The statistical gauge in geometrical verification. Part II. The virtual gauge and verification process

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\textbf{A B S T R A C T}

Currently, most measurement and geometrical verification processes are based on the characterization of distances or angles between geometrical elements. Methods have already been developed for the verification of ISO specifications using the statistical information included in a set of points. These methods are however based on the development of measurement sequences and hence a set of geometrical constructions. The main problem in this verification process is the propagation of uncertainties. With a correct expression of a virtual gauge, it is possible to avoid this propagation; indeed by using a virtual gauge manipulation, initial best-fitted surfaces are never altered. The aim of this paper is to present a new approach using this statistical information to describe the measured part and develop a virtual gauge system for verification. This has been done in two parts. The first one focused on the statistical modeling of the measured surface; this second part will now describe a verification method based on a virtual, i.e. numerical, gauge.

\begin{itemize}
\item Measuring the real part.
\item Determining the features of the real surfaces of the part.
\item Interpreting geometrical specifications.
\item Testing conformance.
\end{itemize}

To deal with the first two problems, a method of statistical determination of surfaces was presented in our first paper. This type of modeling was described in the form of a field of probability of the presence of matter (FPPM) which makes it possible to work out the probability of any point in space being within matter. Part II will now focus on the two remaining points: interpreting tolerances and the conformance test. In order for this test to be reliable and efficient, it is imperative that the model used to interpret specifications be coherent with the one used to characterize surfaces. Methods of verification based on virtual gauges are particularly well adapted when using FPPM. The problem with standard methods of verification is that they rely on series of geometrical constructions which carry out conformance tests based on point/point, point/line, and point/plane distances stemming from measurements or constructions. This accumulation of geometric constructions, with their basis changes, creations of new geometrical features, and projections, is a source of uncertainty propagation. In order to avoid this problem, a method of verification must limit the manipulation of input values and cut back on intermediary geometric constructions when developing tolerance zones.

In this paper, we will first present two essential ideas for checking a part with a virtual gauge based on statistical criterion: the notion of elementary gauge and the notion of an interference probability map (IPM). Then, a method for verifying parts via a statistical criterion using the two notions will be developed. This verification method will enable the metrologist to have a statistical indication of the risk he takes in claiming a part is in compliance or not, in regard to tolerancing.

\textbf{2. State of the art}

As shown in the first part, verification using a three-dimensional measuring machine can be broken down into four parts: best fitting the surface, estimating uncertainty, interpreting geometric specifications, and testing conformance. Fig. 2 shows these different domains.

Different research studies carried out in fields concerning how to determine the features of measured surfaces, as well as how to estimate uncertainty, were also presented. In this second paper, two
new problems will be presented: interpreting a product’s geometrics and different product conformance tests.

2.1. Interpreting the geometrical product specifications

The aim of the GPS method is to give a univocal description of tolerance zones. International standards [1,2] provide geometrical and dimensional tools for tolerancing and verifying parts in a mechanical system. Aiming for coherence between geometrical specifying and verifying, several mathematical-based models have been proposed. One approach was to use a classification of symmetry groups with their impact on functional feature taxonomy, datum definition and parameterization [3]. Another way is to describe the permissible geometrical variations of parts using virtual gauges with internal mobility [4]. Also based on geometrical boundaries, the concept of virtual boundary requirements (VBR), generalized with maximum material conditions (MMC) or least material condition (LMC), is now widely used [5].

2.2. Conformance testing of geometrical specification

Once the best fit is achieved and the tolerance zone established, conformance testing can be carried out. In practice, conformance tests will determine if the estimated derived feature, or the set of points can belong to the tolerance zone. Current measurement software, depending on the range of standardized specifications (ISO standard), is based on the measurement of distances or angles between geometrical elements [6,7].

When no modifier of either the envelope condition, maximum or least material condition is specified, these geometrical quantities can be described in the following way:

- point/point distance,
- point/plane distance,
- point/line distance,
- line/line angle,
- line/plane angle, and
- plane/plane angle.

Other research work has been done on form defect verification (flatness, straightness, cylindricity, etc.) [8–10]. When the modifier of the envelop condition, MMC or LMC, is specified, a gauge should be used to verify geometrical specifications. The gauges are perfect surfaces defined and oriented by the geometrical specification. They materialize (virtually or physically) the limits of the real part variations that are allowed. Gauges with or without internal mobility, furnish a well-adapted solution for this type of specification because they have the same topology as tolerance zones. Physical gauges permit verification with low uncertainties but remain expensive and inflexible. In several cases, they can be unworkable and have no assembly possibilities. When specifying with an envelop zone, a tolerance zone can be built from nominal geometry [11,12] (Fig. 3).

With the development of 3D modelers and the progress made in coordinate measuring machines, the concept of the virtual gauge appeared. Two kinds of conformance tests are currently performed with virtual gauges. First, by giving the appropriate degrees of freedom to the virtual gauge system, the gauges will be able to measure the part [13]. The measurement will be taken by extreme fitting with the set of acquired coordinates. The second way to carry out verifications using the virtual gauge is to check, via distance measuring, that the derived features are inside (or outside depending on specification) the gauge. The binary result provided by this kind of process leads to a poor interpretation of verification. Usually, the least squares best fit method is used to associate a perfect feature to digitized point coordinates, leading to distance measuring, and finally to the evaluation of the form defect. However, the sta-
The statistical information contained in the set of points is generally left unused.

In a previous paper [14], a method was developed for the verification of ISO specifications using the statistical information included in the set of points. This method is based on the development of measurement sequences and hence a set of geometrical constructions. The main problem with this verification process is to propagate uncertainties. With a correct expression of the virtual gauge, it is possible to avoid this propagation; indeed with virtual gauge manipulation, the initial best-fitted surfaces can never be altered. The aim of this paper is to present a new approach using this statistical information to describe the measured part and build a virtual gauge system for verification.

### 3. Modeling and using an elementary gauge

#### 3.1. Definition of an elementary gauge

A virtual gauge is the digital version of a real gauge. Within the framework of standardization, it is easy to use the notion of the virtual gauge for geometrical specification with matter modifier. However, it is possible to use virtual gauges for the whole set of specifications based on the concept of the tolerance zone with or without modifier specifications (maximum material requirement or least material requirement) presented in ISO (1101, 2692, 10578, etc.) and ANSI (Y14.5) standards. A virtual gauge is defined by a set of surfaces with a fixed orientation establishing the boundaries of the tolerance zone. Three types of virtual gauges are required to deal with all geometrical specifications.

**3.1.1. The virtual gauge “go”**

Virtual gauges of the “go” type must be able to be assembled with the specified real element without being, at any point whatsoever, within matter. This type of gauge is applicable with specifications of maximum matter and required envelope type. They guarantee assembly with the part (Fig. 4). We would like to point out that we will not deal with reception on local dimensions.

Further on in this document we will orient the surface of a “go” gauge so that unitary normals on the surface are directed toward the inside of matter.

**3.1.2. The virtual gauge “no go”**

Virtual gauges of the “no go” type must not be able to be assembled with the specified real element. Moreover, they must not be, at any point whatsoever, outside matter. This type of gauge is applicable with specifications of minimum matter. They guarantee maximum clearing or minimum thickness (Fig. 5).

In the previous case, we orient the surface of the gauge “no go” so that unitary normals on the surface are directed toward the outside of matter.

**3.1.3. A virtual gauge “go/no go”**

A gauge “go/no go” is a combination of the two gauge types, “go” and “no go”. It is made up of two surfaces oriented one toward the other. This type of gauge is generally used when checking location, orientation or form without a modifier.

#### 3.2. Modeling an elementary gauge with a Gram matrix

The virtual gauge model for controls using statistical criterion must make the following things possible:

- **Positioning and orienting the gauge**: mastering the position and the orientation of the gauge is obviously required in order to be able to geometrically verify a real element. The orientation and the position of tolerance zones are fixed through geometric specification. The problem of accurate manipulation of the gauge system is much trickier. The system will be displaced in relation to the set of points in order to verify the part. These displacements can either be solid or with distortions if there is internal mobility between gauges.
- **Accurate manipulating**: as a general rule, in a digital control process, there is a problem of optimization which requires a series of displacements. This series of digital manipulations propagates uncertainty in the digital resolution. Depending on the number of iterations during optimization, the propagation of uncertainty can become the same order of magnitude as the tolerance interval.
- **Surface meshing**: a rapid meshing of the surface makes it possible to scan the quantity of digital results generated, which facilitates the optimization process.
- **Expediting the expression of geometrical constraints with other virtual gauges or datum-systems**: this last point is the expression of geometrical constraints between gauges, such as constraints on symmetry, parallelism, equidistance, and orientation.

![Fig. 4. A virtual gauge that “go”.](image4.png)

![Fig. 5. A virtual gauge “no go”.](image5.png)
An approach has been already proposed to satisfy the first requirement on the list. It estimates surfaces using a set of simplexes which are described through a Gram matrix [15] (Fig. 6).

This type of modeling has the advantage of defining both position and orientation of a geometrical object and also of retaining the topology. In our case, we need to keep the exact shape of the virtual gauge (set of parallel planes) which are described through a Gram matrix [15] (Fig. 6).

The parametrical formulation of the plane, cylinder, etc.), a parametric equation based on the local coordinates in a subspace $F$ of $\mathbb{R}^2$. It enables to scan the surface of the virtual gauge. The parametrical equation of the base plane of the IPM, is then considered: $\text{IPM}(s, r) = s \cdot \hat{x} + r \cdot \hat{y}$ where $(\hat{x}, \hat{y})$ is the coordinate system linked to the IPM. A scalar field in the plane of the IPM, will finally be created to represent the probabilities of interference between the part and the gauge. This field is obtained by setting out the following relationship: $\text{ProbInter}(\text{IPM}(u, v), m(u, v)) = \text{Prob}(\vec{C}(u, v))$. The function prob. represents the field of probability of the presence of matter explained in Part I of our work, and ProbInter, the interference probability at one point of the IPM. The parameters l and m are continuous functions that are strictly rising and gener-

where $\vec{C}$ represents the components of the parametric equation. $\vec{L}_C$ is defined as

$$\vec{L}_C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ f(u, v) \\ g(u, v) \\ h(u, v) \end{bmatrix}$$

where $u$ and $v$ are curvilinear coordinates and $F$ is a subspace of $\mathbb{R}^2$ enabling to scan the surface of the virtual gauge. The three functions $f, g$ and $h$ are the parametric equations of the virtual gauge shape. This type of formulation makes it much easier to express the mesh presented in Part I of our work.

3.3. The interference probability map

3.3.1. Principle

The virtual gauge has been defined as the boundary of a tolerance zone. It represents a volume of three-dimensional space in which the real surface must be found. A part will be rejected, if an intersection between the real surface and the virtual gauge exists. The probability of such an intersection can be represented graphically, at any point on the gauge. It is a two-dimensional scalar field (the virtual gauge being a surface) which can be presented in the form of an interference probability map. Then the interference probability will be discussed.

The construction principle for IPMs is as follows:

- The parametrical equation $\vec{C}(u, v)$ of a skew surface modeling a virtual gauge, will first be given, where $u$ and $v$ are the curvilinear coordinates in a subspace $F$ of $\mathbb{R}^2$. It enables to scan the surface of the virtual gauge. The parametrical equation of the base plane of the IPM, is then considered: $\text{IPM}(s, r) = s \cdot \hat{x} + r \cdot \hat{y}$ where $(\hat{x}, \hat{y})$ is the coordinate system linked to the map. A scalar field in the plane of the IPM, will finally be created to represent the probabilities of interference between the part and the gauge. This field is obtained by setting out the following relationship: $\text{ProbInter}(\text{IPM}(u, v), m(u, v)) = \text{Prob}(\vec{C}(u, v))$. The function prob. represents the field of probability of the presence of matter explained in Part I of our work, and ProbInter, the interference probability at one point of the IPM. The parameters l and m are continuous functions that are strictly rising and gener-
ally linear. An IPM is an opened-out version of a virtual gauge (Fig. 8).

IPMs will be the basis for verification with virtual gauges using statistical a criterion. In the next four paragraphs, the different steps that will allow us to calculate and plot IPMs will be presented.

### 3.3.2. Meshing an elementary gauge

It is impossible to find an analytical formulation for the calculation of interference probability and, therefore, of the complete FPPM of a measured part. A numerical analysis is too unwieldy to imagine a continuous description of an IPM. To solve this problem, a discretization of the surface of the virtual gauge will be created with the help of meshing. A continuous and approximate result will then be obtained by interpolation. The parametric equations of the gauge surfaces enable to easily mesh their surfaces. In Section 3.2, we clarified the parametric equation of a gauge and we expressed it as

\[
\vec{CV}(u, v) = \text{parametric equation of a gauge}
\]

The coordinates of the nodes in a mesh \(n\times m\) nodes, will be written in the following way:

\[
\vec{nCV}_{i,j} = \vec{CV}(u_i, v_j) \quad \text{with } (i, j) \in [1 \ldots n] \times [1 \ldots m]
\]

where \(u_i\) and \(v_j\) represent two numerical series representing the curvilinear coordinates of different nodes. They generally are arithmetical series that allow scanning the surface with a constant step (Fig. 9).

The FPPM that characterizes the real surface being checked is expressed in the three-dimensional affine space defined by the coordinate system of the CCM. A projection of the space of the basic gauge toward the space of the CCM must be made. Let us consider that the family of vectors defining the virtual gauge was chosen so that the vectors \(\vec{x}, \vec{y}, \vec{z}\) match the coordinate system of the CCM. The coordinates of a mesh node in the coordinate system of the CCM are obtained from the following scalar products:

\[
\begin{bmatrix}
\vec{x} \\
\vec{y} \\
\vec{z}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix} \cdot 
\begin{bmatrix}
0 \\
0 \\
|OC| \\
f(u_i, v_j) \\
g(u_i, v_j) \\
h(u_i, v_j)
\end{bmatrix}
\]

### 3.3.3. Calculating the interference probabilities in a mesh node

For each mesh node, the interference probability between the virtual gauge and the real surface is calculated. The result is a set of values:

\[
p_{i,j} = \text{prob} \left( \vec{nCV}_{i,j} = \vec{CV}(u_i, v_j) \right) = \text{prob} \left( \vec{nCV}_{i,j} \cdot \vec{x}, \vec{nCV}_{i,j} \cdot \vec{y}, \vec{nCV}_{i,j} \cdot \vec{z} \right)
\]

### 3.3.4. Interpolating interference probabilities

Interpolation is carried out linearly. The four nodes surrounding the point to be calculated have the following probabilities, respectively \(p_{i,j}, p_{i+1,j}, p_{i,j+1}, p_{i+1,j+1}\). We use the following parametrics (Fig. 11).

Probability at point \(M\) is calculated in the following way:

\[
p(M) = \left( \frac{1}{d_3 + d_4} \left( \frac{p_{i,j} \cdot d_2 + p_{i+1,j} \cdot d_1}{d_1 + d_2} \right) \cdot d_4 \right.
\]

\[
\left. + \left( \frac{p_{i,j+1} \cdot d_2 + p_{i+1,j+1} \cdot d_1}{d_1 + d_2} \right) \cdot d_3 \right)
\]

For each node this corresponds to the following integration (Fig. 10).

Next, interpolations remain to be developed in order to calculate IPMs completely.

### 3.3.5. Meshing a cylindrical virtual gauge

Fig. 9. Meshing a cylindrical virtual gauge.

### 3.3.6. Calculating interference probability for a node

Fig. 10. Calculating interference probability for a node.

### 3.3.7. Interpolating interference probabilities

Fig. 11. Parametrics of interpolation.
3.3.5. Plotting and interpreting IPMs

The results are displayed using a scale of grays to make it easier to visualize the different zone types (Fig. 12).

The black areas (interference probability ≈ 1) indicate a quasi-impossible assembly between the gauge and the real surface. In these critical zones, the virtual gauge is behind the estimated mean surface, i.e. inside the matter. The white areas allow us to assert that there is no interference between the gauge and the part. The areas having an interference probability close to 0.5 prohibit making any conclusions. These areas appear in two situations: either the mean surface is very close to the virtual gauge or the measurement uncertainties are too large in relation to the tolerance interval.

3.3.6. Two simple case studies

In order to better understand how to interpret IPMs, several simulated IPMs were drawn taking the case of a verification of envelope requirements. In the example under study, both the estimated mean surface and the associated virtual gauge are cylinders. The radius of the virtual gauge is taken slightly larger than the tested surface so as to obtain every kind of interference in a single example. Two cases were studied. The first one shows an axis displacement between the gauge and the mean surface, but without a parallelism deviation (Fig. 13). The second one is a study of interferences after a rotation between the gauge and the surface (Fig. 14).

3.3.7. Checking envelope requirements

Fig. 13 takes the case of a bore with specifications for an envelope requirement. The envelope requirement was chosen because the most favorable position for the virtual gauge in relation to the part is, of course, coaxiality between the estimated mean surface and the virtual gauge. According to the standard (ISO 8015-1985), the envelope requirement implies that a perfect cylindrical envelope set to the maximum material requirement must not be exceeded (Fig. 13). The distance of any two-point measurement must be lower than the allowed maximum of 30.97 mm (ISO 14 660-1). However, only the envelope constraint will be studied in this example. In the case under study, the diameter of the envelope is 30.93 mm long (Figs. 15 and 16).

Measurements were taken on a standard CCM. In order to show the influence of the number of points on uncertainties, two sets of 16 and 32 points were probed on the bore. Before performing our virtual gauge statistical control, a classical check based on a least squares method and the form defect has been carried out. For the set of 16 points a diameter of 30.987 mm and a form defect of 0.021 mm were calculated. The diameter of the maximum material envelope was estimated to 30.987 – 0.021 = 30.966 mm, which is large enough for the part to be accepted. For the set of 16 points, a diameter of 30.988 mm and a shape defect of 0.018 mm were calculated. The diameter of the maximum material envelope is 30.970 mm. These results do not take into account measurement uncertainties. Checking with a virtual gauge and real surface statistical modeling gives the following results. For a set of 16 points, the following IPM is obtained (Fig. 17).

This IPM shows three different areas. The central zone represents the part of the estimated mean surface which has the lowest
uncertainty. Interference probability is nearly nil, i.e. below 5%, and nearly constant in the whole area. This area is located in the vicinity of the barycenter of the evaluated points. The two other areas located at the limits of the estimated mean surface are regions of larger uncertainty (from 5% to 7.8%). In these areas, uncertainties tend to become, in an asymptotic way, proportional to the distance from the barycenter of probed points. In this example, we notice that there is a symmetrical revolution (revolution symmetry). This is a special case which indicates that the probed points have been evenly distributed in the bore. The covariance matrix of the random vector is a symmetric matrix defined as positive. Uncertainty propagation performed with a Jacobian operator implies that the shape of the iso-probable surface of a cylinder is a hyperboloid with an elliptical base. Thus, if a measurement is taken with concentrations of points in specific areas of the cylinder, a dissymmetry in revolutions will appear.

For the set of 32 points, the following IPM is obtained (Fig. 18).

With 32 points, the uncertainties are piercingly reduced. A maximum of 0.6% of interference probability is found on the extremities. Another striking fact is that the central area, where uncertainties are quasi-constant, is much larger. Even if the significant drop in uncertainties can be attributed to the increased number of points, it is important to note that the maximum deviation on the free side of matter is estimated as smaller (0.018 mm) in the case of 16 points than in the case of 32 points (0.021 mm).

3.3.8. Conclusion

The interference probability map is an essential element of our method of control using a virtual gauge with statistical criterion. It presents two major advantages. First of all, thanks to graphic representations of results, it enables us to have a clear and rapid interpretation of the conformity of a part and the interference probabilities with matter. This allows the metrologist to define an objective function in order to optimize the position of a gauge system in relation to a part when verifications are more complex. The problem of optimization will be taken up in the following section.

4. The method of virtual gauge geometric verification using statistical criterion

The preceding sections showed how to characterize a real surface using a field of probability of presence of matter and then how to build IPMs. However, theses concepts are not sufficient to complete the verification operation. In fact, there are several remaining points to integrate:

- Assembling several gauges.
- Optimizing the position of a system of gauges in relation to the part.

In this section, two efficient tools correlated to the Gram matrix will be introduced to manipulate basic gauges. First of all, a perturbation operator will be considered which allows moving and/or distorting a set of basic gauges. Second, a connection operator will be presented to create virtual gauge systems from a set of elementary gauges. The problem of optimizing the position of a system of gauges in relation to a FPPM will then be described, using a “physical” approach. Finally, a mathematical formulation of the objective function will be given.

4.1. Assembling and manipulating an elementary gauge system

4.1.1. Perturbation operator

A powerful mathematical tool to manipulate Gram matrix, the perturbation matrix, has been developed [16]. Perturbation is a tool which is both flexible and powerful, able to manipulate geometric objects described by a Gram matrix. The idea is to model the distortions of a family of vectors $E$ by using their data. Given a family of vectors $E = \{V_1, V_2, \ldots, V_p\}$, $V_i \in A = \mathbb{R}^n$ and an operator $\tilde{\Omega} : A^p \to A^p$ it is possible to write the following relationship: $\Delta E = \tilde{\Omega} \cdot E$. The family of vectors $\Delta E$ is the perturbation and the operator $\tilde{\Omega}$ represents this perturbation. The result of perturbation is then obtained by simply adding the original family $E$ to the perturbation $\Delta E$:

$E_{n+1} = \Delta E_n + E_n$

$E_{n+1} = \tilde{\Omega} \cdot E_n + E_n$

$E_{n+1} = (\tilde{\Omega} + \Delta) E_n$

Fig. 17. Interference probability map for a set of 16 points.
The use of the perturbation matrix can be applied to the perturbation of a Gram matrix:
\[
\tilde{G}_{n+1} = \tilde{E}_{n+1} \otimes \tilde{E}_{n+1} \\
\tilde{G}_{n} = \left(\tilde{I} + \tilde{\Omega}\right) \cdot \tilde{E}_{n} \otimes \left(\tilde{I} + \tilde{\Omega}\right)^{T} \\
\tilde{G}_{n+1} = \left(\tilde{I} + \tilde{\Omega}\right) \cdot \tilde{G}_{n} \cdot \left(\tilde{I} + \tilde{\Omega}\right)^{T}
\]

In practice, with the perturbation matrix, any linear transformation can be applied to family \( E \). Therefore, this matrix will allow us to make all displacements of virtual gauges in relation to the part.

4.1.2. Connecting two basic gauges

The objective in connecting several gauges is to describe the geometry of the set of these gauges in one single comprehensive matrix. Manipulating the gauges will therefore be greatly simplified during the optimization stage. Moreover, the property of a description with a Gram matrix is to be invariant to a solid displacement. The numerical precision of the geometric model can also be improved. In the preceding section, we showed that every elementary gauge contains 7 vectors. The first three are the 3 vectors making up the construction coordinate system linked to the nominal design. This coordinate system must be chosen so it can be common to all the elementary gauges. This enables to express the connection constraint by specifying the equality between the first three vectors, and the construction coordinate system of each elementary gauge. The two elementary gauges are assembled in three steps (Fig. 19). The first step is the preparation of a comprehensive unconnected matrix from two elementary matrices. Next, a complete specification of the connection, i.e. assembly, must be put together. Finally, the connection can be calculated.

A gauge system is assembled by iterating the assembly of \( n \) elementary gauges. An assembly of \( n \) elementary gauges appears as a series of \( n-1 \) connections. The whole assembly is completed by assembling each gauge, minus the first, one by one.

4.2. Control method

Once the statistical characterization of the parts and the modeling of the virtual gauges have been completed, the last step to be carried out is the control procedure: the conformance test. The goal of the conformance test is to find the relative position of virtual gauges in relation to the part which is the most favorable, i.e. the position which maximizes conformance probability. For every virtual gauge, an IPM can be generated in order to localize the critical point of the virtual gauges/estimated mean surface assembly. The aim is to minimize the greatest probability of interference.

4.2.1. A definition of statistical objective

Given an FPPM representing the dispersion of the estimated mean surface and a series of Gram matrices \( G_{n} \) representing the configuration of the set of virtual gauges, it is then possible to calculate the set of functions \( p \), so that
\[
p_{i,j,k}(\tilde{G}_{n}) = \frac{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^{2}}} e^{-\frac{1}{2\sigma^{2}} \frac{d^{2}}{Total/\hat{R}_{S}}} dx}{2^{\frac{1}{2}} \frac{d^{2}}{Total/\hat{R}_{S}}}
\]

The objective function \( f \) is defined as the interference probability of the point having the greatest interference probability:
\[
f(G_{n}) = \max_{\forall k \in [1, \ldots, r]} \left(\frac{\tilde{G}_{n}}{Total/\hat{R}_{S}}\right)_{i,j,k}
\]

With \( r \) representing the number of virtual gauges, and \( r \times s \) the dimensions of the virtual gauge meshing \( l \).

This function has to be minimized to define the best assembly configuration between the virtual gauge and the fitted surfaces.

4.2.2. Formulating the optimization problem

Fig. 20 shows the organization of the problem to be optimized. It is divided into two units, the virtual gauge system and the set of FPPM representing the estimation of real surfaces. The objective function is defined as the set of maximum probability for each IPM and the free parameters to optimize. The latter defines the connec-

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tion between the coordinate system of the CMM and the coordinate system of the virtual gauge system.

The optimization of the connection between the reference system of the CMM and the virtual gauge system of reference is an optimization of orientation and positioning in the gauge system. This problem will be formulated as a recurring series converging toward the optimal position. The initial value of this series is a virtual gauge system defined as an offset operation in relation to nominal geometry.

The optimization problem is defined in the following way:

\[
\begin{align*}
\bar{f}(\bar{G}) & \geq \bar{f}(\bar{G}_{opt}) \quad \forall \bar{G} \in D \\
\bar{f}(\bar{G}) & = \max\left(p_{i,j,k}(\bar{G})\right) \\
f : \mathbb{R}^{n \times n} & \rightarrow \mathbb{R}
\end{align*}
\]

where \(G_{opt}\) represents the optimal position, \(D\) is the search space defined by the system of geometric specifications, i.e. the set of Gram matrices couples/length vector, defining a virtual gauge system in compliance with the geometric specification (shape, orientation, and position). To optimize the values of \(G\) directly, a rather complex system of constraints for optimization must be defined. To avoid this problem, let us use a recurring series of metric tensors obtained by a perturbation matrix applied to the previous tensor:

\[
\bar{G}_{n+1} = (\bar{I} + \bar{\Omega}_n) \cdot \bar{G}_n \cdot (\bar{I} + \bar{\Omega}_n)^T
\]

with : \(G_0\) defined by offset to the CAD system geometry

Formulating the optimization problem using a series of perturbation matrices allows solving it without introducing any constraint. The perturbation matrices consist of rotations and translations of the gauge elements in relation with their degrees of freedom.

5. Applying a test part to controls

This paragraph will deal with applying statistical criterion virtual gauge controls on a real part. The example under study focuses on three parallel bores each localized by the other, and specified with a MMC requirement. This geometric specification provides a rigid virtual gauge, i.e. without degrees of freedom. The system of virtual gauges is free in relation to the real part. Optimization will focus on 6 free parameters which are 3 translations and 3 rotations of the gauge system in relation to the statistical features of the part. Firstly, an interpretation of the geometric specifications of the part is proposed. The Gram matrix model of the virtual gauge system is thus created. Next, the checking procedures will be given in detail. Finally, the IPMs resulting from the optimization are drawn.

5.1. Definition and geometric specification of the test part

Fig. 21 presents the geometric specification of the part under study.

5.1.1. Interpretations using the ISO standard

The part under consideration contains three real elements known as cylindrical, specified to the MMC and each localized by the other. The diameter of the virtual condition to the maximum of...
matter of each real element must be less than a defined value. The virtual gauge system is made up of three perfect parallel cylinders of a diameter of 19.97 mm for the first one, 19.92 mm for the second one and 19.92 mm for the third. The axes for the first and second cylinder are 60 mm in distance and the distance between the axis of the third cylinder and the plane formed by the axes of the first two cylinders is 100 mm. Moreover, the third cylinder is implicitly centered in relation to the first two in the direction of y (Fig. 21).

5.1.2. Gram matrix modeling

By interpreting the geometric specification of the part under study in Fig. 21 we are now able to lay out the parametrics used for the Gram matrix representation of the gauge system (Fig. 22).

Three families of vectors are found which make up the three basic virtual gauges and their associated Gram matrices:

\[ E_1 = \left\{ \hat{x}, \hat{y}, \hat{z}, \vec{e}_{OC,1}, \vec{e}_{1,1}, \vec{e}_{1,2}, \vec{e}_{1,3} \right\} \Rightarrow G_{elem,1} = E_1 \otimes E_1^T \]

\[ E_2 = \left\{ \hat{x}, \hat{y}, \hat{z}, \vec{e}_{OC,2}, \vec{e}_{2,1}, \vec{e}_{2,2}, \vec{e}_{2,3} \right\} \Rightarrow G_{elem,2} = E_2 \otimes E_2^T \]

\[ E_3 = \left\{ \hat{x}, \hat{y}, \hat{z}, \vec{e}_{OC,3}, \vec{e}_{3,1}, \vec{e}_{3,2}, \vec{e}_{3,3} \right\} \Rightarrow G_{elem,3} = E_3 \otimes E_3^T \]

The connection procedure is used to obtain the global Gram matrix of the virtual gauge. It is of size 15 \times 15 and of rank 3.

5.2. Measuring and checking the part

Each of the three bores is acquired by a set of 16 points probed on a standard CMM. The three elements of the virtual gauge are to be compared to the FPPM of the three best-fitted cylinders (Fig. 23). The following section presents the results after checking.

5.3. Interpreting the results

The checking procedure leads to three interference probability maps (Figs. 24–26). The results show that the part satisfies the geometric specification. On all the maps, three characteristic zones are however recognized. The first zone of interest is the area close to the barycenter of the probed points. In this area, the uncertainties of the fitted surfaces are the lowest. This leads to the smallest interference probabilities. The maps show two other zones at the extremities where the uncertainties rise in proportion to the dis...
tance from the center. The first map shows that the corresponding bore has the largest probability of interference, even if the level of risk remains under 1% (Fig. 24).

The second map (Fig. 25) shows a strong dissymmetry. Since quite low interference probabilities are calculated at a horizontal line of centers, it means that the problem is not linked to the diameter and position of the bore, but to an orientation defect. The maximum interference probability is around 0.5%.

The third IPM (Fig. 26) shows that this bore is the less constraint. At each coordinate of this map, the interference probabilities are very low.

6. Conclusion

This paper uses the statistical approach of CMM measurements which has been proposed in our previous work to characterize any real surface by the field of probability of the presence of matter. This field, based on the concept of random vectors, defines the probability of any point to stay inside the matter. It integrates the random perturbations of the CMM, the uncontrolled factors of the sampling strategy and the form defects of the measured surface. It permitted thus implementing a virtual gauge method which allowed checking the geometrical conformance of a real part defining the risk of acceptance or refusal of the product. For that purpose the skeleton of the virtual gauge has been created in relation to the geometrical specification of the CAD system. It is implemented in the form of Gram matrices which helps describing the connection between the different elementary gauge features. This approach permits also easily distorting the virtual gauge in relation to the degrees of freedom allowed by the geometrical specifications. A perturbation operator is introduced for that purpose. This operator permits also optimizing the position of the virtual gauge to compare it to the FPPM. This optimization procedure provides the risk level taken in accepting or refusing the tested part. Such method avoids analysing only the mean parameters of the best-fitted surfaces and keeps away from any intermediate geometrical construction which amplifies the initial measurement uncertainties. It has been tested successfully on a real case study.

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