

EXPLICIT PRESENTATIONS FOR EXCEPTIONAL BRAID GROUPS

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ABSTRACT. We give presentations for the braid groups associated with the complex reflection groups G_{24} and G_{27} . For the cases of G_{29} , G_{31} , G_{33} and G_{34} , we give (strongly supported) conjectures. These presentations were obtained with VKCURVE, a GAP package implementing Van Kampen’s method.

1. INTRODUCTION

To any complex reflection group $W \subset \mathrm{GL}(V)$, one may attach a braid group $B(W)$, defined as the fundamental group of the space of regular orbits for the action of W on V ([BMR]).

The “ordinary” braid group on n strings, introduced by Artin ([A]), corresponds to the case of the symmetric group \mathfrak{S}_n , in its monomial reflection representation in $\mathrm{GL}_n(\mathbb{C})$. More generally, any Coxeter group can be seen as a complex reflection group, by complexifying the reflection representation. Brieskorn proved in [Bri] that the corresponding braid group can be described by an *Artin presentation*, obtained by “forgetting” the quadratic relations in the Coxeter presentation.

Many geometric properties of Coxeter groups still hold for arbitrary complex reflection groups. Various authors, including Coxeter himself, have described “Coxeter-like” presentations for complex reflection groups. Of course, one would like to have not just a “Coxeter-like” presentation for W , but also an “Artin-like” presentation for $B(W)$.

The problem can be reduced to the irreducible case. Irreducible complex reflection groups have been classified by Shephard and Todd ([ST]): there is an infinite family $G(de, e, r)$ (which contains the infinite families of Coxeter groups), plus 34 exceptional groups G_4, \dots, G_{37} (among them are the exceptional Coxeter groups).

Before this note, presentations were known for all but 6 exceptional groups (see the tables of [BMR]):

- The braid group of $G(de, e, r)$ is computed in [BMR]. The proof makes use of fibration arguments, taking advantage of the fact that $G(de, e, r)$ is monomial.
- The first exceptional groups (G_4 to G_{22}) are 2-dimensional. The spaces of regular orbits are complements of (fairly elementary) complex algebraic curves;

Work on this project was completed during the first author’s stays at Institut Girard Desargues (Lyon) and Independent University of Moscow. Computer resources were provided by the Institut de Mathématiques de Jussieu (Paris) – we thank the computer support team for the quality of the environment. We thank Enrique Artal Bartolo, Jorge Carmona Ruber and Bernard Teissier for useful discussions.

the braid groups have been computed by Bannai ([Ba]), using Zariski/Van Kampen method.

- Among the fifteen higher-dimensional exceptional groups, six are Coxeter groups: Brieskorn's theorem applies to them. In addition, three more groups happen to have orbit spaces isomorphic to orbit spaces of certain Coxeter groups (this was observed by Orlik-Solomon, [OS]).
- The six remaining groups are G_{24} , G_{27} , G_{29} , G_{31} , G_{33} and G_{34} . No presentation for their braid groups are given in [BMR] (except a conjectural one for G_{31}).

In the present note, we describe presentations for first two of the six remaining groups, and conjectural presentations for the last four. The evidence for our conjectures is very strong, and only a minute step of the proof is missing.

2. THE PRESENTATIONS

Before listing the individual presentations, it is worth noting that they share some common features: the number of generators is equal to the rank of the group (except for G_{31} , where an additional generator is needed); the generators correspond geometrically to *generators-of-the-monodromy* (in the sense of [BMR] and [B1]) or equivalently *braid reflections* (this nicer terminology was introduced in [Bro]); the relations are positive and homogeneous; by adding quadratic relations to the presentation, one gets a presentation for the reflection group; the product of the generators, taken in a certain order, has a central power. Existence of such presentations was proved in [B1]. All presentations below satisfy all these properties.

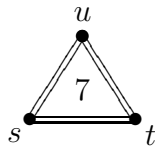
2.1. The 3-dimensional group G_{24} .

Theorem 2.1. *The braid group associated with the complex reflection group G_{24} admits the presentation*

$$\left\langle s, t, u \left| \begin{array}{l} stst = tstst, \text{ } susu = usus, \text{ } tutu = utut, \\ stustus = tustust = ustustu \end{array} \right. \right\rangle.$$

These relations imply that $(stu)^7$ is central.

We suggest to represent this presentation by the following diagram:



Playing with the above presentation, one may obtain other ones, less symmetrical but also interesting. E.g., replacing t by $usts^{-1}u^{-1}$ gives (after simplification)

$$\langle s, t, u | sts = tst, tut = utu, susu = usus, sustustus = ustustust \rangle.$$

Also, replacing t by $susts^{-1}u^{-1}s^{-1}$ yields

$$\langle s, t, u \mid sts = tst, tutu = utut, susu = usus, sutsuts = usutsut \rangle .$$

2.2. The 3-dimensional group G_{27} . For G_{27} , we couldn't find any nice symmetrical presentation, involving only classical braid relations and cyclic three-terms relations.

Theorem 2.2. *The braid group associated with the complex reflection group G_{27} admits the presentations:*

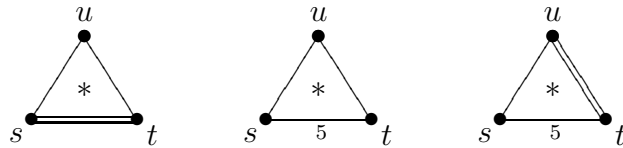
$$\langle s, t, u \mid stst = tst, tut = utu, sus = usu, stustustusts = tstustustust \rangle$$

$$\langle s, t, u \mid ststs = tstst, tut = utu, sus = usu, ststustust = tstustustu \rangle$$

$$\langle s, t, u \mid ststs = tstst, tutu = utut, sus = usu, stustut = ustustu \rangle .$$

In each of these presentations, the element $(stu)^5$ is central.

These presentations could be symbolized by the following diagrams:



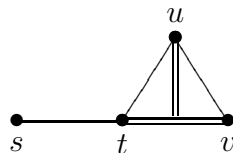
2.3. The 4-dimensional group G_{29} . The presentation for G_{29} given in [BMR] was not conjectured to give (by forgetting the quadratic relations) a presentation for the braid group. Surprisingly, our computations happened to give precisely this presentation.

Conjecture 2.3. *The braid group associated with the complex reflection group G_{29} admits the presentation*

$$\left\langle s, t, u, v \mid \begin{array}{l} sts = tst, tut = utu, uvu = vuv, tvtv = vtvt, \\ su = us, sv = vs, utvutv = tvutvu \end{array} \right\rangle .$$

These relations imply that $(stuv)^5$ is central.

Broué-Malle-Rouquier used the following diagram to symbolize this presentation:



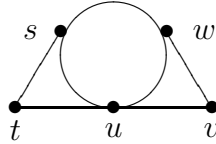
2.4. **The 4-dimensional group G_{31} .** The following conjecture “confirms” the conjectural presentation from [BMR] – but this time there is computational evidence behind.

Conjecture 2.4. *The braid group associated with the complex reflection group G_{31} admits the presentation*

$$\left\langle s, t, u, v, w \left| \begin{array}{l} sts = tst, tut = utu, uvu = vuv, vuv = wvw \\ sv = vs, tv = vt, tw = wt \\ suw = uws = wsu \end{array} \right. \right\rangle.$$

These relations imply that $(stuvw)^6$ is central.

The corresponding Broué-Malle-Rouquier diagram is:



Remark. Since our generators are braid reflections, they map to generating reflections in the reflection group. It is well-known that, even though it is 4-dimensional, G_{31} cannot be generated by less than 5 reflections.

2.5. **The 5-dimensional group G_{33} .** The relations in the presentation below do not coincide with the homogeneous part of the Broué-Malle-Rouquier presentation of G_{33} . However, the relations involving t, u, w coincide with the Broué-Malle-Rouquier relations for the braid group of $G(3, 3, 3)$ (the similar remarks also apply to G_{34}).

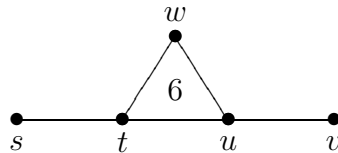
Conjecture 2.5. *The braid group associated with the complex reflection group G_{33} admits the presentation*

$$\left\langle s, t, u, v, w \left| \begin{array}{l} sts = tst, tut = utu, uvu = vuv, wtw = twt, wuw = uwu \\ su = us, sv = vs, tv = vt, ws = sw, wv = vw \\ tuwtuw = uwtuwt = wtuwtu \end{array} \right. \right\rangle.$$

These relations imply that $(stuvw)^9$ is central.

(the relation $uwtuwt = wtuwtu$ is redundant).

We suggest to represent this presentation by the following diagram:



Following [BMR] where a second diagram for G_{33} is given (to account for some parabolic subgroups missing in their first diagram), it is not difficult to obtain the equivalent presentation $\langle s, t, u, v, w | vt = tv, uv = vu, tu = ut, wu = uw, wsw =$

$sus, sus = usu, svs = vsv, sts = tst, vvw = wvw, twt = wtw, twvstw = wvstw >$, which contains a parabolic subdiagram of type D_4 . (A similar diagram may be derived from the conjectural presentation for $B(G_{34})$ given below).

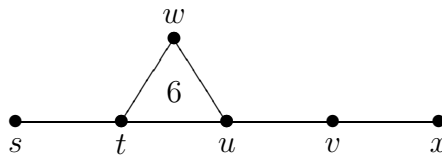
2.6. **The 6-dimensional group G_{34} .**

Conjecture 2.6. *The braid group associated with the complex reflection group G_{34} admits the presentation*

$$\left\langle s, t, u, v, w, x \mid \begin{array}{l} \text{relations of } G_{33} + \\ xvx = vxv, xs = sx, xt = tx, xv = vx, xw = wx \end{array} \right\rangle.$$

These relations imply that $(stuvw x)^7$ is central.

We suggest to represent this presentation by the following diagram:



3. DEFINITIONS AND PRELIMINARY WORK

Our strategy of proof is, basically, brute force. Let V be a \mathbb{C} -vector space of dimension r , and let $W \subset GL(V)$ be a complex reflection group. The algebra $\mathbb{C}[V]^W$ of invariant polynomial functions is isomorphic to a polynomial algebra ([ST]); let f_1, \dots, f_r be homogeneous polynomials such that $\mathbb{C}[V]^W = \mathbb{C}[f_1, \dots, f_r]$.

Let \mathcal{A} be the set of all reflecting hyperplanes. For each $H \in \mathcal{A}$, the pointwise stabilizer W_H of H in W is a cyclic subgroup of order e_H ; choose l_H a linear form with kernel H . Let $V^{\text{reg}} := V - \bigcup_{H \in \mathcal{A}} H$. The *regular orbits space* is V^{reg}/W . We have $\prod_{H \in \mathcal{A}} l_H^{e_H} \in \mathbb{C}[V]^W$, so there is a unique polynomial $\Delta \in \mathbb{C}[X_1, \dots, X_r]$ such that $\prod_{H \in \mathcal{A}} l_H^{e_H} = \Delta(f_1, \dots, f_r)$. We call Δ the *discriminant* of W (with respect to f_1, \dots, f_r). Clearly, V^{reg}/W is isomorphic, as an algebraic variety, to the complement of the hypersurface \mathcal{H} defined in \mathbb{C}^r by the equation $\Delta = 0$.

There is a general method, though not always practically tractable, to compute the fundamental group of such a space. First, choose a 2-plane P such that the embedding $P \cap (\mathbb{C}^r - \mathcal{H}) \hookrightarrow \mathbb{C}^r - \mathcal{H}$ is a π_1 -isomorphism (by a Zariski theorem, this should hold for a *generic* choice of P – how exactly this choice can be made is a difficult issue, which we will discuss later on). Then use the Zariski/Van Kampen method to compute the fundamental group of $P \cap (\mathbb{C}^r - \mathcal{H})$. The computations involved in the second step are far beyond human capabilities (or at least beyond *our* capabilities), especially if one wants to avoid imprecise arguments. Therefore we designed a software package, VKCURVE ([VK]), to carry them by computer.

3.1. General remarks about the implementation. Our computations are performed using the computer algebra software GAP3, which is designed to handle cyclotomic numbers, matrices over these numbers, permutations, presentations, and all sorts of algebraic objects and algorithms involving exact computations. The source of its mathematically advanced functions is public (and in a rather intelligible language) and any user is free to check their validity.

Our package VKCURVE builds on the older package CHEVIE, which implements (among other) complex reflection groups, Coxeter groups and Artin groups.

3.2. Computing the discriminant. For each of the six groups, the discriminant can be recovered from the data given in Appendix B of [OT], where Orlik and Terao explain how to construct the matrix $M \in M_r(\mathbb{C}[X_1, \dots, X_r])$ of logarithmic vector fields (aka. basic derivations) for the quotient singularity (called the discriminant matrix in [OT, 6.67]). The polynomial Δ is simply the determinant of M .

To prevent typos, we actually re-checked all needed computations.

We summarize their method. Let $d_i = \deg f_i$, and let d_1^*, \dots, d_j^* be the codegrees of W . We assume that the degrees are ordered in increasing order (but we *do not* assume the same for codegrees). The matrix M is an $r \times r$ -matrix whose (i, j) entry is an homogeneous invariant polynomial of degree $d_i + d_j^*$.

The six groups have the property that $d_1 < d_2$ so f_1 is unique up to a scalar, and if $H = (\partial_j \partial_i f_1)_{ij}$ is the Hessian matrix of f_1 , then $\det H$ may be chosen as one of the basic invariants f_k (which we assume). Then, if $J = (\partial_j f_i)_{ij}$ is the Jacobian matrix of the f_i , we have the following matrix equation ([OT, (1) p.280]):

$$M = (d_1 - 1)JH^{-1t}JC \tag{1}$$

where C is a matrix of homogeneous invariant polynomials such that $\deg C_{ij} = d_1 + d_j^* - d_i$. Orlik and Terao note that there exists an ordering of the d_j^* such that C is the identity matrix, except for some line q where $C_{iq} = 0$ for $i < q$, $C_{qq} = f_k$, and C_{iq} is a polynomial in $f_1, \dots, f_{k-1}, f_{k+1}, \dots, f_r$ for $i > q$ (the degrees of the entries of C determine the ordering).

Equation (1) is used first to determine C , and then to determine M . It may be used to determine C since it implies the polynomial congruence $0 \equiv JH^tJC \pmod{f_k}$, where H' is the cofactor matrix of H ; each non-zero entry of C is a linear combination of (known from their degree) monomials in the basic invariants, and the above polynomial congruence is sufficient to determine the coefficients of the linear combination.

Example. Sufficient data to construct the matrix of basic derivations for G_{24} is given on p. 284 in [OT]. Note however that the formula given p. 264 in [OT] for its determinant contains

a typo. The correct formula is

$$\begin{aligned} \Delta_{24} = & -2048x^9y + 22016x^6y^3 - 60032x^3y^5 + 1728y^7 - 256x^7z \\ & + 1088x^4y^2z + 1008xy^4z^2 - 88x^2yz^2 + z^3. \end{aligned}$$

To check that such a formula is correct, it suffices to substitute the invariants: the result should be the product of the square of the linear forms defining the reflecting hyperplanes.

In the Appendix, we list basic derivations for all examples (except G_{34} , for which the matrix is too large to be printed...)

4. CHOOSING THE 2-PLANE

4.1. A general strategy. An explicit *genericity* criterion is given in [D], Ch. 4, Theorem 1.17: it suffices that P is transverse to all the strata of a Whitney stratification of the hypersurface. The theorem applies to a projective context. We replace Δ by a homogeneous polynomial $\tilde{\Delta} \in \mathbb{C}[X_0, \dots, X_r]$. The equation $\tilde{\Delta} = 0$ defines a projective hypersurface $\tilde{\mathcal{H}}$; we are interested in the complement $\mathbb{C}^r - \mathcal{H} = \mathbb{C}\mathbb{P}^r - \tilde{\mathcal{H}} \cup \mathbb{C}\mathbb{P}^{r-1}$.

First, we stratify \mathbb{C}^r as follows: for all $k \in \{0, \dots, r\}$, set E_k to be the locus where the matrix M has rank k . This stratification is the quotient modulo W of the stratification of V by the intersection lattice of \mathcal{A} , hence is a Whitney stratification. Moreover, the tangent space of the stratum at a given point is spanned by the columns of M . With the explicit knowledge of this matrix, there is no major difficulty in checking transversality of a given 2-plane.

Example. For G_{31} , one may check that the transversality at infinity is satisfied by the 2-plane of the equations

$$\begin{aligned} z &= y \\ t &= 1 + x \end{aligned}$$

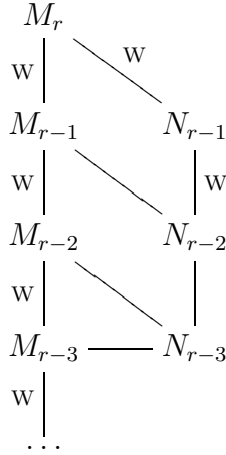
The affine transversality condition for this 2-plane is that, for each value of x and y , the following matrix has rank 4 (the matrix of basic derivations for G_{31} is given in the Appendix):

$$\begin{pmatrix} 8x & 12y+12xy & 20y+\frac{1}{135}xy & 24+24x-\frac{1}{135}y^2 & 1 & 0 \\ 12y & 18x^2-97200y^2+18x^3 & -36-36x+\frac{1}{90}x^3 & -42xy-\frac{1}{90}x^2y & 0 & 1 \\ 20y & -36-72x+60xy^2-36x^2 & -\frac{1}{270}x-\frac{1}{270}x^2-\frac{1}{54}y^2 & \frac{1}{270}y+\frac{1}{270}xy+\frac{1}{54}x^2y & 0 & 1 \\ 24+24x & -42xy-42x^2y-60y^3 & -\frac{1}{54}y-\frac{1}{54}xy-\frac{1}{135}x^2y & \frac{1}{135}xy^2+\frac{1}{54}x^2+20y^2+\frac{1}{54}x^3 & 1 & 0 \end{pmatrix}$$

where the first four columns generate the tangent vector to the local stratum of the discriminant and the last two columns generate the tangent vector space to the 2-plane.

To apply [D, Ch. 4, Theorem 1.17], we also need a stratification of the hyperplane at infinity $\mathbb{C}\mathbb{P}^{r-1}$. Let $\mathcal{H}_\infty := \tilde{\mathcal{H}} \cap \mathbb{C}\mathbb{P}^{r-1}$. We view \mathcal{H}_∞ as an algebraic hypersurface in $\mathbb{C}\mathbb{P}^{r-1}$, defined by the equation $\sqrt{\Delta_\infty} = 0$, where Δ_∞ is the homogeneous part of highest degree of Δ , and $\sqrt{\Delta_\infty}$ is a reduced version of Δ_∞ . We set $N_{r-1} := \mathbb{C}\mathbb{P}^{r-1} - \mathcal{H}_\infty$, $N_{r-3} := (\mathcal{H}_\infty)_{\text{sing}} \cup \overline{M_{r-2}}$ and $N_{r-2} := \mathcal{H}_\infty - N_{r-3}$.

Together, the M_i 's and the N_i 's form a stratification (without border condition), with incidence diagram:



We mark W where we know that the incidence satisfies Whitney's conditions. We have already explained why the first column is a Whitney stratification. It is trivial that M_r is Whitney over N_{r-1} and that N_{r-1} is Whitney over N_{r-2} . By splitting N_{r-3} into smaller strata, we may ensure that everything below N_{r-2} and M_{r-2} is Whitney (see for example the construction explained at the beginning of [GWPL]).

Question 4.1. Does M_{r-1} satisfy Whitney's conditions over N_{r-2} ?

Note that, since Δ_∞ is not reduced, the points of N_{r-2} are not smooth in \mathcal{H} , so the answer is not that trivial. It is a pity that no software is available to answer such a question, on specific examples with explicit equations.

Example. For G_{31} , we represent points of \mathbb{CP}^5 by 5-tuples (h, x, y, z, t) , with either $h = 1$ (affine portion) or $h = 0$ (space at infinity). The strata M_i have explicit equations, using the matrix given in the Appendix. The affine hypersurface \mathcal{H} is given by $h = 1$ and $\Delta(x, y, z, t) = 0$, where Δ is the determinant of the relevant matrix from the Appendix. We have $\Delta_\infty = -\frac{4}{27}x^7z^2t - \frac{8}{81}x^6yz^3$, thus a reduced equation for \mathcal{H}_∞ is $xz(3xt + 2yz)$ (and $h = 0$). One may prove (by means of Gröbner basis) that if a sequence $(1, x_m, y_m, z_m, t_m)_{m \in \mathbb{N}}$ of points in $M_2 \cup M_1 \cup M_0$ converges to $(0, \bar{x}, \bar{y}, \bar{z}, \bar{t})$, then either $(\bar{x}, \bar{y}) = (0, 0)$, or $(\bar{x}, \bar{z}) = (0, 0)$, or $(\bar{t}, \bar{z}) = (0, 0)$. This locus actually coincides with $(\mathcal{H}_\infty)_{\text{sing}}$, thus N_{r-3} is the complement in \mathcal{H}_∞ of this locus (this explains why the particular 2-plane given earlier avoids N_{r-3} : the points at infinity of the 2-plane have the form $(0, x, y, y, x)$, where either $x \neq 0$ or $y \neq 0$). Question 4.1 specializes to: is $\mathcal{H}_{\text{smooth}}$ Whitney over $(\mathcal{H}_\infty)_{\text{smooth}}$?

We may now explain what we have checked, and what is missing to turn our conjectures into theorems:

- For all six examples, our presentations were obtained by applying Van Kampen's method to the algebraic curves obtained with particular 2-planes.

- For all six examples, we have checked that the 2-planes are transversal to the affine strata M_0, \dots, M_r .
- For all examples but G_{34} , we have computed (by means of Gröbner basis) equations for N_{r-1} , N_{r-2} and N_{r-3} , and checked that our 2-planes are also transversal to these strata. Transversality implies that the 2-planes do not intersect N_{r-3} , and therefore remain transversal to the Whitney refinement of N_{r-3} . Therefore, if Question 4.1 had a positive answer, our conjectures would be theorems (except for G_{34}).
- Note that it is easy to check that our 2-planes give generators of the fundamental group, and any homotopy in the 2-plane is a homotopy in \mathbb{C}^r . Therefore, we know for sure that there are presentations for the braid groups obtained by adding relations to our conjectural presentations. On the other hand, we have checked that adding quadratic relations to our conjectural presentations yields actual presentations for the complex reflection group. Any missed relation should be trivial in this quotient.

4.2. A strategy for 3-dimensional groups. Another approach, more algebraic, can be used to find good 2-planes. Although we may start the discussion with any of our examples, it will be conclusive only for 3-dimensional groups. We work with the setting and notations from [B1, Section 2.2]: we have $\Delta \in \mathbb{C}[X_1, \dots, X_r]$ (Δ plays the part of the polynomial P of *loc. cit.*). We distinguish the variable $X := X_r$, we choose a generic (in the sense of *loc. cit.*) line L of direction X . Viewed as a polynomial in only the variable X (with coefficients involving the other variables), P has a discriminant $\text{Disc}(P_X)$. Let $E := \{v \in \mathbb{C}^r \mid P(v) \neq 0, \text{Disc}(P_X)(v) \neq 0\}$. We denote by p the projection $(x_1, \dots, x_r) \mapsto (x_1, \dots, x_{r-1})$. Let $\overline{E} := p(E)$. The map p induces a fibration $E \rightarrow \overline{E}$, whose exact sequence ends as follows:

$$\cdots \longrightarrow \pi_2(\overline{E}, y_0) \longrightarrow \pi_1(L - L \cap \mathcal{H}, x_0) \xrightarrow{\iota_*} \pi_1(E, (x_0, y_0)) \xrightarrow{p_*} \pi_1(\overline{E}, y_0) \longrightarrow 1$$

In our setting, Δ is monic in X (since d_r is regular, it follows from [B1, Lemma 1.6]). It is then easy to construct a section $s : \pi_1(\overline{E}, y_0) \rightarrow \pi_1(E, (x_0, y_0))$ of p_* .

The basespace \overline{E} is the complement in \mathbb{C}^{r-1} of the hypersurface of equation $\text{Disc}(\Delta_X) = 0$. In our setting, $\text{Disc}(\Delta_X)$ is a weighted homogeneous polynomial.

When $r = 3$, this implies that $\pi_2(\overline{E}, y_0) = 0$ (complements of weighted homogeneous curves are $K(\pi, 1)$). We then have a semi-direct product structure

$$\pi_1(E, (x_0, y_0)) \simeq \pi_1(L - L \cap \mathcal{H}, x_0) \rtimes \pi_1(\overline{E}, y_0).$$

To obtain a presentation for $\pi(\mathbb{C}^r - \mathcal{H})$, one starts with a presentation for $\pi_1(E, (x_0, y_0))$, and adds relations forcing elements of $\pi_1(\overline{E}, y_0)$ to become trivial. It is an easy exercise to check, in this setting, that any 2-plane P satisfying:

- (i) the line L is contained in P and,

(ii) the image line $p(P)$ is such that $p(P) \cap \overline{E} \hookrightarrow \overline{E}$ is π_1 -surjective,

is good for our purposes. In our examples, it is easy to construct such planes, since $\text{Disc}(\Delta_X)$ is monic in one the remaining variables. This is how we obtained, for G_{24} and G_{27} , theorems rather than conjectures.

Note that, for other groups, all assumptions used here (including the monicity of $\text{Disc}(\Delta_X)$) remain valid, except that we do not know whether $\pi_2(\overline{E}, y_0) = 0$. Instead of answering Question 4.1, checking that $\pi_2(\overline{E}, y_0) = 0$ would turn our conjectures into theorems.

5. THE PACKAGE VKCURVE

Once a 2-plane P has been chosen, it is enough to feed VKCURVE with the equation of the curve $P \cap \mathcal{H}$ to obtain a presentation of $\pi_1(P - (P \cap \mathcal{H}))$.

Example. For G_{31} , when computing the determinant of M_{31} and evaluating at $z = y$ and $t = 1 + x$, we obtain the following equation for $P \cap \mathcal{H}$:

$$\begin{aligned} \Delta'_{31} = & 746496 + 3732480x - 3111936xy^2 - \frac{93281756}{27}xy^4 + \frac{58341596}{27}xy^6 + 7464960x^2 - 384y^2 - 9334272x^2y^2 \\ & + \frac{17556484}{27}x^2y^4 + \frac{43196}{27}x^2y^6 + 7464576x^3 - \frac{756138248}{81}x^3y^2 + \frac{192792964}{81}x^3y^4 + \frac{16}{81}x^3y^6 + 3730944x^4 \\ & - \frac{139967996}{81}y^4 - \frac{84021416}{27}x^4y^2 + \frac{82088}{27}x^4y^4 + 744192x^5 + \frac{43192}{27}x^5y^2 - \frac{1720}{27}x^5y^4 - \frac{124412}{81}x^6 \\ & + 777600800y^6 + \frac{95896}{81}x^6y^2 - \frac{8}{81}x^6y^4 - \frac{10364}{27}x^7 - \frac{4}{27}x^7y^2 + \frac{4}{27}x^8 - \frac{8}{81}y^8 - \frac{4}{27}x^8y^2 + \frac{4}{81}x^9 \end{aligned}$$

On a 3 GHz Pentium IV, VKCURVE needs about one hour to deal with this example.

Writting VKCURVE was of course the most difficult part of our work. This software accepts as input any quadratfrei polynomial in $\mathbb{Q}[i][X, Y]$ and computes a presentation for the fundamental group of the complement of the corresponding complex algebraic curve. The program does not use floating point computations (even when computing monodromy braids); therefore there is no issue of numerical accuracy and the result is “certified” to be correct (provided that our implementation does not contain mathematical errors...)

The remainder of this section is an overview of the algorithms used in VKCURVE. We rely on the version of Van Kampen’s method exposed in [B2, Procedure 4], where it is decomposed into four steps.

5.1. Implementing steps 1 and 2. Starting with our polynomial $P \in \mathbb{Q}[i][X, Y]$, we view it as a one variable polynomial in $\mathbb{Q}[i][Y][X]$ and compute its discriminant $\Delta \in \mathbb{Q}[i][Y]$. The discriminant Δ may not be reduced; to compute approximations $\tilde{y}_1, \dots, \tilde{y}_r \in \mathbb{Q}[i]$ of its complex roots y_1, \dots, y_r , we apply Newton’s method to the reduced polynomial Δ_0 obtained by dividing Δ by the resultant of Δ and Δ' . As Hubbard, Schleicher and Sutherland proved in their beautiful article [HSS], Newton’s method can be made into a failsafe algorithm producing arbitrarily good approximations of y_1, \dots, y_r .

Since we will re-use them later, we recall a few trivialities about complex polynomials. Let $P \in \mathbb{C}[Z]$. Let $\alpha_1, \dots, \alpha_n$ be the complex roots of P . Let $z \in \mathbb{C}$. If $P'(z) \neq 0$, we set $N_P(z) := z - \frac{P(z)}{P'(z)}$. Considering the first order approximation of P around z , we expect $P(N_P(z))$ to be close to 0. Newton's method consists of starting with $z_0 \in \mathbb{C}$ (chosen randomly, or smartly as in [HSS]) and to construct iteratively $z_{m+1} := N_P(z_m)$, hoping that (z_m) will converge towards a root of P – which indeed happens for “many” choices of z_0 . How may we decide that a given z_n is a “good enough” approximation?

Lemma 5.1. *Assume P has n distinct roots $\alpha_1, \dots, \alpha_n$. Let $z \in \mathbb{C}$, with $P'(z) \neq 0$. Then there exists $\alpha \in \{\alpha_1, \dots, \alpha_n\}$ such that $|z - \alpha| \leq n \left| \frac{P(z)}{P'(z)} \right|$.*

Proof. If $P(z) = 0$, the result is trivial. Otherwise, we have $\frac{P'(z)}{P(z)} = \sum_{i=1}^n \frac{1}{z - \alpha_i}$. Choose i such that for all j , $|z - \alpha_i| \leq |z - \alpha_j|$. By triangular inequality, $\frac{1}{|z - \alpha_i|} \geq \left| \frac{P'(z)}{P(z)} \right| - \sum_{j \neq i} \frac{1}{|z - \alpha_j|} \geq \left| \frac{P'(z)}{P(z)} \right| - (n - 1) \frac{1}{|z - \alpha_i|}$. The result follows. \square

Although elementary, this lemma provides a very inexpensive (in terms of computational time) test for deciding whether a tentative list $\tilde{\alpha}_1, \dots, \tilde{\alpha}_n$ of complex numbers “separates” the roots (*i.e.*, whether there exists $\varepsilon_1, \dots, \varepsilon_n$ such the disks $D(\tilde{\alpha}_i, \varepsilon_i)$ do not overlap and each of them contains a root of P).

Instead of working with the exact Newton's method, we use a truncated version, where $N_P(z)$ is replaced by an approximate $(a + ib)10^k$, where $a, b \in \mathbb{Z}$, and k is an integer slightly smaller than $\log_{10} \left| \frac{P'(z)}{P(z)} \right|$. This is to avoid the very fast increase of the denominators, when the exact method is carried out in $\mathbb{Q}[i]$: the complexity of the exact method is very good from the “abstract” viewpoint (the number of iterations), but in practice really bad (each individual iteration involves costly operations on very big integers). Of course, our modification does not make the method less rigorous, since the test can be performed exactly. The main difference between our implementation and floating point is that k is modified dynamically and has no pre-assigned bound.

Once separating approximates $\tilde{y}_1, \dots, \tilde{y}_r \in \mathbb{Q}[i]$ of the roots of Δ have been obtained, Step 2 of [B2, Procedure 4] is performed as follows: first, we construct the Voronoi cells around $\tilde{y}_1, \dots, \tilde{y}_r$; then, concatenating some of the affine segments bounding the Voronoi cells, we construct, for each i , a loop γ_i representing a meridian around \tilde{y}_i ; it is easy to make sure that we recover a meridian around the actual y_i .

5.2. Step 3: computing monodromy braids. [B2, Procedure 12] decomposes Step 3 into smaller steps a–e. Only Substep *a* is not a straightforward algebraic manipulation – and most of the computational time is spent there. The problem is as follows: let $[y_0, y_1]$ be one of the affine segments involved in the γ_i 's. For $t \in [0, 1]$, denote by

$P_t \in \mathbb{Q}[i][X]$ the polynomial obtained by evaluating P at $Y = (1-t)y_0 + ty_1$. We want to compute the word in Artin generators corresponding to the real projection of the braid obtained by tracking the roots of P_t when t runs over $[0, 1]$.

As we have seen above, we may find $x_1, \dots, x_n \in \mathbb{Q}[i]$ separating the roots of P_0 . Concretely, using Lemma 5.1, we iterate a truncated Newton method until, when we set $\varepsilon_i := \inf_{j \neq i} \frac{|x_i - x_j|}{2}$ (this is a simple way, though not optimal, to ensure that $\forall i, j, |x_i - x_j| > \varepsilon_i + \varepsilon_j$), we have

$$\forall i, \left| \frac{P_0(x_i)}{P_0'(x_i)} \right| < \frac{\varepsilon_i}{n}.$$

For each i , consider the polynomial

$$Q_i := \varepsilon_i^2 |P_t'(x_i)|^2 - n^2 |P_t(x_i)|^2 \in \mathbb{Q}[t].$$

By assumption, we have $\forall i, Q_i(0) > 0$. Whenever $t_0 \in [0, 1] \cap \mathbb{Q}$ is such that $\forall t \in [0, t_0], \forall i, Q_i(t) > 0$, we know that, for $t \in [0, t_0]$, the strings of the monodromy braids will be in the cylinders of radius ε_i around the x_i 's. This fragment of the monodromy braid can be replaced by the constant braid with strings fixed at the positions given by the x_i 's. Set $y'_0 := (1-t_0)y_0 + y_1$, $x'_i := N_{P_{t_0}}(x_i)$. Though the x_i 's already separate the roots of P_{t_0} , the x'_i 's should be “better” approximates. We compute new radii ε'_i separating the x'_i 's and iterate, studying now the monodromy braid over $[y'_0, y_1]$, with initial approximates x'_1, \dots, x'_n . Eventually, we hope that after some number of iterations, $t_0 = 1$ will suit.

The main difficulty is to find an actual t_0 such $\forall t \in [0, t_0], \forall i, Q_i(t) > 0$. On the one hand, we want it to be as large as possible, to avoid unnecessary iterations; on the other hand, computing the largest theoretical value for t_0 , for example using Sturm sequences, is very costly. Finding a good balance is a delicate art. The curious reader may have a look at the source of the VKCURVE function `FollowMonodromy`, where a very naive method is used, together with careful coding and adaptative heuristics (note that, in `FollowMonodromy`, one actually computes a distinct t_0 for each individual string – the above description is simplified for the sake of clarity).

5.3. Step 4: writing and simplifying the presentation. Working with GAP, it is then straightforward to write a presentation. However, this presentation is much more complicated than desirable. Since no “normal form” theory exists for arbitrary group presentations, it is not clear how one can simplify it and obtain one of our “pretty” presentations. Fortunately, some natural heuristics (typically, replace a generator by its conjugate by another generator, try to simplify, iterate in the regions of the tree of all possibilities where the total length of the presentation tends to decrease) happen to be quite effective in dealing with the (highly redundant) presentations obtained with Van Kampen’s method. Playing with these (non-deterministic) heuristics, which are

part of VKCURVE, we obtained quite easily a few “simple” presentations. At this point, in the absence of a general combinatorial theory of generalized braid groups, there is some arbitrary in deciding which one should be retained; in most cases though, one of them clearly emerged as being the “prettiest”.

APPENDIX A. EXPLICIT MATRICES OF BASIC DERIVATIONS

The group G_{24} . With Klein’s matrices (as in [OT]) the first invariant is $f_1 = x^3y + zy^3 + xz^3$. The others are $f_2 = \det(\text{Hessian}(f_1))/108$ and $\text{bord}(f_1, f_2)/36$.

Basic derivations:

$$\begin{pmatrix} 4x & 6y^2 & 14z - 36x^2y \\ 6y & -z & 128xy^2 - 7x^4 \\ 14z & 128xy^3 - 6x^2z - 7x^4y & 287xyz - 35x^3y^2 - 294y^4 + 7x^6 \end{pmatrix}$$

The group G_{27} . With Wiman’s matrices (as in [OT]), the first invariant is $f_1 = -135xyz^4 - 45x^2y^2z^2 + 10x^3y^3 + 9x^5z + 9y^5z + 27z^6$. The others are $f_2 = \det(\text{Hessian}(f_1))/6750$ and $\text{bord}(f_1, f_2)/5400$.

Basic derivations:

$$\begin{pmatrix} 6x & 12y^2 & 30z + 234x^3y \\ 12y & -4z & -34xz - 1362x^2y^2 + 156y^3 + 900x^4y - 270x^6 \\ -34xyz - 1362x^2y^3 + 78x^3z + & 4836xy^4 + 330y^2z - 3349x^2yz - 17727x^3y^3 + & \\ 30z & 156y^4 + 900x^4y^2 - 270x^6y & 2013x^4z + 7110x^5y^2 + 135x^7y - 810x^9 \end{pmatrix}$$

The groups G_{29} and G_{31} . For the data relative to G_{29} and G_{31} , see [M]. The group G_{31} is generated by the matrices T and $Ue^{2i\pi/8}$ in Maschke’s notations.

G_{29} is the subgroup which leaves invariant Φ_1 which we take as the first invariant. Then we choose $(-1/20736) \det(\text{Hessian}(\Phi_1)) = (4F_8 - \Phi_1^2)/3$. We do not choose F_{12} but the simpler $((\Phi_1^3 - 3\Phi_1F_8)/2 + F_{12})/108$. We do not choose F_{20} but the simpler $(F_{20} - F_8F_{12})/1296$.

For G_{31} we choose F_8, F_{12} , then as for G_{29} we choose $(F_{20} - F_8F_{12})/1296$; the fourth is still (as in [OT]) $\det(\text{Hessian}(F_8))/265531392$.

Basic derivations of G_{29} :

$$\frac{1}{80} \begin{pmatrix} 320x & 640y^2 & 960z + 2xy & 1600t + 8yz \\ 640y & 4096000t + 225280yz + 1280xy^2 & 64xz + 4x^2y & -640tx + 16xyz + 1536z^2 \\ 960z & -12800tx - 640xyz & 200t - 5yz + 3x^2z & -10ty + 10tx^2 - 8xz^2 \\ 1600t & -51200tz - 640txy - 1280yz^2 & -10ty + 8tx^2 - 4xz^2 & 72txz - 96z^3 \end{pmatrix}$$

Basic derivations of G_{31} :

$$\frac{1}{270} \begin{pmatrix} 2160x & 3240ty & 5400z + 2xy & 6480t - 2y^2 \\ 3240y & 4860tx^2 - 26244000z^2 & -9720t + 3x^3 & -11340xz - 3x^2y \\ 5400z & 16200xz^2 - 9720t^2 & -tx - 5yz & ty + 5x^2z \\ 6480t & -11340txz - 16200yz^2 & -5ty - 2x^2z & 2xyz + 5tx^2 + 5400z^2 \end{pmatrix}$$

The group G_{33} . We take the matrices and invariants of [Bu, pp 208–209] with the corrections indicated in [O]. The third invariant is taken to be $\det(\text{Hessian}(J_4))/63700992$ where J_4 is the first invariant. Basic derivations of G_{33} :

$$\frac{1}{128} \begin{pmatrix} 512x & 768yz & 1280z + (\frac{-4}{3})xy & 1536t - 4y^2 & 2304u - 4ty \\ 768y & -663552u + 1152x^2z & 768t - 2x^3 & 8064xz - 6x^2y & -6tx^2 + 23040z^2 \\ 1280z & 768tz - 1152ux & (\frac{-1}{3})tx + 3yz & 576u - ty + 9x^2z & 6uy + 36xz^2 - 4t^2 \\ 1536t & -3456uy + 8064xz^2 & 576u + 3ty - 5x^2z & -15xyz + 9tx^2 + 11520z^2 & -42txz + 18u x^2 + 108yz^2 \\ 2304u & -3456tu + 23040z^3 & 6uy - 4xz^2 & 18ux^2 - 12yz^2 & 90uxz - 48tz^2 \end{pmatrix}$$

REFERENCES

- [A] E. Artin, *Theory of braids*, Ann. of Math. (2) **48** (1947), 101-126.
- [Ba] E. Bannai, *Fundamental groups of the spaces of regular orbits of the finite unitary reflection groups of dimension 2*, J. Math. Soc. Japan **28** (1976), 447–454.
- [B1] D. Bessis, *Zariski theorems and diagrams for braid groups*, Invent. Math. **145** (2001), 487–507.
- [B2] D. Bessis, *Variations on Van Kampen’s method*, arXiv:math.GR/0301327.
- [Bri] E. Brieskorn, *Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe*, Invent. Math. **12** (1971), 57–61.
- [Bro] M. Broué, *Reflection groups, braid groups, Hecke algebras, finite reductive groups*, Current developments in mathematics, 2000, 1–107, Int. Press, Somerville, MA, 2001.
- [BMR] M. Broué, G. Malle et R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. reine angew. Math. **500** (1998), 127–190.
- [Bu] H. Burkhardt, *Untersuchungen aus dem Gebiete der hyperelliptischen Modulfunctionen II* Math. Annalen **38** (1891), 161–224.
- [D] A. Dimca, *Singularities and topology of hypersurfaces*, Springer-Verlag Universitext, 1992.
- [GAP] M. Schönert and *al.*, *GAP - Groups, Algorithms and Programming*, Lehrstuhl D für Mathematik, RWTH Aachen, Germany, fourth ed. (1994).
- [GWPL] G. Gibson, K. Wirthmüller, A. du Plessis, E. Looijenga, *Topological stability of smooth mappings*, Lecture Notes in Mathematics **552**, Springer-Verlag, Berlin-New York, 1976.
- [HSS] J. Hubbard, D. Schleicher and S. Sutherland, *How to find all roots of complex polynomials by Newton’s method*, Invent. Math. **146** (2001), 1–33.
- [M] H. Maschke *Ueber die quaternäre, endliche, lineare Substitutionsgruppe der Borchardt’schen Moduln*, Math. Ann. **30** (1887) 496–515.
- [O] P. Orlik, *Basic derivations for unitary reflection groups*, with an appendix by Hiroaki Terao and Yoichi Enta, Contemp. Math **90** (1989) 211–228.
- [OS] P. Orlik, L. Solomon, *Discriminants in the invariant theory of reflection groups*, Nagoya Math. J. **109** (1988), 23–45.
- [OT] P. Orlik, H. Terao, *Arrangements of hyperplanes*, Grundlehren der mathematischen Wissenschaften **300**, Springer-Verlag, 1992.
- [ST] G. C. Shephard, J. A. Todd, *Finite unitary reflection groups*, Canad. J. Math. **6** (1954), 274-304.
- [VK] D. Bessis, J. Michel, *VKCURVE*, software package for GAP3, source and documentation available at:

`\protect\vrule width0pt\protect\href{http://www.math.jussieu.fr/string~jmichel/vkcurve.html}{ht`

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