation

Table 2 shows the mean estimation of different quantiles on 10 trials of the target distance \( D \) estimation with Monte Carlo simulations. Relative error is defined by the ratio between the standard deviation of the estimation and the mean estimation. One remarks that this method is well adapted to estimate 0.5-quantiles with a low relative error even with few samples. Indeed, whatever the number of samples, the same quantile level is estimated. However, rare quantiles are generally underestimated when not enough samples are generated. It is due to the convergence rate of the Monte Carlo algorithm. When more samples are randomly generated, the quantile estimation is more accurate and less underestimate.

Each Monte Carlo simulation takes about 1 second on a dedicated computer cluster. It is thus hardly possible with the code \( \phi \) to generate more than \( 2 \times 10^5 \) samples for one target distance estimation because of large computation time (about 3 days of computing for \( 2 \times 10^5 \) samples). Moreover, the underestimation of rare quantile is also an important issue in safety notably when one estimates the occurrence of missile collateral damage; one indeed might accept an overestimation of the inherent risk but not an underestimation.

We have shown that Monte Carlo simulations are not adapted to such rare quantile estimations and cannot be used with confidence for rare missile CEP estimation. We consequently propose in the following to apply an advanced rare event method called importance splitting and compare its results to Monte Carlo and importance sampling method.

Table 1

<table>
<thead>
<tr>
<th>Group of inputs</th>
<th>Sobol indices (in % of the total value)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Launch aircraft inertial measurement unit</td>
<td>32</td>
</tr>
<tr>
<td>Missile inertial measurement unit</td>
<td>24</td>
</tr>
<tr>
<td>Meteorological conditions</td>
<td>4</td>
</tr>
<tr>
<td>Ramjet combustion process</td>
<td>10</td>
</tr>
<tr>
<td>Missile design and weight</td>
<td>5</td>
</tr>
<tr>
<td>Internal hydraulic cylinders efficiency</td>
<td>12</td>
</tr>
<tr>
<td>Aerodynamic and flight mechanic parameters</td>
<td>13</td>
</tr>
</tbody>
</table>

The physical model \( \phi \) has been deeply tested with computer experiment techniques over the input domain to verify that its output does not deviate from the terrain measurements. The computation algorithms have notably been based on [17,27,9]. To express the uncertainty on the inputs \( X \), one considers that the inputs are realisations of a random variable, defined by its multidimensional probability density function (PDF) \( f_\phi \). The density \( f_\phi \) is a multidimensional Gaussian random variable with mean \( M \) and covariance matrix \( \Gamma \). The output of the simulation code is the target distance and is consequently a random variable with PDF \( g \). Fig. 1 shows the target distance PDF estimated with kernel density estimator.

To characterise the influence of the inputs on the output, global sensitivity analysis has been applied on the computer code. It focuses on the variance of model output \( D \) and more precisely on how the input variability influences the variance output [13,30,14,25,29,26,12]. It enables to determine which parts of the output variance are due to the different inputs with the estimation of Sobol indices. They are a central tool in sensitivity analysis since they give a quantitative and a rigorous overview of how the different inputs influence the output. Table 1 presents the estimation of Sobol indices on the function \( \phi \). Measurement unit of launch aircraft and missile are the most influential group of parameters on the output \( D \).

The objective of this article is to estimate CEPs, that can be defined as quantiles of target distance \( D \). In this article, we aim at estimating rare \( \alpha \)-quantiles of \( D \) that is when \( \alpha \in [1 - 10^{-2}, 1 - 10^{-10}] \). To estimate a \( \alpha \)-quantile of a random variable, the usual method is to consider Monte Carlo simulations.
Table 2
Mean estimation on 10 trials of different $\alpha$-quantiles of target distance with Monte Carlo simulations.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\hat{q}_{MC,10^4}$</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.216</td>
<td>1.2%</td>
</tr>
<tr>
<td>0.90</td>
<td>0.334</td>
<td>1.4%</td>
</tr>
<tr>
<td>0.95</td>
<td>0.373</td>
<td>1.7%</td>
</tr>
<tr>
<td>0.99</td>
<td>0.456</td>
<td>2.6%</td>
</tr>
<tr>
<td>1 − $10^{-4}$</td>
<td>0.601</td>
<td>10.3%</td>
</tr>
<tr>
<td>1 − $10^{-6}$</td>
<td>0.604</td>
<td>8.0%</td>
</tr>
</tbody>
</table>

(b) $10^4$ Monte Carlo simulations

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\hat{q}_{MC,10^4}$</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.216</td>
<td>0.4%</td>
</tr>
<tr>
<td>0.90</td>
<td>0.334</td>
<td>0.4%</td>
</tr>
<tr>
<td>0.95</td>
<td>0.374</td>
<td>0.8%</td>
</tr>
<tr>
<td>0.99</td>
<td>0.459</td>
<td>0.9%</td>
</tr>
<tr>
<td>1 − $10^{-4}$</td>
<td>0.658</td>
<td>8.2%</td>
</tr>
<tr>
<td>1 − $10^{-6}$</td>
<td>0.684</td>
<td>6.1%</td>
</tr>
</tbody>
</table>

(c) $10^4$ Monte Carlo simulations

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\hat{q}_{MC,10^4}$</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.216</td>
<td>0.1%</td>
</tr>
<tr>
<td>0.90</td>
<td>0.334</td>
<td>0.1%</td>
</tr>
<tr>
<td>0.95</td>
<td>0.374</td>
<td>0.2%</td>
</tr>
<tr>
<td>0.99</td>
<td>0.458</td>
<td>0.4%</td>
</tr>
<tr>
<td>1 − $10^{-4}$</td>
<td>0.662</td>
<td>1.5%</td>
</tr>
<tr>
<td>1 − $10^{-6}$</td>
<td>0.760</td>
<td>5.8%</td>
</tr>
</tbody>
</table>

4. Quantile estimation with importance splitting

Since Monte Carlo method are not efficient to estimate rare quantile of the target distance, we propose to apply a very efficient and novel algorithm called Importance Splitting (ISP) [4,5,16,10,5,20]. ISP is an alternative to Monte Carlo methods and to Importance Sampling (IS) [1,2,8,3,32,23,11,19]. We firstly propose to present the principle of importance splitting and then apply it in the following section on missile rare CEP estimation. Finally a comparison with importance sampling result is proposed.

4.1. Overview

Let us define the set $A = \{x \in \mathbb{R}^5 | \phi(x) < q_\alpha\}$. In this article, we aim to determine the density in this set $A$ since one has $P(X \in A) = P(\phi(X) < q_\alpha) = \alpha$. The principle of ISP is to iteratively estimate supersets of the set $A$ with conditional probabilities and then to determine the quantile $q_\alpha$. We also propose to define $A_0 = \mathbb{R}^5 \supset A_1 \supset \cdots \supset A_{k-1} \supset A_k = A$, a decreasing sequence of $\mathbb{R}^5$ subsets with smallest element $A = A_0$. The probability $P(X \in A)$ can be obtained in the following way:

$$P(X \in A) = \alpha = \prod_{k=1}^{n} P(X \in A_k | X \in A_{k-1})$$

where $P(X \in A_k | X \in A_{k-1})$ is the probability that $X \in A_k$ conditionally to $X \in A_{k-1}$. The choice of the sequence $A_k, k = 0, \ldots, n$, is not obvious. Nevertheless, it has been shown that performance estimation is optimal when all the conditional probabilities are equal, that is $P(X \in A_k | X \in A_{k-1}) = p$, with $p$ a constant [15]. Nevertheless, importance splitting technique requires the choice of simple $A_k$ sequence and also the ability to generate samples from the conditional density $f_k$. These topics are considered in the following and possible approaches are proposed.

4.2. Simple $A_k$ sequence

The subset $A_k$ sequence is easily evaluated in the following way. Indeed, it can be defined with $A_k = \{x \in \mathbb{R}^5 | \phi(x) > S_k\}$ for $k = 0, \ldots, n$ with $S = S_n > S_{n-1} > \cdots > S_k > \cdots > S_0$. Determining the sequence $A_k$ is equivalent to choose some values for $S_k$, with $k = 0, \ldots, n$.

The values of $S_k$ for $k = 0, \ldots, n$ can be determined in an adaptive manner to perform valuable results. We set $f_k$, the density of $X$ restricted to the set $A_k$ and $\mu_k$, the density of $\phi(X)$ when $X$ is restricted to the set $A_k$. Generate $X^{(k)}_1, \ldots, X^{(k)}_{N_k}$ samples from the density $f_k$. The Monte Carlo estimator of the $\mu_k$ cumulative distribution function $G_{\mu_k}$ is given by

$$G_{\mu_k} (y) = \frac{1}{N_k} \sum_{i=1}^{N_k} 1_{\phi(X^{(k)}_i) \leq y}$$

The Monte Carlo estimator of the $\beta$-quantile of the density $\mu_k$ is given by:

$$q_{\beta}^{(k)} = \inf \{ y, G_{\mu_k} (y) \geq \beta \}$$

Set $S_{k+1} = q_{\beta}^{(k)}$. The subset $A_{k+1}$ is then defined with $A_{k+1} = \{ x \in \mathbb{R}^5 | \phi(x) > S_{k+1} = q_{\beta}^{(k)} \}$. Let us then estimate $P(X \in A_{k+1} | X \in A_k)$:

$$E_P (P(X \in A_{k+1} | X \in A_k)) = E_{\mu_k} (P(X \in A_{k+1}))$$

$$= E_{\mu_k} (P(\phi(X) > S_{k+1}))$$

By definition of $S_{k+1}$, one has then

$$E (P(X \in A_{k+1} | X \in A_k)) = \beta$$

With this adaptive definition of $S_{k+1}$, a valuable sequence of $A_k$ is then determined. The number $n$ of subset depends on the value of the $\alpha$ and $\beta$ since $P = \beta^{n-1}P(X \in A_n | X \in A_{n-1})$. The number of subsets $n$ can be evaluated at the start of the algorithm. In order to limit the computation time, the parameter $\beta$ has to be adjust depending on the number of samples that will be used.

4.3. Generating samples from the density $f_k$

Unfortunately, generating directly independent samples from the $f_k$ conditional densities is in most cases impossible as they are usually unknown [31,4]. Nevertheless, ISP provides an iterative way to do it, yet in a dependent fashion using a $f_0$-reversible Markovian kernel $M(X, \cdot)$. The most general and arduous way is solving the reversibility equation. Luckily enough, there are some results for multivariate Gaussian PDF. Actually, if $X$ is a centered multivariate Gaussian

$$X \sim \mathcal{N}(0_d, Id) \quad \text{and} \quad M(X, \cdot) \sim \frac{X + c \mathcal{N}(0_d, Id)}{\sqrt{1 + c^2}}$$

is a valuable reversible Markovian kernel [31,4]. One just has to choose an appropriate value for $c$. In this article, $c$ has been defined experimentally in order to minimise the variance of the estimation. $c$ is set to 1.1 in the rest of the article.

All the required steps for importance splitting have been defined in the previous sections. Let us apply this technique on general cases to compare the results of the techniques with Monte Carlo and importance sampling simulations.

4.4. Algorithm

We propose to analyse the different stages of the algorithm ISP to estimate the $\alpha$-quantile $q_\alpha$.

1. Set $k = 0$.
2. Generate $N_k$ samples $X^{(k)}_1, \ldots, X^{(k)}_{N_k}$ from $f_k(X)$. 
Estimate the $\beta$-quantile $q_{\beta}^{(k)}$ of the samples $\phi(X_i^{(k)})$, $\phi(X_{N_k}^{(k)})$.

4. Determine the subset $A_{k+1}$ with $A_{k+1} = \{X \in \mathbb{R}^N \mid (\phi(x)) > q_{\beta}^{(k)} \}$ and the conditional density $f_k$.

5. If $1 - (1 - \beta)^{k+1} < \alpha$, set $k = k + 1$ and go back to stage (2) of the algorithm. Otherwise, estimate the $\alpha$-quantile with the following equation:

$$q_{\alpha} = \inf \left\{ y, G_{\mu_k}(y) \geq 1 - \frac{\alpha}{1 - \beta^k} \right\}$$

where the cumulative distribution function $G_{\mu_k}$ is given by

$$G_{\mu_k}(y) = \frac{1}{N_k} \sum_{i=1}^{N_k} \phi(X_i^{(k)}) \leq y$$

5. Application on target distance quantile estimation

In this subsection, we propose to estimate rare target distance quantiles with importance splitting and compare the results with Monte Carlo simulations and importance sampling simulations. We finally propose to characterise the influence of quantile parameter $\beta$ in importance splitting on the relative error estimation.

5.1. Importance splitting application and comparison with other methods

In this section we apply importance splitting algorithm with the following tuning parameters:

- The number of samples $N_k$ is set to 10000.
- The kernel parameter $c$ is chosen equal to 1.1 but do not influence much the results.
- The quantile parameter $\beta$ is set to 0.8.

One then obtains the importance splitting mean quantile estimation on 10 trials given in Table 3(a). In Table 3(b), we present the mean quantile estimation on 10 trials given by Monte Carlo simulations and finally for comparison purpose, we propose in Table 3(c) the results obtained with parametric adaptive importance sampling optimised with cross-entropy (CE) [3,7,23,24] on a Gaussian parametric model for the sampling parametric densities.

As shown in the previous sections of this article, Monte Carlo quantile estimator underestimate rare quantiles whereas importance splitting and importance sampling algorithm give relatively equivalent quantile values. Nevertheless, importance splitting seems to obtain a lower relative error estimation than importance sampling. Moreover, the performance difference between the two methods increases with the value of $\alpha$. When $\alpha = 1 - 10^{-4}$, splitting and sampling have comparable results but when $\alpha = 1 - 10^{-10}$, it seems that splitting is more adapted. In this kind of applications, importance splitting is a valuable approach to consider when one tries to estimate a rare quantile.

5.2. Influence of the quantile parameter $\beta$ in importance splitting

In this section, we propose to analyse the influence of parameter $\beta$ on importance splitting quantile estimation. In Table 4, we compare the estimation results for $\beta = \{0.2, 0.5, 0.8, 0.9\}$. There is not much difference between the different estimations. Nevertheless, to obtain valuable results, extreme value of value of $\beta$ (near 0 or 1) have to be excluded. The best estimation results are given when $\beta = 0.8$ since the relative error estimation for this value is the weakest. It confirms a study on this subject proposed in [6].

### Table 3

Estimation of different $\alpha$-quantiles of target distance with importance splitting algorithm (a), with Monte Carlo simulations (b) and with importance sampling simulations (c).

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$N$</th>
<th>$\tilde{q}_{\alpha}^{(N)}$</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - $10^{-4}$</td>
<td>60000</td>
<td>0.672</td>
<td>0.6%</td>
</tr>
<tr>
<td>1 - $10^{-5}$</td>
<td>90000</td>
<td>0.840</td>
<td>0.8%</td>
</tr>
<tr>
<td>1 - $10^{-8}$</td>
<td>120000</td>
<td>1.06</td>
<td>1.5%</td>
</tr>
<tr>
<td>1 - $10^{-10}$</td>
<td>150000</td>
<td>1.23</td>
<td>2.5%</td>
</tr>
<tr>
<td>$\beta = 0.2$</td>
<td>$\left{ \begin{array}{ll} 1 - 10^{-4} &amp; 60000 \ 1 - 10^{-5} &amp; 90000 \ 1 - 10^{-8} &amp; 120000 \ 1 - 10^{-10} &amp; 150000 \ \end{array} \right.$</td>
<td>$\left{ \begin{array}{ll} 0.667 \ 0.861 \ 1.04 \ 0.789 \ \end{array} \right.$</td>
<td>$\left{ \begin{array}{ll} 1.4% \ 5.4% \ 10.1% \ 4.3% \ \end{array} \right.$</td>
</tr>
<tr>
<td>$\beta = 0.5$</td>
<td>$\left{ \begin{array}{ll} 1 - 10^{-4} &amp; 60000 \ 1 - 10^{-5} &amp; 90000 \ 1 - 10^{-8} &amp; 120000 \ 1 - 10^{-10} &amp; 150000 \ \end{array} \right.$</td>
<td>$\left{ \begin{array}{ll} 0.689 \ 0.856 \ 1.05 \ 1.23 \ \end{array} \right.$</td>
<td>$\left{ \begin{array}{ll} 0.6% \ 2.1% \ 2.9% \ 3.2% \ \end{array} \right.$</td>
</tr>
<tr>
<td>$\beta = 0.8$</td>
<td>$\left{ \begin{array}{ll} 1 - 10^{-4} &amp; 60000 \ 1 - 10^{-5} &amp; 90000 \ 1 - 10^{-8} &amp; 120000 \ 1 - 10^{-10} &amp; 150000 \ \end{array} \right.$</td>
<td>$\left{ \begin{array}{ll} 0.672 \ 0.840 \ 1.06 \ 1.23 \ \end{array} \right.$</td>
<td>$\left{ \begin{array}{ll} 0.6% \ 0.8% \ 1.5% \ 2.5% \ \end{array} \right.$</td>
</tr>
</tbody>
</table>

### Table 4

Estimation of different $\alpha$-quantiles of target distance with importance splitting for different values of $\beta$.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\alpha$</th>
<th>$N$</th>
<th>$\tilde{q}_{\alpha}^{(N)}$</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>$\left{ \begin{array}{ll} 1 - 10^{-4} &amp; 60000 \ 1 - 10^{-5} &amp; 90000 \ 1 - 10^{-8} &amp; 120000 \ 1 - 10^{-10} &amp; 150000 \ \end{array} \right.$</td>
<td>$\left{ \begin{array}{ll} 0.669 \ 0.856 \ 1.04 \ 1.39 \ \end{array} \right.$</td>
<td>$\left{ \begin{array}{ll} 1.1% \ 2.2% \ 3.5% \ 11% \ \end{array} \right.$</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>$\left{ \begin{array}{ll} 1 - 10^{-4} &amp; 60000 \ 1 - 10^{-5} &amp; 90000 \ 1 - 10^{-8} &amp; 120000 \ 1 - 10^{-10} &amp; 150000 \ \end{array} \right.$</td>
<td>$\left{ \begin{array}{ll} 0.689 \ 0.856 \ 1.05 \ 1.23 \ \end{array} \right.$</td>
<td>$\left{ \begin{array}{ll} 0.6% \ 2.1% \ 2.9% \ 3.2% \ \end{array} \right.$</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>$\left{ \begin{array}{ll} 1 - 10^{-4} &amp; 60000 \ 1 - 10^{-5} &amp; 90000 \ 1 - 10^{-8} &amp; 120000 \ 1 - 10^{-10} &amp; 150000 \ \end{array} \right.$</td>
<td>$\left{ \begin{array}{ll} 0.672 \ 0.840 \ 1.06 \ 1.23 \ \end{array} \right.$</td>
<td>$\left{ \begin{array}{ll} 0.6% \ 0.8% \ 1.5% \ 2.5% \ \end{array} \right.$</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>$\left{ \begin{array}{ll} 1 - 10^{-4} &amp; 60000 \ 1 - 10^{-5} &amp; 90000 \ 1 - 10^{-8} &amp; 120000 \ 1 - 10^{-10} &amp; 150000 \ \end{array} \right.$</td>
<td>$\left{ \begin{array}{ll} 0.753 \ 0.909 \ 1.05 \ 1.26 \ \end{array} \right.$</td>
<td>$\left{ \begin{array}{ll} 1.3% \ 2.6% \ 4.9% \ 5.4% \ \end{array} \right.$</td>
<td></td>
</tr>
</tbody>
</table>

6. Conclusion

Estimating missile collateral effect zone is an important issue to evaluate possible missile error consequences on the population. The missile is modelled in this article with a computer simulation code with one output, the target distance. To evaluate the missile safety, one has estimated different quantiles of the target distance. Usual Monte Carlo approach is shown to be invaluable since it underestimates the quantile value. In this article, we have thus presented a quite novel alternative to estimate rare event called importance splitting which consists in the estimation of consecutive conditional probabilities. This method is applied with success...
to the estimation of rare quantile of the target distance with a low estimation relative error.

References