Boundary Geometric Control of Nonlinear Diffusion Systems

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Abstract: The paper addresses the boundary control of a nonlinear diffusion system submitted to Neumann actuation. The control law is designed in the framework of geometric control theory using directly the nonlinear partial differential equation model without any previous reduction. First, an equivalent linear model using the Cole-Hopf transformation is obtained, then the manipulated variable is inserted in the state equation of this equivalent linear model by means of a Dirac delta function to make the boundary condition homogeneous. Based on the resulting final model, the control law is derived using the characteristic index notion and the closed loop stability is demonstrated using concepts from the powerful semigroup theory. The control law performance is evaluated through numerical simulation by considering a nonlinear heat conduction control problem. Copyright © 2013 IFAC.

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1. INTRODUCTION

Control of nonlinear distributed parameter systems (DPSs) is a challenging field (Chen et al., 1999; Christofides, 2001). Compared with the control theory of the linear DPSs, which has attained a certain level of maturity thanks to semigroup theory (Pazy, 1983), many interesting questions concerning control theory of nonlinear DPSs remain open (Chen et al., 1999; Christofides, 2001; Padhi and Faruque Ali, 2009). Among these questions, boundary control occupies an important place and constitutes an active research area (Vazquez and Krstic, 2004; Krstic and Smyshlyaev, 2008). In practice, boundary control is more convenient for real implementation and, compared to interior control, it represents an economic approach. Nevertheless, from a control point of view, boundary control introduces technical complexities since the control operator is unbounded (Curtain and Zwart, 1995; Emirşlİow and Townley, 2000).

The design of a boundary control can be performed following two approaches (Ray, 1989; Christofides, 2001). The first one is early lumping, which consists in performing a spatial discretization of the partial differential equations (PDEs) to derive a set of ordinary differential equations (ODEs) which constitute an approximation of the original PDE model, and the controller design is performed in the framework of the control theory of lumped parameter systems (LPSs). It must be noted that through early lumping, the fundamental control theoretical properties (controllability, observability and stability) are lost, which generally leads to high dimensional controllers (Ray, 1989; Christofides, 2001). The second approach, termed as late lumping, uses directly the PDEs model for the controller design (Christofides and Daoutidis, 1996; Shang et al., 2005; Maidi and Corriou, 2011b). The approximation is performed only for implementation purpose. Late lumping allows us to avoid the distributed nature loss of the PDE system and to take full advantage of their natural properties.

Recently geometric control has proved to be very successful as a control approach of linear and quasi-linear DPSs. Christofides and Daoutidis (1996) developed a distributed control law for first-order quasilinear hyperbolic systems, based on the concept of the characteristic index and illustrated its performances by a successful application to the control of a tubular reactor. Using the concept of characteristic index, successful applications are addressed by Gundepudi and Friedly (1998), Wu and Liou (2001) and Maidi et al. (2009, 2010). Control based on the geometric theory is advantageous as the PDE model can be used in control design without any approximation, which allows to preserve the fundamental control theoretical properties since the distributed nature of the system is taken into account, thus increasing the performance (Ray, 1989; Christofides, 2001). In addition, geometric control allows to take advantage of the full potential of an existing control theory for LPSs that offers full powerful controller design techniques.

The examination of the DPSs control literature based on geometric control shows that most proposed control methodologies deal with distributed control for linear distributed parameter systems. An attempt to extend the
geometric approach to nonlinear systems, with boundary actuation and punctual controlled variable, is proposed by Maidi and Corriou (2011a). Nevertheless, the difficulty for the geometric control of nonlinear systems is that the closed loop stability is difficult to prove and needs some sophisticated mathematical tools from functional analysis. To overcome this difficulty in the case of a nonlinear diffusion system with distributed control, an equivalent linear model obtained using the Cole-Hopf transformation is considered and the stability is demonstrated by Maidi and Corriou (2012) thanks to semigroup theory.

The investigations of the present work are intended as a contribution to the boundary control of nonlinear diffusion systems, which is one of the most important classes encountered in a wide variety of practical applications (Demetriou and Kazantzis, 2005). Thus, a design approach of the control law that enforces the desired performance and stability is developed in the case of Neumann actuation. In Section 2, the boundary control problem is formulated. The principle of the Cole-Hopf transformation used to derive the equivalent linear model is presented in Section 3. Section 4 is devoted to the proposed control design approach while Section 5 illustrates the performance of the controller in the case of a heating problem of a steel rod with a non constant thermal conductivity. Finally, concluding remarks are provided in Section 6.

2. BOUNDARY CONTROL PROBLEM

The class of DPS, considered here, is modeled by the one-dimensional nonlinear PDE according to the following state-representation

\[ \rho c_p \frac{\partial x(z, t)}{\partial t} = \frac{\partial}{\partial z} \left( k(x(z, t)) \frac{\partial x(z, t)}{\partial z} \right) \quad \text{in } \Omega \times [0, t[ \]

accompanied by the boundary conditions

\[ -k(x(0, t)) \frac{\partial x(z, t)}{\partial z} \bigg|_{z=0} = u(t) \quad \text{(2)} \]
\[ -k(x(l, t)) \frac{\partial x(z, t)}{\partial z} \bigg|_{z=l} = 0 \quad \text{(3)} \]

and the initial condition

\[ x(z, 0) = x^*(z), \quad \text{in } \Omega \quad \text{(4)} \]

where \( x(z, t) \) denotes the state, \( z \in \Omega = [0, l] \subset \mathbb{R} \) is the spatial domain, \( t \in [0, \infty[ \) is the time variable. Without loss of generality and to simplify the presentation, the terminology from heat conduction is adopted. Consequently, \( k(x(z, t)) > 0, \rho, c_p \) respectively denote the thermal conductivity, density and heat capacity. \( x^*(z) \) is a given initial profile. \( u(t) \in L^2([0, \infty[; \mathbb{R}) \) is the manipulated heat flux. The state-space is defined as \( L^2(\Omega) \), which is the space of the square-integrable functions on \( \Omega = [0, l] \), endowed with the usual norm \( \| \cdot \|_2 \) and the inner product \( \langle , \rangle \) (Atkinson and Han, 2009).

The controlled variable is defined as

\[ y(t) = C x(z, t) = \int_0^l c(z) x(z, t) \, dz; \quad C \in \mathcal{L}(L^2(\Omega); \mathbb{R}) \]

where \( C \) is a bounded linear operator, \( \mathcal{L} \) is the space of all linear bounded operators from \( L^2(\Omega) \) into \( \mathbb{R} \). \( c(z) \) is a known smooth function chosen according to the control objectives and control design specifications (Christofides, 2001). In the following, it is assumed that \( c(z) \in H^2(\Omega), H^2(\Omega) \) being a Hilbert space defined as the Sobolev space of order 2 (Atkinson and Han, 2009), \( c(z) \) verifies the following conditions

\[ c(0) \neq 0 \text{ and } c(l) = 0 \quad \text{(6)} \]
\[ \dot{c}(0) = \dot{c}(l) = 0 \quad \text{(7)} \]

with \( \dot{c}(z) = dc(z)/dz \).

3. COLE-HOPF TRANSFORMATION

The Cole-Hopf transformation consists in converting the nonlinear diffusion equation into a linear one if the thermal diffusivity defined by the following ratio

\[ \alpha = \frac{k(x(z, t))}{\rho c_p} \]

is approximately constant. This hypothesis is accepted in many cases since the variation of \( \alpha \) with \( x(z, t) \) is much less important than that of \( k(x(z, t)) \), so that this approximation is reasonable (Carslaw and Jaeger, 1959), therefore \( \alpha \approx \text{constant} \).

To linearize the nonlinear diffusion equation (1) using the Cole-Hopf technique, one seeks a transformation of the form (Vadasz, 2010)

\[ x(z, t) = h(w(z, t)) \]

where \( h(x(z, t)) \) is a continuous bijective function (one-to-one function or mapping). By using the transform (9), the evaluation of the derivatives of the left hand side of (1) gives

\[ \frac{\partial h(w(z, t))}{\partial t} = \frac{\partial h(w(z, t))}{\partial w(z, t)} \frac{\partial w(z, t)}{\partial t} \]

and for the right hand side of (1), one obtains

\[ \frac{\partial}{\partial z} \left( k(h(w)) \frac{\partial h(w(z, t))}{\partial z} \right) = M \left( \frac{\partial w(z, t)}{\partial z} \right)^2 \]
\[ + k(h(w)) \frac{dh(w)}{dw} \frac{\partial^2 w(z, t)}{\partial z^2} \]

with

\[ M = k(h(w)) \frac{d^2 h(w)}{dw^2}(z, t) \]
\[ + k(h(w)) \left( \frac{dh(w)}{dw} \right)^2 \]

To make the right hand side of (11) linear, the term \( M \) is set equal to zero. The term \( M \), which a differential equation, can be expressed in the following integrable form

\[ \frac{d}{dw(z, t)} \left[ k(h(w(z, t))) \frac{dh(w(z, t))}{dw(z, t)} \right] = 0 \]

Integrating (13) gives

\[ k(h(w(z, t))) \frac{dh(w(z, t))}{dw(z, t)} = c_1 \]

hence

\[ \int k(h(w(z, t))) dw(w(z, t)) = \int c_1 dw(z, t) \]
\[ = c_1 w(z, t) + c_2 \]

or equivalently

\[ w(z, t) = h^{-1}(x(z, t)) = \frac{1}{c_1} \int k(x(z, t)) dx(z, t) - \frac{c_2}{c_1} \]

where \( h^{-1} \) is the inverse function of \( h \), which is continuous.
The equivalent punctual control problem is given as this transformation can be found in Maidi and Corriou (2011a) to get an homogeneous form by inserting the manipulated homogeneous boundary condition (18) will be transformed according to (16), mathematically it is easier to consider new boundary conditions can be easily deduced and by taking into account equation (14), the following 

\[
\begin{align*}
\frac{\partial w(z, t)}{\partial z} \bigg|_{z=0} &= -\frac{u(t)}{c_1} \\
\frac{\partial w(z, t)}{\partial z} \bigg|_{z=t} &= 0
\end{align*}
\]

and the controlled output becomes 

\[
y(t) = Ch(w(z, t)) = \int_0^l c(z) h(w(z, t)) \, dz
\]

To determine the control law of the original system (1)–(5), the equivalent linear model (17)–(20) will be used in the design procedure proposed in the following section.

4. BOUNDARY CONTROLLER DESIGN

To design the control law that ensures the desired performances of the output \( y(t) \), the equivalent linear model will be used. The control law will be derived in the framework of geometric control using the notion of the characteristic index introduced by Christofides and Daoutidis (1996), which is a generalization to DPSs of the concept of relative order used in LPSs (Isidori, 1995). For a boundary control problem, as the manipulated input \( u(t) \) is applied at the boundary, it is easy to show that the characteristic index does not exist. To overcome this problem, the inhomogeneous boundary condition (18) will be transformed under homogeneous form by inserting the manipulated variable \( u(t) \) into the state equation (17) with a Dirac delta function (Ray, 1989; Maidi and Corriou, 2011a) to get an equivalent punctual control problem. More details about this transformation can be found in Maidi and Corriou (2011a).

4.1 Equivalent punctual control problem

The equivalent punctual control problem is given as

\[
\begin{align*}
\frac{\partial w(z, t)}{\partial t} &= \alpha \frac{\partial^2 w(z, t)}{\partial z^2} + \alpha \frac{\partial z}{c_1} \delta(z) u(t) \\
\frac{\partial w(z, t)}{\partial z} \bigg|_{z=0} &= 0
\end{align*}
\]

where \( \delta(z) \) is the Dirac delta function. The right boundary condition (19) is unchanged and, by an appropriate choice of \( c_2 \), the initial condition is 

\[
w(z, 0) = 0
\]

Note that the linear operator \( \alpha \delta(z)/c_1 \) is unbounded (Curtain and Zwart, 1995; Emirslow and Townley, 2000), i.e. it maps out of the considered state space \( L^2(\Omega) \), thus to have a bounded operator, it is assumed that the punctual actuator is spanned over a finite portion of spatial domain \( \varepsilon \), that is, \( \delta(z) \) is approximated as

\[
\delta(z) = \begin{cases} 
1/\varepsilon & \text{for } 0 \leq z \leq \varepsilon \\
0 & \text{for } \varepsilon < z \leq t
\end{cases}
\]

thus \( \delta(z) \in L^2(\Omega) \).

4.2 Control law design

Based on the notion of characteristic index, the controller is designed. Hence, the first derivative of the controlled output (20) is

\[
\begin{align*}
\frac{dy(t)}{dt} &= C \left\{ \frac{dh(w(z, t))}{dt} \frac{\partial w(z, t)}{\partial t} \right\} \\
&= \frac{c_1}{\alpha \rho c_p} CA w(z, t) + \frac{c_1}{\alpha \rho c_p} CB u(t)
\end{align*}
\]

where \( A = \alpha \partial^2 / \partial z^2 \) and \( B = \alpha \delta(z)/c_1 \).

The manipulated input \( u(t) \) appears linearly in the first time derivative (25) of the output \( y(t) \). The development of the second term of the right-hand side of equation (25) leads to

\[
\begin{align*}
\frac{c_1}{\alpha \rho c_p} CB u(t) &= \frac{c_1}{\alpha \rho c_p} \int_0^l c(z) \delta_\varepsilon(z) u(t) \, dz \\
&= \frac{c_1(0)}{\rho c_p} u(t)
\end{align*}
\]

According to condition (6), \( c(0) \neq 0 \), hence the characteristic index is \( \sigma = 1 \), which suggests requesting the following input-output response for the closed loop system (between the controlled output \( y(t) \) and the reference input \( y^d(t) \)).

\[
\tau \frac{dy(t)}{dt} + y(t) = y^d(t)
\]

where \( \tau \) is the desired closed loop time constant.

Thus, substituting (25) into equation (28), the state feedback control law can be easily deduced as follows

\[
u(t) = \frac{1}{c(0)} \left\{ \frac{\rho c_p}{\tau} [y^d(t) - y(t)] - \frac{c_1}{\alpha} CA w(z, t) \right\}
\]

Let us express the control law \( u(t) \) according to the state \( x(z, t) \). Remember that the expression (12) is made equal to zero by transformation (9), thus considering equations (11) and (14), the term \( Aw(z, t) \) of the control law (29) can be written as

\[
Aw(z, t) = \alpha \frac{\partial^2 w(z, t)}{\partial z^2} = \alpha \frac{\partial}{\partial z} \left( k(x(z, t)) \frac{\partial x(z, t)}{\partial z} \right)
\]

and the control law (29) takes the following form

\[
u(t) = \frac{1}{c(0)} \left\{ \frac{\rho c_p}{\tau} [y^d(t) - y(t)] - C \left\{ \frac{\partial}{\partial z} \left( k(x(z, t)) \frac{\partial x(z, t)}{\partial z} \right) \right\} \right]\]
Remark 2. The proposed design approach of the control law is developed for controlling the output \( y_p(t) \) defined as the spatial weighted average. The proposed control law is still applicable in the case of a punctual output \( y_p(t) \) by adopting the control strategy proposed by Maidi et al. (2010).

### 4.3 Closed loop stability

In Maidi and Corriou (2011a), a certainty equivalence nonlinear punctual control problem is derived using a linearization procedure but the control problem remains nonlinear, thus proving the closed loop stability is a difficult task. On the other hand, by using the Hole-Cop transformation, the resulting equivalent punctual control problem is linear, consequently the closed loop stability can be easily proved using some concepts from semigroup theory.

The closed loop equivalent system is given by

\[
\frac{\partial w(z, t)}{\partial t} = (A + F) w(z, t) + \frac{\rho c_p}{\tau (c(0))} B Y(t) \tag{32}
\]

\[ y(t) = Ch(w(z, t)) \tag{33}\]

where \( Y(t) = y^d(t) - y(t) \) and \( F = (-\delta(z)/c(0)) C A \).

Thus, this system can be written in the form of the following interconnected system and w-system

\[
\frac{\partial y(z, t)}{\partial t} = -\delta(z) Y(t) \tag{35}
\]

To verify the closed loop stability, it is equivalent to verify that the cascade interconnection is stable, i.e. to verify the closed loop stability of each subsystem (Christofides and Daoutidis, 1996; Maidi and Corriou, 2011b, 2012).

The y-system is exponentially stable since the time constant \( \tau > 0 \), consequently

\[
|Y(t)| \leq K_Y |Y(0)| e^{-\omega_Y t}, \quad K_Y \geq 1 \text{ and } \omega_Y > 0 \tag{36}
\]

with \( Y(0) = y^d(0) - y(0) \).

For the w-subsystem, the operator \( A + F \) generates a semigroup if the zero dynamics associated with the open-loop system (32)–(33), obtained by constraining the reference input \( y^d(t) \) and the output \( y(t) \) to zero \( y(t) = y^d(t) = 0 \), given as

\[
\frac{\partial w(z, t)}{\partial t} = (A + F) w(z, t) \tag{37}
\]

\[ y(t) \equiv 0 \tag{38}\]

is exponentially stable (Christofides and Daoutidis, 1996).

The operator \( A + F \) generates a semigroup if the operator \( A \) is a generator of a semigroup and the operator \( F \) is bounded on the considered state space \( L^2(\Omega) \) (Pazy, 1983, Theorem 1.1, page 76).

The operator \( A \) with the boundary conditions (19) and (22), and the initial condition (23) generates an exponentially stable semigroup \( U(t) \) (El Jai and M. Amouroux, 1990; Curtain and Zwart, 1995), that is,

\[
\|U(t)\|_2 \leq K e^{-\omega t} \tag{39}\]

with stability constants \( K = 1 \) and \( \omega = \alpha (\pi/\ell)^2 > 0 \).

The operator \( F \) is bounded if there exists a constant \( C \) such that

\[
\|Fa(w, t)\|_2 \leq C \|a(w, t)\|_2 \tag{40}\]

The left hand side of (40) can be developed as follows

\[
\|Fa(w, t)\|_2 = \|\left(\delta_c(z)/c(0)\right) \delta(z) C A w(z, t)\|_2 \tag{41}\]

then by taking into account the conditions (6) and (7) and following the same development as in Maidi and Corriou (2012), one obtains

\[
\|Fa(w, t)\|_2 \leq (\alpha/c(0))^2 \gamma \|a(w, t)\|_2 \tag{42}\]

As \( c(z) \in H^2(0, l) \), the constant \( C \) of (40) exists and is equal to \( \gamma^{1/2}/c(0) \), which means that \( F \) is bounded and \( \|F\|_2 = \gamma^{1/2}/c(0) \).

According to the perturbation theorem (Pazy, 1983, Theorem 2.3, page 132), the operator \( A + F \) generates a semi-group \( W(t) \) such that

\[
W(t) \leq K e^{-\omega_K t} \tag{43}\]

The semigroup \( W(t) \) is exponentially stable if \( \omega_K = \omega - K \gamma^{1/2}/c(0) > 0 \). Consequently, the exponential stability of the zero dynamics is related to the choice of the function \( c(z) \) since \( \gamma \) depends on it. Thus, having chosen a function \( c(z) \) that ensures the stability condition, the operator \( A + F \) generates an exponentially stable semigroup \( W(t) \).

In this case, the state \( w(z, t) \) of the closed loop system (32)–(33) verifies (Curtain and Zwart, 1995)

\[
\|w(z, t)\|_2 \leq \|w(0, t)\|_2 e^{-\omega w t} \tag{44}\]

Substituting \( |Y(\xi)| \) by its expression (36) in (45) gives

\[
\|w(z, t)\|_2 \leq \|w(0, 0)\|_2 e^{-\omega w t} + N e^{-\omega w t} \int_0^t e^{((\omega - \omega_Y) \xi)} d\xi \tag{46}\]

with \( N = \alpha \rho c_p \|\delta_c(z)\|_2 K_Y |Y(0)|/(c_1 \tau c(0)) \).

Then, if \( \omega_K < \omega \),

\[
\|w(z, t)\|_2 \leq \|w(0, 0)\|_2 e^{-\omega w t} + N t e^{-\omega w t} \leq \|w(0, 0)\|_2 e^{-\omega w t} + N t e^{-\omega w t} \tag{47}\]

where \( 0 < \omega < \omega_K \). Thus, the closed loop system is exponentially stable.

Now, if \( \omega < \omega_K \),

\[
\|w(z, t)\|_2 \leq \|w(0, 0)\|_2 e^{-\omega w t} + N e^{-\omega w t} \left[ e^{((\omega - \omega_Y) t)} - 1 \right]/(\omega - \omega_Y) \leq \|w(0, 0)\|_2 e^{-\omega w t} + N e^{-\omega w t} \left[ e^{(\omega - \omega_Y) t} - e^{-\omega w t} \right]/(\omega - \omega_Y) \tag{48}\]

\[ \leq \|w(0, 0)\|_2 e^{-\omega w t} + N e^{-\omega w t} \omega - \omega_Y \tag{48}\]

\[ \leq \|w(0, 0)\|_2 e^{-\omega w t} + N e^{-\omega w t} \omega - \omega_Y \tag{48}\]
According to (31), the control law is given as

\[ K \cdot T \leq w(z, 0) e^{-\omega_T t} + N \frac{e^{-\omega_T t}}{\omega_T - \omega} \]  

(49)

In each case, the closed loop system is exponentially stable. This implies that \( w(z, t) \) is exponentially stable, thus \( \lim_{t \to \infty} w(z, t) = 0 \). Consequently, according to (9), it follows that

\[ \lim_{t \to \infty} x(z, t) = \lim_{t \to \infty} h(w(z, t)) = \lim_{w \to 0} h(w) \]  

(50)

Since \( h(w) \) is a continuous bijective function, hence the limit (50) exists, i.e. the state \( x(z, t) \) is stable and the closed loop system

\[ \frac{\partial x(z, t)}{\partial t} = \frac{1}{\rho c_p} \frac{\partial}{\partial z} \left( k(x(z, t)) \frac{\partial x(z, t)}{\partial z} \right) \]  

(51)

with left and right boundary conditions

\[ -k(x(0, t)) \frac{\partial x(z, t)}{\partial z} \bigg|_{z=0} = \frac{1}{c(0)} \left[ \frac{\rho c_p}{\tau} [y^d(t) - y(t)] - \int_0^t c(z) \frac{\partial}{\partial z} \left( k(x(z, t)) \frac{\partial x(z, t)}{\partial z} \right) dz \right] \]  

\[ -k(x(l, t)) \frac{\partial x(z, t)}{\partial z} \bigg|_{z=l} = 0 \]  

(52)

is internally stable.

5. APPLICATION EXAMPLE

Let us consider the temperature control problem of a one-dimensional steel rod (Fig. 1). The manipulated variable is a heat flux \( q''(t) \) applied at \( z = 0 \), while at \( z = l \), the rod is assumed to be perfectly insulated. The corresponding model in terms of deviation variables is given as

\[ \rho c_p \frac{\partial T(z, t)}{\partial t} = \frac{\partial}{\partial z} \left( k(T(z, t)) \frac{\partial T(z, t)}{\partial z} \right) \]  

(53)

\[ -k(T(0, t)) \frac{\partial T(z, t)}{\partial z} \bigg|_{z=0} = q''(t) \]  

(54)

\[ -k(T(l, t)) \frac{\partial T(z, t)}{\partial z} \bigg|_{z=l} = 0 \]  

\[ T(z, 0) = 0 \]  

(55)

\[ T(0, t) = 0 \]  

(56)

The thermo-physical properties of the steel (Taler and Duda, 2006) are

\[ \rho = 7884 \text{ kg m}^{-3}, \quad c_p = 520 \text{ J kg}^{-1} \text{ K}^{-1}, \]  

and the thermal conductivity \( k \) is a nonlinear function of temperature given as

\[ k(T(z, t)) = 14.6 + 1.27 \times 10^{-2} T(z, t) \]  

(57)

where \( T \) is expressed in °C and \( k \) in W m\(^{-1}\) K\(^{-1}\).

The objective is to control the spatial weighted average temperature \( T_m(t) \), along the rod, defined as

\[ T_m(t) = \int_0^l [1 + \cos(\pi z)] T(z, t) dz \]  

(58)

to following a desired set-point \( T_m^d(t) \), whose units are K m.

According to (31), the control law is given as

\[ q''(t) = \frac{1}{2} \frac{\rho c_p}{\tau} \left[ T_m^d(t) - T_m(t) \right] - \int_0^t [1 + \cos(\pi z)] \frac{\partial}{\partial z} \left( k(T(z, t)) \frac{\partial T(z, t)}{\partial z} \right) dz \]  

(59)

For simulation purpose of the closed loop system, the method of lines (Wouwer et al., 2001) is applied with evaluation of the spatial partial derivatives by means of finite differences based on \( N = 200 \) discretization points. The integral terms in the control law are evaluated numerically using the trapezoidal method. The control law parameter \( \tau \) is taken equal to 5 min and the control is held constant over the sampling period equal to 2 min. To avoid the consequences due to brutal set point changes, the desired set point \( T_m^d(t) \) has been filtered by a first order filter with a time constant equal to \( \tau_f = 5 \) min. To evaluate the performance of the control strategy, a set point step corresponding to \( T_m^d(t) = 50 \text{ K m} \) of the temperature is specified at \( t = 20 \) min. The performance of the controller is shown by Figs. 2 and 3. The simulation results show clearly that the temperature \( T_m(t) \) tracks the imposed set point \( T_m^d(t) \) with a first order dynamic behavior, and the control moves of the manipulated heat flux \( q''(t) \) are physically acceptable (Fig. 3). Note that a negligible steady state is observed on the response of the output due to the approximation of the infinite dimensional controller (Balas, 1986; Christofides and Daoutidis, 1996).

6. CONCLUSION

In this paper, a boundary control law of a one-dimensional nonlinear diffusion system is developed in the framework of geometric control based on the notion of the characteristic index. First, a linear equivalent model of the boundary control problem is derived using the Cole-Hopf transformation, then, by means of a Dirac delta function, this linear boundary control is converted to a punctual control problem by assuming that the actuator is spanned over a portion of the spatial domain. Then, it is demonstrated that, by an adequate choice of the output operator, the characteristic index between the manipulated and controlled variables is equal to one, which ensures a first order behavior of the controlled output in closed loop. It is also shown that the closed loop stability is related to the definition of the output operator and the conditions under which the closed loop stability is guaranteed are derived. The performance of the developed controller is evaluated by simulation considering the control problem of the temperature of a steel rod by manipulating the heat flow at the boundary. The obtained results show that the developed controller behaves adequately to track the imposed set point.

![Fig. 1. A one-dimensional heated steel rod.](image-url)
The proposed design approach can be extended to the case of boundary control with Dirichlet actuation. Nevertheless, proving the closed loop stability is difficult since the manipulated variable will be inserted by means of the Dirac delta function derivative to get the equivalent punctual control. Therefore, generalized derivative (i.e. the derivative in the sense of distributions) is needed for the Dirac delta distribution and also for its approximation, which will introduce mathematical difficulties to demonstrate the closed loop stability. This problem is under the investigation of the authors.

Fig. 2. Evolution of the controlled temperature $T_m(t)$.

Fig. 3. Evolution of the manipulated variable $q''(t)$.

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