Optimality conditions for quasiconvex programs

Nguyen Thi Hong Linh\textsuperscript{*} and Jean-Paul Penot\textsuperscript{†}

Abstract

We present necessary and sufficient optimality conditions for a problem with a convex set constraint and a quasiconvex objective function. We apply the obtained results to a mathematical programming problem involving quasiconvex functions.

Key words: convex programming, normal cone, optimality conditions, quasiconvex function, quasiconvex programming, subdifferential.

Mathematics Subject Classification: 90C26, 26B25, 52A41

1 Introduction

It is the purpose of this note to present some optimality conditions for constrained optimization problems under generalized convexity assumptions. We essentially deal with quasiconvex functions, i.e. functions whose sublevel sets are convex. Such functions form the main class of generalized convex functions and are widely used in mathematical economics. We do not assume the data of the problem are smooth. Thus, we replace the derivatives appearing in the classical results by subdifferentials. We only use the adapted subdifferentials of quasiconvex analysis, namely the Plastria subdifferential and the infradifferential or Gutiérrez subdifferential. Because these subdifferentials are useful for algorithmic purposes ([31], [16]) and have links with duality ([13, Prop. 6.1], [26], [25], [27]), their use in problems in which quasiconvexity properties occur seems to be sensible, albeit their calculus rules are not as rich as in the case of convex analysis ([20], [29]). In [13, Prop. 6.1] Martínez-Legaz presented a result of the Kuhn-Tucker type using these subdifferentials; in [14, Thm. 4.1] and [15, Prop. 6.3] variants of these subdifferentials are used. In each of these results, a strict quasiconvexity assumption (and a Slater condition) is imposed. Such an assumption lays aside the important convex case. Here we essentially assume the functions are quasiconvex and we deduce the results from optimality conditions for problems with set constraints. We also point out the link with the differentiable case.

\textsuperscript{*}Department of Mathematics, University of Natural Sciences, Ho Chi Minh City, Vietnam honglinh98t1@yahoo.com

\textsuperscript{†}Laboratoire de Mathématiques, CNRS UMR 5142, Faculté des Sciences, Av. de l’Université 64000 PAU, France jean-paul.penot@univ-pau.fr
We do not make a comparison with results using the all-purpose subdifferentials of nonsmooth analysis (see [1], [18]). The reason is that these subdifferentials are local, whereas the ones we use are of global character; intermediate notions are presented in [2], [17] and [21]. On the other hand, our necessary conditions refine conditions using normal cones or subdifferentials related to normal cones to sublevels sets as the Greenberg-Pierskalla subdifferential ([8]) as in [22]. Numerous papers deal with optimality conditions for constrained problems under invexity conditions (see [3], [4], [5], [30], [32] for example). A link with the case we are dealing with here, which is essentially the quasiconvex case, could be found by assuming that the Gutiérrez or Plastria subdifferentials are nonempty at each point. However, we do not wish to impose such a restrictive condition.

2 Preliminaries: Gutiérrez and Plastria functions

In the sequel $X$ is a normed vector space. We denote by $N(C, x)$ the normal cone at $x \in C$ to a convex subset $C$ of $X$ by

$$N(C, x) := \{x^* \in X^* : \forall u \in C, \langle x^*, u - x \rangle \leq 0\}.$$ 

It is the polar cone of the tangent cone $T(C, x)$ which is the closure of $\mathbb{R}_+(C - x)$.

A function $f : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ is said to be $x$ if for each $x \in X$ its sublevel set

$$S_f(\varpi) := \{x \in X : f(x) \leq f(\varpi)\}$$

is convex, or equivalently if for each $r \in \mathbb{R}$ the strict sublevel set $S_f^<(f, r) := \{x \in X : f(x) < r\}$ is convex.

Recall that the lower subdifferential, or Plastria subdifferential of a function $f : X \to \mathbb{R}$ on a Banach space $X$ at some point $\varpi$ of its domain $\text{dom } f := \{x \in X : f(x) \in \mathbb{R}\}$ is the set

$$\partial^< f(\varpi) := \{x^* \in X^* : \forall x \in S_f^<(\varpi), f(x) - f(\varpi) \geq \langle x^*, x - \varpi \rangle\},$$

where $S_f^<(\varpi) := S_f^<(f, f(\varpi))$ is the strict sublevel set of $f$ at $\varpi$. We will also use the following variant, called the infradifferential or Gutiérrez subdifferential:

$$\partial^\leq f(\varpi) := \{x^* \in X^* : \forall x \in S_f(\varpi), f(x) - f(\varpi) \geq \langle x^*, x - \varpi \rangle\},$$

If no point of the level set $L_f(\varpi) := f^{-1}(f(\varpi))$ is a local minimizer of $f$, we have $\partial^< f(\varpi) = \partial^\leq f(\varpi)$. This equality also holds when $f$ is radially continuous (i.e. continuous along lines) and semistrictly quasiconvex in the sense that when $f(x) < f(y)$ one has $f((1-t)x + ty) < f(y)$ for any $t \in (0, 1)$; in particular, this equality holds for convex continuous functions. In spite of the fact that the preceding constructions have a close similarity with the Fenchel subdifferential, they differ by significant features. In particular $\partial^< f(\varpi)$ and $\partial^\leq f(\varpi)$ are unbounded or empty (they are shady in the sense that they are closed, convex and stable by homotheties with rate greater than 1).
We say that $f$ is a Plastria function at $\overline{x}$ if its strict sublevel set $S_f^< (\overline{x})$ is convex and such that

$$N(S_f^< (\overline{x}), \overline{x}) = \mathbb{R}_+ \partial^< f(\overline{x}).$$

(1)

We say that $f$ is a Gutiérrez function at $\overline{x}$ if its sublevel set $S_f(\overline{x})$ is convex and such that

$$N(S_f(\overline{x}), \overline{x}) = \mathbb{R}_+ \partial^\leq f(\overline{x}).$$

(2)

Since $\partial^< f(\overline{x})$ and $\partial^\leq f(\overline{x})$ are shady in the sense that they are stable under multiplication by any $t \in [1, \infty)$, relations (1) and (2) are equivalent to $N(S_f^< (\overline{x}), \overline{x}) = [0, 1] \partial^< f(\overline{x})$ and $N(S_f(\overline{x}), \overline{x}) = [0, 1] \partial^\leq f(\overline{x})$ respectively. These conditions being rather stringent, it may be useful to replace $f$ by its extension by $+\infty$ outside some ball (see [16] for the case of nonconvex quadratic functions). However we provide three criteria. The first one deals with convex transformable functions, an important class of quasiconvex functions.

**Proposition 1** Suppose $f$ is a proper convex function and $\overline{x} \in \text{dom } f := f^{-1}(\mathbb{R})$ is such that $f(\overline{x}) > \inf f(X)$ and $\mathbb{R}_+(\text{dom } f - \overline{x}) = X$. Then $f$ is a Gutiérrez function and a Plastria function:

$$N(S_f(\overline{x}), \overline{x}) = \mathbb{R}_+ \partial^\leq f(\overline{x}) = \mathbb{R}_+ \partial f(\overline{x}) = \mathbb{R}_+ \partial^< f(\overline{x}) = N(S_f^< (\overline{x}), \overline{x}).$$

More generally, if $f := h \circ g$, where $g : \mathbb{R} \to \mathbb{R}_\infty$ is a convex function and $h : T \to \mathbb{R}_\infty$ is an increasing function on some interval $T$ of $\mathbb{R}_\infty$ containing $g(X)$, with $h(+\infty) = +\infty$, then $f$ is a Gutiérrez function and a Plastria function at $\overline{x}$ provided $f(\overline{x}) > \inf f(X)$. Moreover, $\mathbb{R}_+ \partial^< f(\overline{x}) = \mathbb{R}_+ \partial^\leq g(\overline{x}) = \mathbb{R}_+ \partial^\leq g(\overline{x}) = \mathbb{R}_+ \partial^\leq f(\overline{x})$.

**Proof.** By [28, Prop. 5.4] one has $N(S_f(\overline{x}), \overline{x}) = \mathbb{R}_+ \partial f(\overline{x})$; here we use the fact that under the assumption $\mathbb{R}_+(\text{dom } f - \overline{x}) = X$, we have $N(\text{dom } f, \overline{x}) = \{0\}$. Since $\mathbb{R}_+ \partial f(\overline{x}) \subset \mathbb{R}_+ \partial^\leq f(\overline{x}) \subset N(S_f(\overline{x}), \overline{x})$, we get equality. Moreover, if $\overline{x}^* \in \partial^< f(\overline{x})$ and $x \in S_f(\overline{x})$, taking $z \in X$ such that $f(z) < f(\overline{x})$ and $t \in (0, 1)$ we have $x_t := (1 - t)x + tz \in S_f^< (\overline{x})$, hence

$$(1 - t)f(x) + tf(z) \geq f(x_t) \geq f(\overline{x}) + \langle \overline{x}^*, x - \overline{x} \rangle$$

and taking the limit as $t \to 0$, we get $f(x) \geq f(\overline{x}) + \langle \overline{x}^*, x - \overline{x} \rangle$, hence $\overline{x}^* \in \partial^< f(\overline{x})$. Moreover, the preceding argument shows that $S_f(\overline{x})$ is contained in the closure of $S_f^< (\overline{x})$, so that $N(S_f(\overline{x}), \overline{x}) = N(S_f^< (\overline{x}), \overline{x})$.

Now suppose $f := h \circ g$ as in the second part of the statement. Since $h$ is increasing, we have $S_f(\overline{x}) = S_g(\overline{x})$, $S_f^< (\overline{x}) = S_g^< (\overline{x})$. Setting $\overline{g} := g(\overline{x})$, since $\partial^< h(\overline{g}) \subset (0, +\infty)$, using [29, Prop. 3.5], we have

$$N(S_f^< (\overline{x}), \overline{x}) = N(S_f^< (\overline{x}), \overline{x}) = \mathbb{R}_+ \partial^\leq g(\overline{x}) = \mathbb{R}_+ \partial^\leq h(\overline{g}) \partial^\leq g(\overline{x}) \subset \mathbb{R}_+ \partial^\leq f(\overline{g})$$

hence equality, the reverse inclusion being obvious. The proof that $f$ is a Gutiérrez function is similar. \qed
The second one is a slight improvement of previous results in [31], [13], [20]. It uses the now classical notion of calmness: \( f : X \to \mathbb{R} \) is said to be \textit{calm} with rate \( c \) at \( w \in X \) if \( f(w) \) is finite and if

\[
\forall x \in X \quad f(x) - f(w) \geq -c \|x - w\|.
\]

Such a condition is obviously satisfied if \( f \) is Lipschitz with rate \( c \) or if \( f \) is Stepanovian (or stable) with rate \( c \) at \( w \) in the sense that \( |f(x) - f(w)| \leq c \|x - w\| \) for any \( x \in X \).

\textbf{Proposition 2} Assume that \( f \) is radially continuous on \( X \) and calm with rate \( c \in \mathbb{R}_+ \) at each point of the level set \( L_f(\bar{x}) := f^{-1}(f(\bar{x})) \) and that \( S_f(\bar{x}) \) and \( S_f^\infty(\bar{x}) \) are convex. Then

\[
N(S_f^\infty(\bar{x}), \bar{x}) \cap \partial^\infty f(\bar{x}) \cap \partial f(\bar{x}),
\]

\[
N(S_f(\bar{x}), \bar{x}) \cap \partial f(\bar{x}) \cap B_X
\]

If moreover \( N(S_f^\infty(\bar{x}), \bar{x}) \neq \{0\} \), then \( f \) is a Plastria function while if \( N(S_f(\bar{x}), \bar{x}) \neq \{0\} \) and a Gutiérrrez function at \( \bar{x} \).

The condition \( N(S_f(\bar{x}), \bar{x}) \neq \{0\} \) (or \( N(S_f^\infty(\bar{x}), \bar{x}) \neq \{0\} \)) is a rather mild condition when \( X \) is finite dimensional. However, when \( X \) is infinite dimensional, it may occur that a closed convex set \( C \neq X \) is such that \( N(C, \bar{x}) = X^\ast \) for some \( \bar{x} \in C \).

\textbf{Proof.} Let us first prove that whenever \( \bar{x}^\ast \in N_f^\ast(\bar{x}) := N(S_f^\infty(\bar{x}), \bar{x}) \) satisfies \( \|\bar{x}^\ast\| := \lambda c \) for some \( \lambda > 1 \), then \( \bar{x}^\ast \in \partial^\infty f(\bar{x}) \). Since \( N_f^\ast(\bar{x}) \) is a cone and \( \partial^\infty f(\bar{x}) \) is \( w^\ast \)-closed, that will show that \( N_f^\ast(\bar{x}) \cap \partial^\infty f(\bar{x}) \subseteq \partial^\infty f(\bar{x}) \cap \partial f(\bar{x}) \) and that equality holds. Let \( \bar{y}^\ast := c^{-1}\lambda^{-1}\bar{x}^\ast \). Given \( x \in [f < f(\bar{x})] \) we have by assumption \( t := \langle \bar{y}^\ast, x - \bar{x} \rangle < 0 \). Let us choose \( v \in X \) such that \( \|v\| \leq \lambda \) and \( \langle \bar{y}^\ast, v \rangle = 1 \) and set \( w := x - \bar{x} - tv \). Then \( \langle \bar{y}^\ast, w \rangle = 0 \). Let us show that \( f(\bar{x} + w) \geq f(\bar{x}) \). In order to do so, we pick \( z \in X \) such that \( \langle \bar{y}^\ast, z \rangle > 0 \); then for any \( s > 0 \), we have \( \langle \bar{y}^\ast, w + sz \rangle > 0 \), thus \( f(\bar{x} + w + sz) \geq f(\bar{x}) \). By radial continuity, \( f(\bar{x} + w) \geq f(\bar{x}) \).

Since \( f \mid [x, \bar{x} + w] \) is continuous and \( f(x) < f(\bar{x}) \leq f(\bar{x} + w) \), there exists \( x' \in [x, \bar{x} + w] \) such that \( f(x') = f(\bar{x}) \) and \( f(x'') < f(\bar{x}) \) for all \( x'' \in [x, x'] \). Then, since \( \lambda > 1, \|\bar{y}^\ast\| = 1 \) and \( t = \langle \bar{y}^\ast, x - \bar{x} \rangle \), we have

\[
f(x) - f(\bar{x}) = f(x) - f(x') \geq -c \|x - x'\| \geq -c \|x - (\bar{x} + w)\|
\]

\[
d = -c \|tv\| = ct \|v\| \geq \lambda c \langle \bar{y}^\ast, x - \bar{x} \rangle = \langle \bar{x}^\ast, x - \bar{x} \rangle.
\]

Thus \( f(x) - f(\bar{x}) \geq \langle \bar{x}^\ast, x - \bar{x} \rangle \) holds for all \( x \in [f < f(\bar{x})] \) and \( \bar{x}^\ast \in \partial^\infty f(\bar{x}) \).

Assume now that \( \bar{x}^\ast \in N(S_f(\bar{x}), \bar{x}) \) is such that \( \|\bar{x}^\ast\| \geq c \); then \( \bar{x}^\ast \in N(S_f^\infty(\bar{x}), \bar{x}) \), hence \( \bar{x}^\ast \) is such that \( f(x) - f(\bar{x}) \geq \langle \bar{x}^\ast, x - \bar{x} \rangle \) for each \( x \in S_f^\infty(\bar{x}) \), while if \( f(x) = f(\bar{x}) \), then

\[
f(x) - f(\bar{x}) = 0 \geq \langle \bar{x}^\ast, x - \bar{x} \rangle.
\]

This proves that \( \bar{x}^\ast \in \partial^\infty f(\bar{x}) \).
Now, given \( \bar{x}^* \in N(S_f^c(\bar{x}), \bar{x}) \), \( \bar{x}^* \neq 0 \), we can find \( r > 0 \) such that \( \bar{x}^* = r\bar{w}^* \) for some \( \bar{w}^* \in N(S_f^c(\bar{x}), \bar{x}) \) satisfying \( \|\bar{w}^*\| \geq c \). Thus \( \bar{x}^* \in \mathbb{R}_+\partial^< f(\bar{x}) \). Since \( \partial^< f(\bar{x}) \) is nonempty by what precedes, we also have \( 0 \in \mathbb{R}_+\partial^< f(\bar{x}) \), hence \( N(S_f^c(\bar{x}), \bar{x}) \subset \mathbb{R}_+\partial^< f(\bar{x}) \). The reverse inclusion being obvious, we get \( N(S_f^c(\bar{x}), \bar{x}) = \mathbb{R}_+\partial^< f(\bar{x}) \). The equality \( N(S_f^c(\bar{x}), \bar{x}) = \mathbb{R}_+\partial^< f(\bar{x}) \) is obtained similarly. \( \square \)

The third criterion uses a differentiability assumption and brings some supplement to [19, Prop. 15].

**Proposition 3** Suppose \( f \) is quasiconvex, differentiable at \( \bar{x} \) with a non null derivative. If \( \partial^< f(\bar{x}) \) is nonempty, then \( f \) is a P’{\lowercase{la}}stria function at \( \bar{x} \) and there exists some \( \bar{
abla} \geq 1 \) such that \( \partial^< f(\bar{x}) = [\bar{x}, \infty) f'(\bar{x}) \). If \( \partial^= f(\bar{x}) \) is nonempty then \( f \) is a Gutiérrez function at \( \bar{x} \) and there exists some \( \bar{
abla} \geq 1 \) such that \( \partial^\leq f(\bar{x}) = [\bar{x}, \infty) f'(\bar{x}) \).

**Proof.** Let us first prove that if \( f \) is quasiconvex, differentiable at \( \bar{x} \) with a non null derivative one has \( N(S_f^c(\bar{x}), \bar{x}) = \mathbb{R}_+ f'(\bar{x}) \). We first observe that

\[
f'(\bar{x})^{-1}((-\infty,0)) \subset T(S_f^c(\bar{x}), \bar{x}) \subset T(S_f(\bar{x}), \bar{x}) \subset f'(\bar{x})^{-1}((-\infty,0)).
\]

Since \( f'(\bar{x}) \neq 0 \), we can find some \( w \in X \) with \( f'(\bar{x})w < 0 \). Then, for any \( v \in f'(\bar{x})^{-1}((-\infty,0]) \) and any sequence \( (v_n) \to v \), we have \( v_n := v + r w \in f'(\bar{x})^{-1}((-\infty,0]) \subset T(S_f^c(\bar{x}), \bar{x}) \), and since \( (v_n) \to v \) and \( T(S_f^c(\bar{x}), \bar{x}) \) is closed, we get \( v \in T(S_f^c(\bar{x}), \bar{x}) \). Thus \( T(S_f^c(\bar{x}), \bar{x}) = T(S_f(\bar{x}), \bar{x}) = f'(\bar{x})^{-1}((-\infty,0]) \). Then, the Farkas lemma ensures that

\[
N(S_f^c(\bar{x}), \bar{x}) = N(S_f(\bar{x}), \bar{x}) = \mathbb{R}_+ f'(\bar{x}).
\]

Let us now assume that \( 0 \notin \partial^< f(\bar{x}) \neq \emptyset \) and let us pick some \( \bar{x}^* \in \partial^< f(\bar{x}) \). By what precedes, for any \( v \in X \) such that \( f'(\bar{x})v \leq 0 \) we can find sequences \( (t_n) \to 0_+, (v_n) \to v \) such that \( f(\bar{x} + t_n v_n) < f(\bar{x}) \) for each \( n \). Then we get

\[
(\bar{x}^*, v) \leq \lim_{n} \frac{1}{t_n} (f(\bar{x} + t_n v_n) - f(\bar{x})) = f'(\bar{x})v \leq 0,
\]

so that, by the Farkas lemma again, there exists some \( r \in \mathbb{R}_+ \) such that \( \bar{x}^* = r f'(\bar{x}) \). In fact, the preceding inequalities (taken with some \( v \) such that \( f'(\bar{x})v < 0 \)) show that \( r \geq 1 \). Let \( \bar{r} := \inf \{ r \in \mathbb{R} : \exists \bar{x}^* \in \partial^< f(\bar{x}), \bar{x}^* = r f'(\bar{x}) \} \). We have \( \bar{r} \geq 1 \) by what precedes, and, by closedness of \( \partial^< f(\bar{x}) \), \( \bar{x}^* = \bar{r} f'(\bar{x}) \) for some \( \bar{x}^* \in \partial^< f(\bar{x}) \). Thus, \( \partial^< f(\bar{x}) \subset [\bar{r}, \infty) f'(\bar{x}) \) and since \( \bar{r} f'(\bar{x}) = \bar{x}^* \in \partial^< f(\bar{x}) \) and \( [\bar{r}, \infty) \partial^< f(\bar{x}) \subset \partial^< f(\bar{x}) \), we get \( \partial^< f(\bar{x}) = [\bar{r}, \infty) f'(\bar{x}) \).

It follows that \( \mathbb{R}_+ \partial^< f(\bar{x}) = \mathbb{R}_+ f'(\bar{x}) \).

The proof for \( \partial^\leq f(\bar{x}) \) is similar. \( \square \)

The following example shows that one may have \( \bar{r} > 1 \).

**Example.** Let \( X = \mathbb{R} \) and for \( c < 0 \) let \( f \) be given by \( f(x) = c x^2 / 3 \) for \( x \in (-\infty, c) \), \( f(x) = x^3 / 3 \) for \( x \geq c \). Then, for \( \bar{x} = 1 \), we have \( \partial^< f(\bar{x}) = \partial^\leq f(\bar{x}) = [\bar{r}, \infty) \) with \( \bar{r} = \max(1, (1 + c + c^2)/3) \).
A localization of the preceding concepts may enlarge the range of the optimality conditions which follow. Let us define the local normal cone to $C$ at $\bar{x}$ as

$$N_{loc}(C, \bar{x}) := \bigcup_{r>0} N(C \cap B(\bar{x}, r), \bar{x}),$$

where $B(\bar{x}, r)$ denotes the open ball with center $\bar{x}$ and radius $r$. When $C$ is convex, we have $N_{loc}(C, \bar{x}) = N(C, \bar{x})$. We also define the local Gutiérrez subdifferential and the local Plastria subdifferential of $f$ at $\bar{x}$ by

$$\partial_{\leq}^{\bar{x}} f \left( \bar{x} \right) := \{ x^* \in X^* : \exists r > 0, \forall x \in S_f^\leq(\bar{x}) \cap B(\bar{x}, r), f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle \},$$

$$\partial_{<}^{\bar{x}} f \left( \bar{x} \right) := \{ x^* \in X^* : \exists r > 0, \forall x \in S_f^<(\bar{x}) \cap B(\bar{x}, r), f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle \}$$

respectively. We say that $f$ is locally a Plastria function at $\bar{x}$ if there exists some $r > 0$ such that $S_f^<(\bar{x}) \cap B(\bar{x}, r)$ is convex and if

$$N_{loc}(S_f^<(\bar{x}), \bar{x}) = \mathbb{R}_+ \partial_{<}^{\bar{x}} f(\bar{x}).$$

Locally Gutiérrez functions can be defined similarly.

### 3 Optimality conditions for constrained problems

In the present section we consider the minimization problem

$$(C) \quad \text{minimize } f(x) \text{ subject to } x \in C$$

where $f : X \to \mathbb{R}$ is a function on the n.v.s. $X$ and $C$ is a convex subset of $X$.

**Proposition 4** Suppose $f$ is an u.s.c. Plastria function at $\bar{x}$ and $\bar{x}$ is a solution to $(C)$ but is not a local minimizer of $f$. Then one has

$$0 \in \partial_{<} f(\bar{x}) + N(C, \bar{x}).$$

**Proof.** Since $f$ is quasiconvex and u.s.c., the strict sublevel set $S_f^\leq(\bar{x})$ is open and convex; it is nonempty since $\bar{x}$ is not a minimizer of $f$. Since $\bar{x}$ is a solution to $(C)$, this sublevel set is disjoint from $C$. Thus, the Hahn-Banach separation theorem yields some $c \in \mathbb{R}$ and $u^*$ in the unit sphere of $X^*$ such that

$$\langle u^*, x - \bar{x} \rangle \geq c \geq \langle u^*, w - \bar{x} \rangle \quad \forall w \in S_f^\leq(\bar{x}), \forall x \in C.$$  (4)

Taking $x = \bar{x}$, we see that $c \leq 0$. Moreover, since $\bar{x}$ is not a local minimizer of $f$, there exists a sequence $(w_n) \to \bar{x}$ such that $w_n \in S_f^\leq(\bar{x})$ for each $n$. Therefore $c = 0$. Then we have $u^* \in N(S_f^\leq(\bar{x}), \bar{x}) = \mathbb{R}_+ \partial_{<} f(\bar{x})$ and since $u^* \neq 0$, we can find $\bar{x}' \in \partial_{<} f(\bar{x})$ and $r \in \mathbb{R}_+$ such
that \( \overline{x} = ru^* \). On the other hand, the first inequality of (4) means that \(-u^* \in N(C, \overline{x}) \). Thus, \( \overline{x} + ru^* = 0 \) and (3) is satisfied. \( \square \)

**Example.** The example (taken from [22]) of the function \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = x \) for \( x \in (-\infty, 0) \), \( f(x) = 0 \) for \( x \in [0, 1) \), \( f(x) = x - 1 \) for \( x \in (0, +\infty) \) and \( \overline{x} = 1 \) shows that the assumption that \( \overline{x} \) is not a local minimizer cannot be dispensed with in the preceding statement.

Now let us give a sufficient condition. Observe that no assumption is required on \( f \) besides finiteness at \( \overline{x} \).

**Proposition 5** Suppose that \( f : X \to \mathbb{R} \cup \{\infty\} \) is an arbitrary function and \( f \) is finite at \( \overline{x} \) and satisfies relation (3). Then \( \overline{x} \) is a solution to \((C)\).

**Proof.** Let \( \overline{x}^* \in \partial^< f(\overline{x}) \) be such that \(-\overline{x}^* \in N(C, \overline{x}) \). Assume that \( \overline{x} \) is not a solution to \((C)\): there exists some \( x \in C \) such that \( f(x) < f(\overline{x}) \). Then one has, by the definitions of \( \partial^< f(\overline{x}) \) and \( N(C, \overline{x}) \),

\[
0 > f(x) - f(\overline{x}) \geq \langle \overline{x}^*, x - \overline{x} \rangle,
\]

\[
\langle \overline{x}^*, x - \overline{x} \rangle \geq 0,
\]

a contradiction. \( \square \)

Let us observe that the preceding sufficient condition can also be derived from the one in [22, Prop. 2.1] which uses the Greenberg-Pierskalla subdifferential

\[
\partial^* f(\overline{x}) := \{\overline{x}^* \in X^* : \forall x \in S_f^<(\overline{x}) \langle \overline{x}^*, x - \overline{x} \rangle < 0\}
\]

since \( \partial^< f(\overline{x}) \subset \partial^* f(\overline{x}) \). On the other hand, the necessary condition in Proposition 4 is more precise than the necessary condition in [22, Prop. 2.2].

A slight supplement to the preceding results can be given. It deals with strict solutions to \((C)\), i.e. points \( \overline{x} \in C \) such that \( f(\overline{x}) < f(x) \) for each \( x \in C \setminus \{x\} \). For the sufficient condition we assume that \( C \) is strictly convex at \( \overline{x} \) in the sense that \( \langle \overline{x}^*, x - \overline{x} \rangle < 0 \) for every \( x \in C \setminus \{\overline{x}\} \) and \( \overline{x}^* \in N(C, \overline{x}) \setminus \{0\} \). Observe that if \( N(C, \overline{x}) \setminus \{0\} \) is nonempty (in particular if \( C \) is a convex subset of a finite dimensional space) and if \( C \) is strictly convex at \( \overline{x} \), then \( \overline{x} \) is an extremal point of \( C \) (i.e. \( C \setminus \{\overline{x}\} \) is convex).

**Proposition 6** Given a function \( f : X \to \mathbb{R} \cup \{\infty\} \) finite at \( \overline{x} \) and a subset \( C \) of \( X \) which is strictly convex at \( \overline{x} \), the following relation implies that \( \overline{x} \) is a strict solution to \((C)\) or a global minimizer of \( f \) on \( X \):

\[
0 \in \partial^< f(\overline{x}) + N(C, \overline{x}). \tag{5}
\]

Conversely, when \( X \) is finite dimensional, \( C \) is a convex subset of \( X \) not reduced to \( \{\overline{x}\} \), \( \overline{x} \) is an extremal point of \( C \) and \( f \) is a Gutiérrez function at \( \overline{x} \), relation (5) is necessary in order that \( \overline{x} \) be a strict solution to \((C)\) or a global minimizer of \( f \) on \( X \).
Proof. Suppose relation (5) holds and \( C \) is strictly convex at \( \overline{x} \). If \( \overline{x} \) is not a global minimizer of \( f \) on \( X \) there exists some \( \overline{x}^* \in \partial f(\overline{x}) \) such that \( -\overline{x}^* \in N(C, \overline{x}) \) and \( \overline{x} \neq 0 \). Then, if \( x \in C \setminus \{ \overline{x} \} \) is such that \( f(x) \leq f(\overline{x}) \) we have \( \langle \overline{x}, x - \overline{x} \rangle \leq f(x) - f(\overline{x}) \leq 0 \) since \( \overline{x}^* \in \partial f(\overline{x}) \) and \( \langle -\overline{x}^*, x - \overline{x} \rangle < 0 \) since \( -\overline{x}^* \in N(C, \overline{x}) \setminus \{ 0 \} \), a contradiction. Thus \( \overline{x} \) is a strict solution to \( (C) \).

When \( \overline{x} \) is a strict solution to \( (C) \), the sets \( C \setminus \{ \overline{x} \} \) and \( S_f(\overline{x}) \) are disjoint. If moreover \( f \) is a Gutiérrez function at \( \overline{x} \) and \( \overline{x} \) is an extremal point of \( C \) but is not a global minimizer of \( f \) on \( X \), and \( C \neq \{ \overline{x} \} \), these sets are convex and nonempty. Thus, when \( X \) is finite dimensional, a separation theorem yields some \( c \in \mathbb{R} \) and \( u^* \) in the unit sphere of \( X^* \) such that

\[
\langle u^*, x - \overline{x} \rangle \geq c \geq \langle u^*, w - \overline{x} \rangle \qquad \forall w \in S_f(\overline{x}), \forall x \in C \setminus \{ \overline{x} \}.
\]

Since \( x \) can be arbitrarily close to \( \overline{x} \), we have \( c \leq 0 \). On the other hand, since we can take \( w = \overline{x} \), we have \( c \geq 0 \), hence \( c = 0 \). Thus \( -u^* \in N(C, \overline{x}) \) and \( u^* \in N(S_f(\overline{x}), \overline{x}) = \mathbb{R}_+ \partial f(\overline{x}) \) since \( f \) is a Gutiérrez function at \( \overline{x} \). Since \( u^* \neq 0 \), one can find \( r > 0 \) and \( \overline{x}^* \in \partial f(\overline{x}) \) such that \( u^* = r \overline{x}^* \) and \( -\overline{x}^* \in N(C, \overline{x}) \), so that relation (5) holds. When \( \overline{x} \) is a global minimizer of \( f \) on \( X \), we have \( 0 \in \partial f(\overline{x}) \cap (-N(C, \overline{x})) \).

Now, let us give conditions for local minimization.

**Proposition 7** Suppose \( f \) is an u.s.c. locally Plastria function at \( \overline{x} \) and \( \overline{x} \) is a local solution to \( (C) \) but is not a local minimizer of \( f \). Then one has

\[
0 \in \partial f_{loc}^\infty(\overline{x}) + N(C, \overline{x}).
\]

Conversely, for any function \( f \) finite at \( \overline{x} \) which satisfies relation (7), \( \overline{x} \) is a solution to \( (C) \).

Proof. By assumption, we can find \( r > 0 \) such that \( \overline{x} \) is a minimizer of \( f \) on \( C \cap V \), where \( V := B(\overline{x}, r) \). Taking a smaller \( r \) if necessary and setting \( f_V(x) = f(x) \) if \( x \in V \), \( f_V(x) = +\infty \) if \( x \in X \setminus V \), we may assume that \( f_V \) is a Plastria function at \( \overline{x} \). Then relation (7) follows from Proposition 4.

The converse assertion follows from the sufficient condition and the observation that if \( \overline{x}^* \in \partial f_{loc}^\infty(\overline{x}) \) then there is some neighborhood \( V \) of \( \overline{x} \) such that \( \overline{x}^* \in \partial f_V(\overline{x}) \).

4 Necessary condition for the mathematical programming problem

Let us consider now the case the constraint set \( C \) is defined by a finite family of inequalities, so that problem \( (C) \) turns into the mathematical programming problem

\[
(M) \quad \text{minimize } f(x) \text{ subject to } x \in C := \{ x \in X : g_1(x) \leq 0, \ldots, g_n(x) \leq 0 \}.
\]

Let us first consider the case of a single constraint.
Lemma 8 Let $\overline{x}$ be a solution to $(\mathcal{M})$ in which $g_1 = \ldots = g_n = g$ and $\overline{x}$ is not a local minimizer of $f$. Assume that $f$ is a Plastria function at $\overline{x}$, that $g$ is u.s.c. at $\overline{x}$ and a Gutiérrez function at $\overline{x}$. Then $g(\overline{x}) = 0$ and there exists some $y \in \mathbb{R}_+$ such that

$$0 \in \partial^\prec \{f(\overline{x}) + y \partial^\leq g(\overline{x})\}.$$ 

\textbf{Proof.} By Proposition 4, there exists $\overline{x}^* \in \partial^\prec \{f(\overline{x})\}$ such that $-\overline{x}^* \in N(C, \overline{x})$. If $g(\overline{x}) < 0$, since $g$ is u.s.c. at $\overline{x}$, $\overline{x}$ belongs to the interior of $C$, hence $\overline{x}$ is a local minimizer of $f$, and our assumption discards that case. Thus $g(\overline{x}) = 0$, and since $g$ is a Gutiérrez function at $\overline{x}$, we have $N(C, \overline{x}) = \mathbb{R}_+ \partial^\leq g(\overline{x})$. Thus there exists $y \in \mathbb{R}_+$ such that $-\overline{x}^* \in y \partial^\leq g(\overline{x})$. \qed

Now let us turn to the general case. We will use the following lemma.

Lemma 9 Let $(g_i)_{i \in I}$ be a finite family of quasiconvex Gutiérrez functions at some $\overline{x} \in X$. For $i \in I$, let $C_i := g_i^{-1}((-\infty, 0])$. Assume $g_i$ is u.s.c. at $\overline{x}$, $g_i(\overline{x}) = 0$ for each $i \in I$ and either

(a) there exist some $k \in I$ and some $z \in C_k$ such that $g_i(z) < 0$ for each $i \in I \setminus \{k\}$ (Slater condition), or

(b) $C_i$ is closed for each $i \in I$ and $\mathbb{R}_+ \left( X - \prod_{i \in I} C_i \right) = X^I$, where $\Delta$ is the diagonal of $X^I$.

Then, $g := \max_{i \in I} g_i$ is a Gutiérrez function at $\overline{x}$ and one has

$$\mathbb{R}_+ \partial^\leq g(\overline{x}) = \sum_{i \in I} \mathbb{R}_+ \partial^\leq g_i(\overline{x}). \quad (8)$$

\textbf{Proof.} In case (a) we have $C_k \cap \left( \bigcap_{i \in I \setminus \{k\}} \text{int} C_i \right) \neq \emptyset$, hence, for $C := \bigcap_{i \in I} C_i$, we get

$$N(C, \overline{x}) = \overline{\text{co}} \left( \bigcup_{i \in I} N(C_i, \overline{x}) \right) = \sum_{i \in I} N(C_i, \overline{x}).$$

In case (b), this relation also holds by . Thus, since $\partial^\leq g_i(\overline{x}) \subset \partial^\leq g(\overline{x})$ and $\partial^\leq g(\overline{x})$ is convex,

$$\mathbb{R}_+ \partial^\leq g(\overline{x}) \subset N(C, \overline{x}) = \sum_{i \in I} N(C_i, \overline{x}) = \sum_{i \in I} \mathbb{R}_+ \partial^\leq g_i(\overline{x}) \subset \mathbb{R}_+ \partial^\leq g(\overline{x}),$$

so that $g$ is a Gutiérrez function at $\overline{x}$ and relation (8) holds. \qed

Proposition 10 Let $\overline{x}$ be a solution to $(\mathcal{M})$ and $\overline{x}$ is not a local minimizer of $f$. Assume that $f$ is a Plastria function at $\overline{x}$, $g_1, \ldots, g_n$ are Gutiérrez functions at $\overline{x}$ and u.s.c. at $\overline{x}$. 

Let $I := \{ i \in \mathbb{N} : g_i(\overline{x}) = 0 \}$, $C_i := g_i^{-1}(\{-\infty, 0\})$ and assume that one of the assumptions (a) or (b) of Lemma 9 is satisfied. Then, there exist some $y_1, \ldots, y_n \in \mathbb{R}_+$ such that

$$0 \in \partial^\leq f(\overline{x}) + y_1 \partial^\leq g_1(\overline{x}) + \ldots + y_n \partial^\leq g_n(\overline{x}),$$

for $i = 1, \ldots, n$, $y_i g_i(\overline{x}) = 0$.

Proof. Let $h := \max_{1 \leq i \leq n} g_i$, let $D := h^{-1}(\{-\infty, 0\})$. Let $I := I(\overline{x}) := \{ i \in \{1, \ldots, n\} : g_i(\overline{x}) = h(\overline{x}) \}$. Then, for $i \in \{1, \ldots, n\}\setminus I$ the point $\overline{x}$ belongs to the interior of $C_i := g_i^{-1}(\{-\infty, 0\})$, so that for any $x \in C := g^{-1}(\{-\infty, 0\})$ and any $t > 0$ small enough we have $\overline{x} + t(x - \overline{x}) \in D$. It follows that $N(D, \overline{x}) = N(C, \overline{x})$. By Proposition 4 there exists some $\overline{x}^* \in \partial^\leq f(\overline{x})$ such that $-\overline{x}^* \in N(D, \overline{x}) = N(C, \overline{x})$. Let us set $g := \max_{i \in I} g_i$. Then $g$ is u.s.c. at $\overline{x}$ and is a Gutiérrez function at $\overline{x}$ by Lemma 8. Then, by relation (8), there exist $y_i \in \mathbb{R}_+$, $\overline{y}_i \in \partial^\leq g_i(\overline{x})$ such that $-\overline{x}^* = y_1 \overline{y}_1 + \ldots + y_n \overline{y}_n$ and the result is proved. 

Proposition 1 shows that the preceding statement encompasses the classical result for convex mathematical programming.

A link with the classical Karush, Kuhn and Tucker Theorem is delineated in the next statement.

**Corollary 11** Suppose the assumptions of the preceding proposition are satisfied and that $f, g_1, \ldots, g_n$ are differentiable at $\overline{x}$ with non null derivatives. Then there exist some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_+$ such that

$$f'(\overline{x}) + \lambda_1 g'_1(\overline{x}) + \ldots + \lambda_n g'_n(\overline{x}) = 0,$$

$$\lambda_i g_i(\overline{x}) = 0$$

for $i = 1, \ldots, n$.

Proof. By Proposition 3 and the preceding result, there exist some $r \geq 1, y_i \in \mathbb{R}_+$ and some $\overline{y}_i \in \partial^\leq g_i(\overline{x})$ for $i = 1, \ldots, n$ such that

$$rf'(\overline{x}) + y_1 \overline{y}_1 + \ldots + y_n \overline{y}_n = 0;$$

also $\overline{y}_i = s_i g'_i(\overline{x})$ for some $s_i \geq 1$. Setting $\lambda_i = r^{-1} s_i y_i$, we get the result.

Let us give a simple sufficient condition for the mathematical programming problem $(M)$.

**Proposition 12** If $\overline{x} \in C$ is such that there exist $y_i \in \mathbb{R}_+$ for $i = 1, \ldots, n$ such that the following conditions are satisfied, then $\overline{x}$ is a solution to problem $(M)$:

$$0 \in \partial^\leq f(\overline{x}) + y_1 \partial^\leq g_1(\overline{x}) + \ldots + y_n \partial^\leq g_n(\overline{x}),$$

$$g_1(\overline{x}) \leq 0, \ldots, g_n(\overline{x}) \leq 0,$$

$$y_1 g_1(\overline{x}) = 0, \ldots, y_n g_n(\overline{x}) = 0.$$
Proof. Suppose on the contrary that there exists some \( x \in C \) such that \( f(x) < f(\bar{x}) \). Let \( x^* \in \partial_< f(\bar{x}) \), \( x^*_i \in \partial \leq g_i(\bar{x}) \) for \( i = 1, \ldots, n \) be such that

\[
x^* + y_1 x^*_1 + \ldots + y_n x^*_n = 0.
\]

Let \( I(\bar{x}) := \{ i \in \{1, \ldots, n \} : g_i(\bar{x}) = 0 \} \). Then for \( i \in I(\bar{x}) \), by the definitions of \( \partial_< f(\bar{x}) \), \( \partial \leq g_i(\bar{x}) \) we have, since \( f(x) < f(\bar{x}) \), \( g_i(x) \leq 0 = g_i(\bar{x}) \),

\[
(\bar{x}, x - \bar{x}) \leq f(x) - f(\bar{x}),
\]

\[
(\bar{x}_i, x - \bar{x}) \leq g_i(x) - g_i(\bar{x}) \quad i = 1, \ldots, n.
\]

Multiplying each side of the last inequality by \( y_i \) and adding the obtained sides of the obtained relations to the first one we get, since \( y_i = 0 \) if \( i \in \{1, \ldots, n\} \setminus I(\bar{x}) \),

\[
0 = \langle x^*, x - \bar{x} \rangle + \sum_{i=1}^n y_i \langle x^*_i, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) + \sum_{i=1}^n y_i (g_i(x) - g_i(\bar{x}))
\]

\[
\leq f(x) - f(\bar{x}),
\]

a contradiction.

References


