Spherical wavelet transform for ODF sharpening

I. Kezele *, M. Descoteaux, C. Poupon, F. Poupon, J.-F. Mangin

*Corresponding author. Tel.: +33 1 69 28 90 77; fax: +33 1 69 08 79 80.
E-mail address: irina.kezele@cea.fr (I. Kezele).

Article info

Article history:
Received 9 February 2009
Received in revised form 18 December 2009
Accepted 7 January 2010
Available online 1 February 2010

Keywords:
Diffusion MRI
q-Ball imaging
Spherical deconvolution
Orientation distribution function (ODF)
Spherical wavelets
Multiresolution analysis
“A trous” algorithm

A B S T R A C T

The choice of local HARDI reconstruction technique is crucial for discerning multiple fiber orientations, which itself is of substantial importance for tractography, and reliable and accurate assessment of white matter fiber geometry. Due to the complexity of the diffusion process and its milieu, distinct diffusion compartments can have different frequency signatures, making the HARDI signal spread over multiple frequency bands. Therefore, we put forth the idea of multiscale analysis with localized basis functions, ensuring that different frequency ranges are probed. With the aim of truthful recovery of fiber orientations, we reconstruct the orientation distribution function (ODF), by incorporating a spherical wavelet transform (SWT) into the Funk–Radon transform. First, we apply and validate our proposed SWT method on real physical phantoms emulating fiber bundle crossings. Then, we apply the SWT method to a real brain data set. The analysis of the real data set suggests that different angular frequencies may capture different information, thus stressing the importance of multiscale analysis. For both phantom and real data, we compare the SWT reconstruction with state-of-the-art q-ball imaging and spherical deconvolution reconstruction methods. We demonstrate the algorithm efficiency in diffusion ODF denoising and sharpening that is of particular importance for applications to fiber tracking (especially for probabilistic approaches), and brain connectome mapping. Also, the algorithm results in considerable data compression that could prove beneficial in applications to fiber bundle segmentation, and for HARDI based white matter morphometry methods.

© 2010 Elsevier B.V. All rights reserved.

1. Introduction

In diffusion MRI, the measured signal attenuation is proportional to residual spin phase incoherence that results directly from the component of spin motion in the direction of applied diffusion gradients. Since the spin motion depends on the geometry of the substrate, the signal inherently encodes this dependence. A number of methods for local diffusion modeling that estimate the full three-dimensional diffusion probability distribution (i.e., diffusion propagator) have been reported in the literature. The state-of-the-art technique, diffusion spectrum imaging (DSI) (Wedeen et al., 2005) uses the direct relation between the average diffusion propagator and diffusion signal via Fourier transform (Callaghan, 1991), and samples densely the Cartesian grid in q-space. A DSI derivative, hybrid diffusion imaging (HYDI) (Wu and Alexander, 2007), uses a lighter q-space sampling scheme on multiple q-shells to interpolate and regrid the whole Cartesian volume.

Recently, a number of methods appeared that propose to model the diffusion signal in a general manner, that would not depend on the geometry of the substrate (Ozarslan et al., 2009; Assemalal et al., 2009; Descoteaux et al., 2009b). These methods rely on diffusion signal decomposition using orthogonal bases and special functions. The two latter methods also propose an analytic, linear and compact reconstruction of diffusion propagator.

Apart from the direct q-space imaging, the mathematical models that explain the diffusion signal (Stanisz et al., 1997; Assaf and Basser, 2005; Vestergaard-Poulsen et al., 2007) have also been introduced. While the former poses some constraints on diffusion gradient pulse duration and regularity of the imaged structure, signal models can be derived with more relaxed assumptions, and in particular can accommodate finite pulse widths. For certain simple geometries (e.g., parallel planes, spheres or cylinders), the solution to diffusion equation is analytic (Neuman, 1974; Murday and Cotts, 1968). This can be readily exploited in modeling of signals in tissues approximated by combinations of finite numbers of simple geometric elements. In addition, mathematical models can incorporate a variety of parameters related to diverse microstructure features, such as: cell density and radii, cell-membrane permeability, intrinsic diffusivities, and transverse relaxation constants (Assaf and Basser, 2005; Stanisz et al., 1997; Neuman, 1974; Stanisz, 2003). However, to our knowledge, none of these have been thus far exploited for fiber tracking. While the full three-dimensional propagator is of great interest to studies of microstructural architecture, its angular part, orientation distribution function (ODF), obtained through angular q-space sampling methods on...
single q-shells, like, high-angular resolution diffusion imaging (HARDI), is what fiber tracking methods require.

It is now accepted that between one third and two thirds of white matter (WM) voxels in common HARDI datasets contain evidence for multiple fiber configurations (Pierpaoli and Basser, 1996; Behrens et al., 2003; Descoteaux, 2008). It is thus important to incorporate more informative models of local diffusion into tractography algorithms, to capture this complex tissue architecture. Several HARDI reconstruction models can resolve non-trivial fiber configurations such as crossings, splits, fannings. However, the power of these methods greatly depends on the ODF, or fiber orientation distribution (FOD) reconstruction algorithm. The choice of HARDI reconstruction technique is therefore of crucial importance.

To render the tractography algorithms truthful and reliable, the reconstruction method is expected to provide sharp and accurate descriptions of the ODF/FOD. It is particularly important to get sharp ODF/FODs when using probabilistic tractography techniques that sample the directions around peaks. Sharper ODF/FODs yield better direction samplings (i.e. the samples are more concentrated around the peaks).

Since the directions of maximal diffusion (that are implied by average principal fiber orientations inside a voxel) are unknown a priori, HARDI explores the space in a uniform manner, and thus despite its obvious gain over DTI, it still suffers from a certain amount of information redundancy. A number of approaches for “simplification” and regularization of Apparent Diffusion Coefficient profiles (ADCps) resulting from HARDI by projection onto predefined bases have been demonstrated (Frank, 2002; Alexander et al., 2002; Ozarslan and Mareci, 2003; Descoteaux et al., 2006).

The preliminary work was oriented towards spherical harmonics (SHs) (Frank, 2002; Alexander et al., 2002), that offered a suitable spherical basis for high ADCp compression as well as some fast algorithms for ODF reconstruction (Hess et al., 2006; Descoteaux et al., 2007). In addition, some preliminary work on ODF sharpening in SH basis was also presented (Descoteaux et al., 2005). However, the SH decomposition is not localized and is therefore not fully adapted to the problem of extracting ODF maxima corresponding to white matter fiber orientations. Nonetheless, SHs have been successfully used in conjunction with some more local bases, e.g., in spherical deconvolution (SD) methods for FOD estimation (Tournier et al., 2004, 2007), that probe the signal locally, thanks to the finite (i.e., local) spatial support of the convolution kernel, which allows for sparse representations. A number of other signal-adapted bases were described in the literature (Jian and Vemuri, 2007; Kaden et al., 2007; Dell’Acqua et al., 2008).

The problem with SD is the low-pass nature of the kernel that renders the method extremely susceptible to noise. Incorporating additional constraints, such as positivity and regularity (Tournier et al., 2007; Jian and Vemuri, 2007), is thus imperative for reliable FOD reconstruction. A good side of these deconvolution techniques is that they can yield accurate FOD estimates even at relatively low FOD reconstruction. A good side of these deconvolution techniques, e.g., in spherical deconvolution (SD) methods for FOD estimation, has been successfully used in conjunction with some more local bases, corresponding to white matter fiber orientations. Nonetheless, SHs have fully adapted to the problem of extracting ODF maxima corresponding to white matter fiber orientations. However, the SH decomposition is not localized and is therefore not fully adapted to the problem of extracting ODF maxima corresponding to white matter fiber orientations. Nonetheless, SHs have successfully been used in conjunction with some more local bases, e.g., spherical deconvolution (SD) methods for FOD estimation (Tournier et al., 2004, 2007), that probe the signal locally, thanks to the finite (i.e., local) spatial support of the convolution kernel, which allows for sparse representations. A number of other signal-adapted bases were described in the literature (Jian and Vemuri, 2007; Kaden et al., 2007; Dell’Acqua et al., 2008).

The preliminary work was oriented towards spherical harmonics (SHs) (Frank, 2002; Alexander et al., 2002), that offered a suitable spherical basis for high ADCp compression as well as some fast algorithms for ODF reconstruction (Hess et al., 2006; Descoteaux et al., 2007). In addition, some preliminary work on ODF sharpening in SH basis was also presented (Descoteaux et al., 2005). However, the SH decomposition is not localized and is therefore not fully adapted to the problem of extracting ODF maxima corresponding to white matter fiber orientations. Nonetheless, SHs have successfully been used in conjunction with some more local bases, e.g., spherical deconvolution (SD) methods for FOD estimation (Tournier et al., 2004, 2007), that probe the signal locally, thanks to the finite (i.e., local) spatial support of the convolution kernel, which allows for sparse representations. A number of other signal-adapted bases were described in the literature (Jian and Vemuri, 2007; Kaden et al., 2007; Dell’Acqua et al., 2008).

The problem with SD is the low-pass nature of the kernel that renders the method extremely susceptible to noise. Incorporating additional constraints, such as positivity and regularity (Tournier et al., 2007; Jian and Vemuri, 2007), is thus imperative for reliable FOD reconstruction. A good side of these deconvolution techniques is that they can yield accurate FOD estimates even at relatively low b-values, as well as they do not necessitate multiple b-value acquisition schemes. This is also true for other recent HARDI reconstruction methods such as PAS-MRI (Jansons and Alexander, 2002) and DOT (Ozarslan et al., 2006).

We opt for a wavelet-based, localized multiscale analysis. Contrary to other techniques, this technique superimposes scale-related components of the data and improves considerably filtering of noise-peaks (inherent to, for example, SD methods), by removing the noise in all scale-related components per se (Starck and Murtagh, 2006). We use the spherical wavelet transform (SWT) of (Starck et al., 2006) directly on the diffusion ODF (Kezele et al., 2008), and we incorporate the approximate relation between the HARDI signal and the ODF into the wavelet analysis, via the FRT (Tuch, 2004). In our approach, we explicitly search for the relevant and localized diffusion information on multiple scales. Our algorithm is an extension of the well known “à trous” algorithm from 2D images to functions defined on the sphere (Starck et al., 2006). It is an undecimated (hence, redundant) wavelet transform, with cubic B-spline scaling function, defined on the sphere. The transformation redundancy actually helps us to avoid Gibbs aliasing inherent to orthogonal or bi-orthogonal basis (Starck and Murtagh, 2006). Also, although the explicit data reconstruction may not be mandatory for a number of applications (e.g., the ODF estimation, as it will be shown in the text that follows), the positive side of the algorithm is that it is fully invertible (unlike some other spherical wavelet algorithms (Starck et al., 2006)). Further, it allows inclusion of different data conditioned constraints for the reconstruction.

In parallel, and independently of our work (Kezele et al., 2008), in Michailovich and Rathi (2008), the authors build a ridgelet frame to decompose HARDI signal and upon this decomposition they transform the basis functions with the Funk–Radon transform (FRT) to calculate the diffusion ODF.

We apply our method to both physical diffusion phantoms (Po- upon et al., 2008) with 90° and 45° fiber-crossings and real in vivo human brain data. Throughout the paper we also make a special effort to compare our SWT results with state-of-the-art analytical q-ball imaging (Descoteaux et al., 2007) and spherical deconvolution (Tournier et al., 2007) reconstructions, and to demonstrate the added value of SWT-based reconstruction that, with its sensitivity even to low angular separation, and its inherent sparseness may prove quite beneficial for applications to fiber tracking, fiber bundle segmentation, or morphometry methods based on HARDI derived features.

The paper is organized as follows. In Section 2, we review the SWT of Starck et al. (2006) and develop the extension for HARDI and ODF sharpening. We also present the real datasets used to compare and validate our wavelet ODF decomposition. Then, in Section 3, we compare the ODF profiles reconstructed from our SWT, analytical q-ball and spherical deconvolution on the physical phantoms and in vivo data. We discuss our results and point to limitations and perspectives of this work in Section 4.

2. Methods

2.1. Spherical wavelet transform – "à trous" algorithm

For our application, we employed an isotropic transform, with cubic B-spline scaling function. This function is shown to be very close to Gaussian, converges to zero rapidly, and in addition, it fulfills the dilation equation (Starck et al., 2006). The algorithm is derived directly from the Fast Fourier Transform (FFT)–based wavelet transform (Starck and Murtagh, 2006), and being defined on the sphere, it relies on the SH transform. The scaling function (see Fig. 1, panels a–c), \( \Phi_l(\theta, \phi) \), where \( l \) is the cutoff frequency, and where \( (\theta, \phi) \) follow the physics convention (i.e., \( \theta \) is longitude and \( \phi \) is azimuth) exhibits azimuthal symmetry, and thus, its SH transform does not depend on the phase \( m \):

\[ \sqrt{2} \sum_{m=-l}^{l} c_{lm} \Phi_l(\theta, \phi) \cos(m\phi) \]

\[ \sqrt{2} \sum_{m=-l}^{l} c_{lm} \Phi_l(\theta, \phi) \sin(m\phi) \]

\[ \sqrt{2} \sum_{m=-l}^{l} c_{lm} \Phi_l(\theta, \phi) \]

Fig. 1. Symmetrized scaling functions (a, b, c) and associated wavelets (d, e, f), as defined in Eqs. (1) and (4). The maximum SH order is \( l = 16 \). (a) \( j = 0 \), \( l = 8 \); (b) \( j = 1 \), \( l = 4 \); (c) \( j = 2 \), \( l = 2 \); (d) \( j = 0 \), \( l = 16 \); (e) \( j = 1 \), \( l = 8 \); (f) \( j = 2 \), \( l = 4 \).
\( \Phi_0(\theta, \phi) = \Phi_c(\theta) = \sum_{l=0}^{\ell} \Phi_c(l, 0) Y_{l,0}(\theta, \phi), \)  

(1)

where \( Y_{lm} \) is the SH of order \( l \) and phase \( m \), and \( \Phi_c \) is the SH transform of \( \Phi_c \).

The SH representation greatly simplifies the convolution with the spherical function \( h(\theta, \phi) \) to be analyzed, which reduces to:

\[
\hat{c}_0(l, m) = (\hat{\Phi}_c \ast h)(l, m) = \sqrt{\frac{4\pi}{2l+1}} \hat{\Phi}_c(l, 0) \hat{h}(l, m),
\]

(2)

where operator \( \ast \) stands for the convolution, \( \hat{c}_0 \) are the SH coefficients of the resulting convolution, and \( \hat{h}(l, m) \) are the SH coefficients of spherical function \( h \).

The multiresolution decomposition of \( h \) is performed on a dyadic scale, by convolving \( h \) with the rescaled versions of the scaling function \( \Phi_c \) (dyadically dividing the cutoff frequency \( l_c \) by powers of 2): \( c_j = \Phi_{2^{-j}l_c} \ast h \), where \( j (j = 0, \ldots, J) \) is the scale, and \( \Phi_{2^{-j}l_c} \) is a rescaled \( \Phi_c \) with \( 2^j \) times lower cutoff frequency. The decomposition can be done recursively as \( c_{j+1} = c_j \ast f_j \) (setting \( c_0(\theta, \phi) = h(\theta, \phi) \)), where \( f_j \) represents a low-pass filter associated to each scale \( j \). This \( f_j \) is defined in terms of the scaling function \( \Phi_c \) and its SH transform \( \hat{f}_j \) is defined as follows:

\[
\hat{f}_j(l, m) = \begin{cases} \hat{\Phi}_{2^{-j}l_c}(l, m) / \hat{\Phi}_{l_c}(l, m) & \text{if } l < 2^{j-1}l_c \text{ and } m = 0 \\ 0 & \text{else} \end{cases}
\]

(3)

Following the approach of the “à trous” algorithm, the wavelet coefficients are then defined as the difference between two consecutive low-pass filtered versions of \( h \): (cf. Fig. 1, panels d–f):

\[
w_{j+1} = c_{j+1} - c_j.
\]

(4)

This relation implies that the high-pass filter \( g \) to obtain the wavelet coefficients directly, is given in SH basis by:

\[
\hat{g}_j(l, m) = 1 - \hat{f}_j(l, m),
\]

where \( \hat{f}_j \) is defined in SH, as in Eq. (3) above, at each scale \( j \). The scaling function itself is defined in SH space:

\[
\hat{\Phi}_j = \frac{3}{2} B_3 \left( \frac{2}{l_c} \right),
\]

where

\[
B_3(x) = \frac{1}{12} (3|x - 2|^3 - 4|x - 1|^3 + 6|x|^3 - 4|x + 1|^3 + |x + 2|^3).
\]

In direct space, the scaling function \( \Phi \) is proportional to the \( \text{sinc}^2 \left( \frac{x}{l_c} \right) \) (where \( x = 1/l_c \) (Starck and Murtagh, 2006), and is as such closely approximated by a Gaussian, and converges to zero rapidly.

Finally, Eq. (4) above implies a straightforward reconstruction scheme of the original signal \( c_0 \):

\[
c_0(\theta, \phi) = c_j(\theta, \phi) + \sum_{j=1}^{J} w_j(\theta, \phi).
\]

(6)

### 2.1.1. Wavelet coefficients filtering

To meet the request on data sparsity, which directly leads to its compression, we introduce an additional step that follows after the signal decomposition of Eq. (6). This step concerns wavelet coefficients shrinkage (or filtering), which is shown to be extremely effective in data denoising and contrast sharpening (Mallat, 1999). Wavelet filtering is, in general, a non-linear transformation of wavelet coefficients, at each analyzed scale. It is well known that with the aim of sparse signal representation, the filtering should be done by minimizing the L0 norm of these coefficients. It is also known that the L0 minimization leads to a NP-hard problem, and that the L1 norm minimization results in the sparsest solution, closest to the one obtained by L0 minimization. It is worthwhile noting that if the wavelet basis were orthogonal, filtering based on hard thresholding would provide us with the exact solution to the L0 minimization (Starck and Murtagh, 2006).

Since the noise of diffusion MRI in each direction follows Rician distribution, it is difficult to estimate the exact distribution of noise on wavelet coefficients that is, at each scale, given as a convolution of noise over different directions with the wavelet band-pass filter. Hence, to define a \( (1-x)\% \) (e.g. 95\%) threshold for wavelet coefficient shrinkage (where \( x \) is the significance level), one possibility would be to perform permutations of signal coefficients along different directions, calculate the wavelet coefficients of this synthesized signals, order them increasingly by their magnitude, and locate the value at \( (1-x)\% \). The average of this value from all permutations defines the \( (1-x)\% \) threshold. At each wavelet scale only those coefficients whose magnitude is larger than the threshold are preserved. We tested this approach for 1-voxel experiments on phantom data. However, permutation tests are highly costly in computation time (e.g., no less than \( 10^7 \) permutations are to be generated at each voxel), and thus are not practically applicable to larger data sets. For these reasons, we define the threshold with respect to the percentage error in wavelet synthesis. In the wavelet synthesis relation given in Eq. (4), we keep the subset of wavelet coefficients with highest magnitude so that the overall synthesized ODF percentage error with respect to ODF synthesized using all the wavelet coefficients does not exceed 1.5\% for phantoms, and 3\% (on the average) for real data. This thresholding scheme gives best trade-off between compression and denoising. No more than 10 wavelet coefficients per scale are needed. This way, the signal is closely preserved while denoised and compressed. The number of coefficients for its final representation is highly reduced (e.g., for a maximum SH order of 8 or 16, instead of 45 or 153 SH coefficients, respectively, we have only 20 wavelet coefficients (10 per scale) to faithfully represent the spherical signal).

### 2.2. Spherical wavelet transform of the ODF

Our goal is to come up with a sharp and sparse representation of the diffusion ODF, with the capacity for resolving multiple fiber bundle configurations, even with low separation angles, and even for low SNR. Such a depiction of diffusion process is expected to boost the power of fiber tracking applications that rely strongly on the quality of local models, and their ability to delineate distinct diffusion compartments along close directions. In addition, we search for sparse signal representations, to: diminish the uncertainty in recovered fiber track directions, and to obtain compact representations for lighter computational loads. Implicitly, compact signal models result also in a small number of interesting features that may prove useful for applications like fiber bundle segmentation or for morphometry studies based on HARDI derived descriptors.

If we assume that the ODF is given as a finite sum of probability distribution functions on the sphere, where each of these functions describes the angular probability of finding a WM fiber bundle along a predefined set of directions (centered at a prescribed direction), then a natural decomposition of the ODF would be onto a basis of such functions. Typically, for voxels with only a single fiber bundle oriented along a certain direction \( d \), its contribution to the ODF in the observed voxel can be obtained analytically (Descoteaux et al., 2009a; Descoteaux, 2008) based on the Gaussian diffusion assumption. However, due to complex WM architecture, including different types of fiber-crossings and similar non-trivial fiber configurations, as well as due to limited capacity of imaging tools to resolve such configurations, the bandwidth of the elementary ODF-building function cannot be assumed uniform across the imaged WM. For that reason, we favor multiscale approach for ODF analysis.
We thus apply the spherical wavelet transform (SWT) to the diffusion ODF. However, we would like to work directly on HARDI signal. This is possible by incorporating the aforementioned approximate relation between the signal and its ODF via the Funk–Radon transform. Starting with the spherical scaling function \( \Phi \) and the associated wavelet function \( W \), we derive in this section, step-by-step, and through FRT, the multiscale spherical filters \( \Phi_b \) and \( W_b \), that enable us to perform the SWT of ODF by convolving the HARDI signal with the derived filters directly. We thus avoid any explicit ODF calculations.

Since both the HARDI signal and its ODF are symmetric and real functions on the sphere, they can be represented by a modified SH basis, \( Y_n \), that is symmetric and real, taking into account only even order \( l \), as defined in Descoteaux et al. (2006, 2007). If we unfold the recursive multiscale scheme described in Section 2.2, then the scaling functions \( \Phi \) convolve the ODF, successively, at each scale \( j, j = 0, \ldots, j \), to produce the low-pass components of the ODF. Referring to Eq. (4), the associated wavelet functions, \( W_j \), are now simply given by:

\[
W_j = \Phi_{j+1} - \Phi_j,  \tag{7}
\]

and are also convolved with the ODF at each scale \( j \), to provide the ODF band-pass details.

The signal symmetry also allows to symmetrize the cubic B-spline scaling function \( \Phi \), and its associated wavelet function \( W \). As the two functions are real, they can be likewise represented by the same symmetric and real SH basis, \( Y_n \).

Letting \( \mathbf{u} \) represent a unit direction on the sphere and \( N \) the maximal order of modified SH basis corresponding to the chosen cutoff frequency \( L_c \), we first express the convolution on the sphere between the ODF, \( \Psi \), and the scaling function \( \Phi \) as

\[
(\Psi \ast \Phi)(\mathbf{u}) = \sum_{n=1}^{N} \tilde{\Psi}_n \tilde{\Phi}_n Y_n(\mathbf{u}).  \tag{8}
\]

Then, FRT of the signal, expressed as great circle integrals over perpendicular directions to \( \mathbf{u} \) on the unit sphere \( \Omega \), is solved analytically (Descoteaux et al. 2007) yielding:

\[
\Psi(\mathbf{u}) = \int_{\mathbb{S}^{2}} \delta(\mathbf{u}, \mathbf{v}) h(\mathbf{v}) d\mathbf{v} \rightarrow \tilde{\Psi} = 2\pi P_0(0) \tilde{H}_s,  \tag{9}
\]

where \( \delta \) is spherical Dirac delta function, \( h \) the HARDI signal defined on the sphere, \( P_0 \) the Legendre polynomial of order 0, corresponding to order \( n \) of the modified SH basis \( Y_n \), and, \( \Psi, \Phi, \Phi_b, \) and \( H_s \) stand for the \( n \)th component of the spherical harmonic transform of \( \Psi, \Phi, \) and \( \Phi_b \) respectively. Eqs. (8) and (9) imply the following identity:

\[
(\Psi \ast \Phi)(\mathbf{u}) = \sum_{n=1}^{N} 2\pi P_0(0) \tilde{H}_s \tilde{\Phi}_n Y_n(\mathbf{u}).  \tag{10}
\]

By grouping the SH coefficients of scaling function \( \Phi \) with Legendre polynomials, \( P_0(0) \) (scaled by \( 2\pi \)), of the corresponding order:

\[
(\Psi \ast \Phi)(\mathbf{u}) = \sum_{n=1}^{N} 2\pi P_0(0) \tilde{\Phi}_n \tilde{H}_s Y_n(\mathbf{u}),  \tag{11}
\]

we obtain the following identity:

\[
(\Psi \ast \Phi)(\mathbf{u}) = \sum_{n=1}^{N} \tilde{\Phi}_n \tilde{H}_s Y_n(\mathbf{u}).  \tag{12}
\]

\( \tilde{\Phi}_n \) represents the SH coefficients of a derived filter \( \Phi_b \), that represents (refer to the relation given by Eq. (9)), the Funk–Radon transform of the scaling function \( \Phi \). This relation implies that we can calculate the convolution of the ODF \( \Psi \) and the scaling function \( \Phi \) by calculating the convolution of the HARDI signal \( h \) and the derived filter \( \Phi_b \).

The derived low-pass function \( \Phi_b \), that we employ to transfer the SWT analysis from diffusion ODF to HARDI signal directly, is illustrated in Fig. 2 (panels a–c). Shown are three different scales that correspond to modified SH cutoff frequencies of 8, 4, and 2 respectively. It is interesting to note that the Gaussian profiles of \( \Phi_b \) scaling functions match naturally HARDI response functions for single fiber bundle orientation inside the voxel (Tournier et al., 2004, 2007; Michailovich and Rathi, 2008). Different fiber characteristics give rise to different frequency content of these response functions. Fig. 2 shows the \( \Phi_b \) that correspond to HARDI response functions for diffusion within the bundles oriented along the z-axis. Similarly, simple rotations of \( \Phi_b \) kernels accommodate single orientation fiber bundle response functions along other directions.

In parallel to our work (Kezele et al., 2008), in Michailovich and Rathi (2008), the authors derive a ridgelet framework, starting with the spherical wavelets, derived from Gauss–Weierstrass kernels of a specified bandwidth following the work of Candes and Donoho (1999). The authors demonstrate that the ridgelet generating function is the Funk–Radon transform of the spherical wavelet scaling function. Hence, our derived filter \( \Phi_b \) is very similar to the ridgelet generating function of Michailovich and Rathi (2008).

Similarly to the \( \Phi_b \) definition, we define the wavelet-derived filters \( W_b \) as FRT of the spherical wavelet functions from Eq. (7):

\[
W_{bn} = 2\pi P_0(0) \tilde{W}_n  \tag{13}
\]

Then, from the linearity of Funk–Radon transform, and Eq. (7) it directly follows:

\[
W_{bn+1} = \Phi_{bn} - \Phi_{bn+1},  \tag{14}
\]

which together with Eq. (12) results in:

\[
(\Psi \ast W_j)(\mathbf{u}) = \sum_{n=1}^{N} \tilde{W}_n \tilde{H}_n Y_n(\mathbf{u}).  \tag{15}
\]

The wavelet-derived function \( W_b \) is illustrated in Fig. 2, panels d–f. Shown are three different scales that correspond to SH cutoff frequencies of 16, 8, and 4, respectively. These wavelet-derived functions are related to scaling function-derived filters \( \Phi_b \) (Fig. 2, panels a–c) through Eq. (14).

Similar to the relation in Eq. (6), we have, combining Eqs. (7)–(15):

\[
(\Psi \ast W_j)(\mathbf{u}) = (\Psi \ast \Phi_j)(\mathbf{u}) + \sum_{j=1}^{j} (\Psi \ast W_j)(\mathbf{u})
\]

\[
= (h \ast \Phi_0)(\mathbf{u}) + \sum_{j=1}^{j} (h \ast \Phi_0)(\mathbf{u}) = \Psi(\mathbf{u})_l + \Psi(\mathbf{u})_l,  \tag{16}
\]

where \( \Psi_l \) and \( \Psi_h \) denote the low and high frequency components of \( \Psi \). The final step in this ODF decomposition is wavelet coefficient filtering, as described in Section 2.1.1. This way, we obtain a sparse and denoised representation of the ODF. Using the convolution theorem it is straightforward to show that the spherical wavelet coefficients of the ODF can be obtained by convolving the HARDI signal with the same wavelet basis and integrating the result along the great circles perpendicular to the given ODF directions.

Despite a high level of ODF wavelet compression, we see that the low frequency part of the ODF is still present in the ODF reconstruction of Eq. (16). In practice, we typically limit this low

\[ ^1 \text{Note that we do not use the recursive scheme presented in Section 2.1 in order not to cancel out the } 2\pi P_0(0) \text{ term from } \Phi_b. \]
frequency component to $l_c = 2$ (the SH order corresponding to the 2nd-rank tensor) that, effectively, carries no relevant information on fiber bundle angular distribution. Recall that our main goal is to extract a small number of interesting ODF components that would facilitate WM fiber tracking and similar applications. Thus, instead of representing our sparse angular distribution function by both low and high frequency component, we rather focus on the "high" frequency part only, effectively neglecting the lowest frequency ODF contribution. In other words, we focus on the sharp ODF, $\Psi_H$, as opposed to the ODF $\Psi$ in its entirety. In this respect, our sharp ODF is expected to resemble a FOD that is in essence a sort of a high-pass filtering adapted to data. This link between the two methods can be better appreciated if we redefine the high-frequency part of the ODF ($\Psi(\tilde{u})_H$) from Eq. (16) above, changing the band-pass filters $W_{l_j}$ with high-pass filters $W_{Hj}$ as the following:

![Fig. 2. Funk–Radon transform of the scaling functions (a, b, c) and associated wavelets (d, e, f) (see Fig. 1), as defined in Eqs. (11), (13) and (14). These functions are used to transfer the ODF wavelet analysis directly onto the HARDI signal (refer to Section 2.2). The maximum SH order is $l_c = 16$. In (a) $j = 0$, $l_c = 8$; (b) $j = 1$, $l_c = 4$; (c) $j = 2$, $l_c = 2$; (d) $j = 0$, $l_c = 16$; (e) $j = 1$, $l_c = 8$; (f) $j = 2$, $l_c = 4$.](image1)

![Fig. 3. ODF and FOD reconstruction for the 90° and 45° crossing phantoms in the centermost voxel. Shown are in rows 1 and 2: the full analytical ODF (sh-q-ball), maximum SH order $l_c = 16$; (a, d) wavelet coefficients of the first resolution scale (highest frequencies of $\Psi_H$ from Eq. (16), $j = 2$, maximum SH order $l_c = 16$ for the 90°-, and $l_c = 14$ for the 45°-crossing phantom) after thresholding; (b, e) wavelet coefficients of the second resolution scale (lower frequencies of $\Psi_H$ from Eq. (16), $j = 1$, maximum SH order $l_c = 8$ for the 90°-, and $l_c = 7$ for the 45°-crossing phantom) after thresholding; (c, f) the sharp ODF ($\Psi_H$ from Eq. (16)) reconstructed with wavelet coefficients at both scales after thresholding; the last row illustrates the spherical deconvolution (SD) profiles, from filtered SD (fSD) and constrained regularized SD (CSD) with SH order $l_c = 16$ for the 90°-, and $l_c = 14$ for the 45°-crossing phantom.](image2)
\[ \Psi_{(\mathbf{u})} = \sum_{j=1}^{J} (h * W_{Hj})(\mathbf{u}) = \sum_{j=1}^{J} (h * R_j^{-1})(\mathbf{u}) \]  

where \( W_{Hj} = R_j^{-1} \) stands for the high-pass inverse of the low-pass filter \( R_j \) at scale \( j \), with \( R_j \) defined similar to the convolution kernel of Tournier (Tournier et al., 2004, 2007) (scale \( j \) determines the filters’ cutoff frequencies). For this resemblance between the methods, we compare our results with both the analytical SH q-ball method (Descoteaux et al., 2007), and the FOD reconstructions from spherical deconvolution (SD) reconstructions (Tournier et al., 2004, 2007).

2.3. Data acquisition

We will first demonstrate our SWT method on two real physical phantoms emulating two fiber bundles crossing at 90° and 45°, respectively (Poupon et al., 2008). The phantom data were acquired on a 1.5T Signa MR system (GE Healthcare, Milwaukee), TE/TR = 130 ms/4.5 s, 12.0 s (45° and 90° phantom, respectively), BW = 200 KHz. To enhance the SNR (keeping SNRmin > 4), large voxel dimensions were used (FOV = 32 cm, matrix size of 32 × 32). We analyze the data acquired at a \( b \)-value of \( b = 2000 \text{ s mm}^{-2} \), along 4000 uniformly distributed orientations (see Fig. 3). For comparisons with real brain data sets, the directions were subsampled to 200, uniformly distributed (using a geodesic interpolation algorithm (Tuch, 2004)).

The SWT is also applied to a real in vivo data sets of a healthy volunteer. The data set was acquired on a 1.5T Signa MR system (GE Healthcare, Milwaukee), TE/TR = 100.2 ms/19 s, BW = 200 KHz, FOV = 24 cm on a 128 × 128 matrix, TH = 2 mm, 60 axial slices, \( b = 0 \text{-} 3000 \text{ s mm}^{-2} \) (SNR ≈ 2), 200 uniformly distributed orientations. This dataset is a part of the publicly available HARDI database (Poupon et al., 2006).

3. Results

3.1. Spherical wavelet transform of the ODF on the physical phantoms

We applied our method for spherical wavelet decomposition of ODF to the 90°- and 45°-crossing real physical phantoms. To enable voxel dimensions were used (FOV = 32 cm, matrix size of 32 × 32). We analyze the data acquired at a \( b \)-value of \( b = 2000 \text{ s mm}^{-2} \), along 4000 uniformly distributed orientations (see Fig. 3). For comparisons with real brain data sets, the directions were subsampled to 200, uniformly distributed (using a geodesic interpolation algorithm (Tuch, 2004)).

The SWT is also applied to a real in vivo data sets of a healthy volunteer. The data set was acquired on a 1.5T Signa MR system (GE Healthcare, Milwaukee), TE/TR = 100.2 ms/19 s, BW = 200 KHz, FOV = 24 cm on a 128 × 128 matrix, TH = 2 mm, 60 axial slices, \( b = 0 \text{-} 3000 \text{ s mm}^{-2} \) (SNR ≈ 2), 200 uniformly distributed orientations. This dataset is a part of the publicly available HARDI database (Poupon et al., 2006).

Fig. 4. Estimates of the peak angular separation for SWT-based ODF sharpening method. Shown are the results from the 3 × 3 voxels ROI (first row) of both 90°- and 45°-crossing phantoms. SH q-balls (max SH order = 16, \( j = 0.006 \)) are shown in the second row. Sharp ODFs by SWT method with peak angular separation at each voxel are shown in the fourth row. Maximum SH order was \( l_x = 16 \) for the 90°-, and \( l_y = 14 \) for the 45°-crossing phantom; \( j = 2 \) wavelet scales. “X” represents the voxels with no crossing detected.
the subsequent comparison of results with the results of similar analysis performed on the real data of two healthy volunteers, we subsampled the directions to the same set of 200 directions as our real dataset (Section 2.3). Also, to adapt our analysis frequency bands to the frequency content of the data, the original cubic B-spline kernel was twice dilated in the frequency space. The cutoff frequency was set to \( l_c = 16 \) for the 90°-crossing phantom. Due to noisy decomposition onto the SH basis (seen in both SWT and SD-based methods), the cutoff frequency was set to \( l_c = 14 \) for the 45°-crossing phantom. The datasets were decomposed onto 2 resolution scales; scale 1 corresponding to maximum SH orders of \( l_c = 16 \), and \( l_c = 14 \), for the 90°- and 45°-crossing phantom, respectively; and similarly, scale 2 corresponding to maximum SH order \( l_c = 8/7 \), respectively. We analyzed first the central 10 x 10 x 14 mm voxel in both phantoms.

For the 90°-crossing phantom, since the configuration is quite simple, only the four most significant wavelet coefficients at each scale were kept. For the 45°-crossing phantom, since the separation angle is quite low, the 10 most significant wavelet coefficients at each scale were kept. For both scales, this thresholding resulted in less than 1.5% error when the overall sum of the lowest frequency ODF component and two thresholded wavelet scales was compared to the estimated full SH analytical q-ball. To regularize SH coefficients estimates for q-ball analysis, we employed Laplace–Beltrami smoothing of \( \lambda = 0.006 \) as explained in (Descombes et al., 2007). Fig. 3 depicts the results for the central voxel of 90° and 45° crossing phantoms. For comparison purposes, we also reconstructed analytical q-ball as well as two types of ODFs (following the algorithms of Tournier et al. (2004, 2007)) with filtered spherical deconvolution (FSD) and with regularized constrained spherical deconvolution (CSD), at different order \( l \) (depicted in Fig. 3). The low-pass filter for FSD was \([1 \ 1 \ 1 \ 0.5 \ 0.1 \ 0.02 \ 0.002 \ 0.0005 \ 0.0001]\) applied to SH order \( l = [0 \ 2 \ 4 \ 6 \ 8 \ 12 \ 14 \ 16] \). For CSD, we used a regularization parameter \( \lambda = 1 \), and the threshold for positivity constraint \( \tau = 0.1 \). The maximum SH order was set \( l_c = 16 \) for the 90°, and \( l_c = 14 \) for the 45° phantom. The deconvolution kernel for FSD and CSD was estimated from a manually defined ROI in the single fiber part of the ex vivo phantom.

With the purpose of quantitative evaluation, we compared the peak angular separation for SWT-based sharpening method to the peak angular separation of the two SD techniques, for the 3 x 3 voxels central ROI of the 90°-crossing and 45°-crossing phantoms. For the 90°-crossing phantom, five most significant coefficients per scale are kept; for the 45°-crossing phantom, this number was set at 10. All other parameters for SWT, and the two SD methods were as above. The calculated angles are shown in tabular form in Figs. 4 and 5, overlaid on the grid of the ROIs’ sharp-ODFs calculated by SWT at each voxel for both phantoms (Fig. 4), and on the grid of ROIs’ FODs for the two phantoms, in the case of the two SD methods (Fig. 5).

Denoising and sharpening obtained very good results, for both phantoms, while reducing the number of necessary coefficients for ODF presentation from 153 to: 8 for 90°-crossing phantom, and to 20 for 45°-crossing phantom. Directions of maximal diffusion are distinguishable for all but the two edge ROIs voxels for the 45°-crossing phantom, with the error in the recovered separation angle within the sampling angular resolution limit for both 90°- and 45°-crossing phantom. The inability to discern the two lobes at those two edge voxels is likely the consequence of the joint effects of unequal contribution of the two diffusion compartments and Bessel blurring (Tuch, 2004), due to Funk–Radon based approximation of the radial projection of the full diffusion probability density function. The approaches to minimize the letter will be the part of the future work (see Section 4).

Overall we reach a valuable result, demonstrating the SWT method’s capacity to resolve the fiber bundle crossings at angles as low as 45°, at a single q-shell, that standard q-ball transform cannot achieve. For 90°-crossing phantom, q-ball transform manages to extract the principle diffusion/fiber directions, however, the result is more blurred compared to our sharp ODF. On 45°-crossing phantom, the directions of maximal diffusion are no longer discernable.

The CSD results in sharper FODs than the ODFs of our SWT method. The peak angular separation is also closer to the ground truth for the former (Figs. 4 and 5). The CSD deconvolution kernel was fully adapted to the phantom data. Also, the CSD is a non-linear regularized method. Although, the SWT produces less sharp ODFs, the good side is that is linear. Further, no regularization needs to be imposed. Although the two methods, SWT and SD are differently founded, the former being based on signal projections and convolution, and the latter on reverse process of signal deconvolution, there are some similarities, especially between the SWT and CSD. The high-pass filtering inherent to SD with the imposed constraint on the number of non-zero directions is very similar in logic to SWT-based sharpening with wavelet filtering to account for signal sparsity. This is the reason why both CSD and SWT outperform the fSD that does not take into account any forms of sparsity constraints. The FODs appear blurred, with numerous negative values of important magnitude, due to which the principal fiber directions are visibly less prominent. Obviously, fSD lacks an additional condition to cope with the noise at higher frequencies.
SH orders (that are needed to discriminate signals along directions with “low” separation angles).

3.2. Spherical wavelet transform of the ODF on real in vivo data

Similar analysis was performed on a ROI of a real data set, only for these experiments, the cutoff frequency was set to \( l_c = 8 \) to accommodate the SH coefficients’ estimate to the number of available measuring directions, and to diminish the high-frequency noise as much as possible (given the rather poor SNR in real data sets). We analyzed the data at two scales, and kept 10 most significant wavelet coefficients at each scale (with the average reconstruction error of 2.6%). To regularize SH coefficients estimates we employed Beltrami–Laplace smoothing as explained in Descoteaux et al. (2007). For real data, the negative values are hard thresholded to zero. For the first data set, along 200 directions, we also

Fig. 6. q-Ball ODF and sharp ODF reconstructions on real data. From left to right, and from top to bottom: analyzed ROI; zoom at ROI on the fused diffusion RGB and anatomical MRI image; full ODF calculated analytically (Descoteaux et al., 2007) in the SH space with \( l = 8 \); (a) full ODF \( \Psi \) reconstructed as a sum of the \( \Psi_c \) and the thresholded \( \Psi_h \) (Eq. (16)) (the maximum number of the significant coefficients was 10 at both scales); (b) wavelet coefficients of the first resolution scale (highest frequencies of \( \Psi_h \) from Eq. (16), \( j = 2 \), maximum \( l_c = 8 \)) after thresholding; (c) wavelet coefficients of the second resolution scale after thresholding (lower frequencies of \( \Psi_h \) from Eq. (16), \( j = 1 \), maximum \( l_c = 4 \)); (d) the sharp ODF (\( \Psi_h \) from Eq. (16)) reconstructed with wavelet coefficients at both scales after thresholding; (e) example WM voxels showing how different frequency scales capture complementary directional information (see text for details).
show some typical WM voxels for which the multiscale analysis proves to be particularly important.

The results of the analysis on the real data set are shown in Figs. 6 and 7. The overall sum of the thresholded wavelet coefficients at both scales and the lowest signal component (Fig. 6a) differs from the estimated SH analytical \( q \)-ball by 2.6% on the average. Denoising is already visible, even with no wavelet shrinkage. Our sharp ODF (Fig. 6d) extracts highly relevant information on fiber compartments. With only 10 coefficients per scale we manage to capture the most significant diffusion/fiber directions at each voxel. Panel e of Fig. 6 suggests the importance of the multiscale approach: different scales may assess complementary information on the underlying diffusion milieu, i.e., distinct diffusion compartments may give rise to different frequency content of the signal (Fig. 6e). (Note that the shown ODFs are min–max scaled.) The first row of panel f, corresponds to the ROI voxel lying exactly on the border between WM and cortical mantle (position \((7, 1)\) (row, column) in panels a–d). The two principal diffusion directions appear to be related to fibers of corona radiata: passing tangentially to the cortical mantle, and protruding into the cortical mantle. The second and third row are associated with the two more central WM ROI positions: \((7,8)\) (likely the cortico-spinal tract and corona radiata crossing), and \((7,9)\) (the crossing of calosal fibers, and radiation fibers from corpus calosum) (in panels a–e), respectively. Spherical deconvolution results on the same two resolution scales are shown in Fig. 7. The results of SWT and CSD look quite similar. (For fair comparisons of the results of SWT-based reconstruction method, and the SD method, we keep the maximum SH order of the CSD technique at the same value as for SWT.) The ground true being currently unavailable, it is not possible to compare the accuracy of the two methods, however, the visual inspection suggests that the SWT outperforms both SD methods in real data. Future work will include more extensive analysis on phantom data emulating larger number of crossing bundles and with different crossing angles. As for ISD, very low SNR in this real data set appears to be quite challenging, and the technique fails to capture the local tissue architecture.

4. Discussion

In this study, we employed a linear spherical wavelet transform for multiscale decomposition of the diffusion ODF. We tested and compared our method with respect to state-of-the-art \( q \)-ball and spherical deconvolution reconstructions on real physical phantoms and real brain data. The experiments allowed us to demonstrate the algorithm efficiency to represent the relevant features with only a very small number of wavelet coefficients. The spherical wavelet transform thus provides a high level of data compression and denoising. For example, a reconstruction of the \( q \)-ball or spherical deconvolution in SH basis of order 16 required 153 coefficients whereas our reconstruction used at most 20 spherical wavelet coefficients (10 per scale).

More importantly, our spherical wavelet transform of the ODF has a desired sharpening effect when focusing on the high-frequency part of the decomposition. This proves to be efficient for extracting pertinent angular information, concealed by signal spread out over frequency scales that cannot be seen with classical ODF alone. The results at some voxels of the real data set suggest that different angular frequency scales may encode complementary diffusion information (i.e., distinct diffusion compartments), and consequently, complementary information on tissue architecture. This may prove valuable for WM fiber tracking, particularly in places where WM fibers cross at relatively small angles. This could also potentially lead to new multiscale scalar/vector measures sensitive to different micro-architectural properties of biological tissues than classical DTI and HARDI-based scalar measures (such as FA, GFA, and others).

As aforementioned, CSD acts like a high-pass filter, with an additional constrain on the maximum number of non-negative directions, and in that sense approaches the philosophy of wavelet analysis with signal sparsity constraint. However, CSD is computed with a constrained regularization at the cost of a non-linear optimization that is computationally more expensive than the ISD or our SWT. A possible advantage of SD techniques lies in data adapted deconvolution kernel. Even so, and especially in real data
that are much less uniform than the manufactured phantoms, the estimated deconvolution kernel may not be equally optimal for all the investigated regions. It is possible that it should be defined differently depending on the region of the brain. Furthermore, SD methods assume a deconvolution kernel that is data-dependent and thus, “b-value dependent”, whereas our approach does not. Although the CSD with the deconvolution kernel fully adapted to the data at hand outperforms the SWT, the experiments on real data suggest that the SWT is no less capable of coping with noise than the CSD. We find the results on real data rather encouraging. Thanks to the multiscale approach and denoising at different resolution levels, our ODF sharpening technique performs quite well, even at maximum SH order of only eight, and even with no explicit constraints put on the signal reconstruction from its significant wavelet coefficients. Incorporating data adapted constraints will also be part of our future work. Perhaps more care needs also to be taken when defining wavelet scaling functions, and render them more adaptive to the data, which is one of the issues that we plan to address in the future work.

Some very recent work (Tristán-Vega et al., 2009) also dealt with the Bessel blurring, inherent to all reconstruction methods that use Funk–Radon transform, as is the presented SWT. Future work will be oriented to algorithmic modifications to include the proposed techniques to correct for this blurring. Also, even though there are practical advantages of SWT application to single q-shell DWI acquisitions, like the possibility to use it in typical clinical settings, future work will look at expanding the SWT to analyse the signals on multiple shells in q-space (Canales-Rodríguez et al., 2009; Aganj et al., 2009). It has been shown recently that an antipodally symmetric spherical function (ASSF) can be equivalently described in the SH basis, in the symmetric high-order Cartesian tensor basis constrained to the sphere, and in the homogeneous polynomial basis constrained to the sphere (Ghosh et al., 2008; Bloy and Verma, 2008). By computing the stationary points of constrained polynomial functions, the maxima and minima of ASSF can be deduced. The ASSF maxima correspond to fiber orientations in the underlying diffusion substrate. It is still not understood though whether extracting the ODF maxima by these methods would outperform the methods that search for the fiber orientations directly, like our SWT-sharpening technique, or SD techniques described in Tournier et al. (2004, 2007). Another point worthwhile checking in these comparisons would be the level of technical requirements for each technique.

To conclude, the purpose of this report was to stress the potential and importance of spherical, local, and multiscale bases in HARDI studies. The spherical wavelet basis was shown to have promising properties that might outperform the popular spherical harmonics basis, as demonstrated by the experiments on phantoms and real in vivo human brain data. Our results suggest that the high frequency decomposition of the diffusion ODF can reveal interesting angular structure, especially when multiple angular frequency scales are analyzed. The next step would now be to incorporate this multiscale spherical wavelet analysis in tractography algorithms. The multiscale property of SWT may open new avenues of research of diffusion that is inherently multiscale in human tissues in vivo.

Acknowledgments

The authors would like to thank Bertrand Thirion, Philippe Ciu-ciu, Daniel Alexander, Pierrick Abrial, and Jean-Luc Starck for helpful discussions. This work was supported by a Marie Curie Actions individual postdoctoral fellowship, EU.

References


