ABSTRACT

In this paper, we define a coordinate system on a smooth surface associated to a set of sample points from the surface. This problem is well understood if the domain of the coordinate system is restricted to the convex hull of the sample set. Notably, Sibson proposed the so-called natural neighbor coordinates. Boissonnat and Cazals show that if the point set is taken from a smooth surface, the natural neighbor coordinates have nice properties when the sampling density of the surface tends to infinity. We suggest a system of coordinates that is defined everywhere on the surface, is continuous, and is local on the surface, even if the sampling density is finite. Moreover, it is inherently \((d - 1)\)-dimensional while the previous systems are \(d\)-dimensional. No assumption is made about the ordering, the connectivity or topology of the sample points nor of the surface. We illustrate our results with an application to interpolation over a surface.

General Terms

Algorithms, Design, Theory

Categories and Subject Descriptors

I.3.5 [Computing Methodologies]: Computational Geometry and Object Modeling—Curve, surface, solid, and object representations; G.1.1 [Mathematics of Computing]: Interpolation; F.2.2 [Theory of Computation]: Nonnumerical Algorithms and Problems—Geometrical problems and computations

1. INTRODUCTION

Surfaces represented by a set of unordered sample points are encountered in many application areas such as computer graphics, computer aided design (CAD) and reverse engineering, image processing, and scientific computation. Many algorithms that are applied to sampled surfaces rely on the definition of a local neighborhood on the surface. Reconstructing the surface from the sample points is one way to respond to this demand. However, it might be unnecessary and also too costly to establish a global approximation of the surface. Differently, our method defines, for any point on a sampled surface, a set of coordinates associated with some neighboring sample points. If the sampling is sufficiently dense, this coordinate system is provably local on the surface and has good continuity properties. Moreover, it can be computed efficiently because locality allows efficient filtering methods. We do not impose any restriction on the genus of the surface, the number of connected components, nor any other global features of the surface. Uniform sampling is neither required, and we allow the sampling density to be related to the local curvature of the surface.

1.1 Related Work

In this section, we describe how our work is related to previous work. It is divided in two parts: first, we outline the work on natural neighbor coordinate systems and, second, we give an introduction to scattered data interpolation on a surface, which is the application we develop at the end of the paper.

1.1.1 Natural neighbor coordinate systems

Natural neighbor interpolation has been introduced by Sibson [26] to interpolate multivariate scattered data. Given a set of points \(A = \{A_1, \ldots, A_n\}\), the natural neighbor coordinate system associated to \(A\) is defined from the Voronoi diagram of \(A\). Various papers ([26], [17], [24], [11],[21]) show that it satisfies the following definition by Brown [11].

**Definition 1.1 ([11]).** A system of coordinates over \(U \subseteq \mathbb{R}^d\) associated with \(A\) is a set of continuous functions \(s_i : U \rightarrow \mathbb{R}, i = 1..n\), such that for all \(X \in U\),

1. \(X = \sum_{i=1}^{n} s_i(X)A_i\) (local coordinate property).
2. For any \(i \leq n, s_i(A_j) = \delta_{ij}\), where \(\delta_{ij}\) is the Kronecker symbol.
3. \(\sum_{i=1}^{n} s_i(X) = 1\).

The major drawback to applying the natural neighbor coordinate system to points issued from a surface comes from the fact that its definition is limited to the convex hull of the sample points. To avoid this problem, a common solution consists of adding a box enclosing the object. Obviously, this solution causes problems, e.g. the choice of the size of the bounding box, the number of sample points taken from it, artifacts arising from the bounding box points, and the augmented computation cost. In [11], Brown has enlarged the coordinate definition outside the convex hull to cover the union of the Delaunay balls, which is still too much restrictive in many applications.
A second drawback is that a point is likely to have neighbors that are far away from the point. Boissonnat and Cazals have shown that the sum of the coordinates associated to those neighbors that are far away tend to zero when the sampling density increases [7]. However, even though the influence of the far neighbors is small, the fact that Sibson’s coordinates are not local is a drawback not only in the beauty and rigor of the result, but also affects the time required to compute the coordinates and in the exactness of an interpolation scheme.

For points issued from a sphere, Brown proposes a solution in [10]. While the above discussed coordinates are defined with respect to the $d$-dimensional Voronoi diagram of $A$, Brown defines natural neighbor coordinates with respect to the geodesic Voronoi diagram on the sphere. This definition generalizes Sibson’s coordinates in a straightforward manner. Therefore, the basic properties of definition 1.1 are fulfilled, except the local coordinate property (i) which cannot be fulfilled since points on the sphere do not belong to the convex hull of their neighbors. The obvious difficulty in enlarging Brown’s approach to general surfaces is that geodesic Voronoi diagrams are much more complicated than Euclidean diagrams and difficult to compute [22]. Moreover, in many applications the surface is not known and neither is the geodesic Voronoi diagram.

In this paper, we suggest another system of coordinates for points on a surface. It is closely related to natural neighbor coordinates, yet instead of considering the geodesic Voronoi diagram on the surface, as Brown, or the Euclidean $d$-dimensional Voronoi diagram of the sample set, as Sibson, it is defined in the intersection of the tangent plane of each surface point with the Euclidean Voronoi diagram of the sample set. If the tangent planes are not given as part of the input, they can be easily estimated from the sample points. The resulting coordinate system is local on the surface, it is defined everywhere in the neighborhood of $S$, and it is inherently $(d-1)$-dimensional.

### 1.1.2 Scattered data interpolation on a surface

In the last part of the paper, we apply the new coordinate system to interpolate a function defined on a surface. More exactly, we want to approximate $\Phi : S \to \mathbb{R}$ where $S \subset \mathbb{R}^d$ is a smooth surface, knowing a sample set $\{(A_i, z_i) : A_i \in S, z_i = \Phi(A_i)\}$ and a query point $X \in S$ on which we want to interpolate $\Phi$.

This problem, which is also called ‘scattered data fitting’ or ‘surface on surface’ problem, arises in a variety of settings. For example in geodesy, geophysics, and meteorology, $S$ is some model of the earth, and the function to interpolate from a number of discrete measurements represents temperature, rainfall, pressure, etc. In other contexts, $S$ might be a complicated surface, e.g., the surface of some mechanical piece in CAD, a molecular surface or the wing of an airplane [6]. Several methods exist to solve this problem. One of the most popular is to enlarge the definition of splines to treat the case of a non-planar parameter domain. This was first done for the spherical case in [2]. With this achievement, it suffices to partition a general surface $S$ into a collection of non-overlapping surface patches, i.e. geodesic triangles, and to define a globally smooth interpolation function as piecewise polynomials on the triangles that are carefully accorded at the boundaries. See [23] for an introduction to splines on surfaces.

Other methods are radial basis functions, variational methods or multi-resolution methods. See [18] for a survey of the principal methods for scattered data fitting on the sphere. Foley et al propose in [20] to map $S$ to the sphere, to apply a known interpolation method on the sphere, e.g. [19], and to apply the inverse mapping back from the sphere to the surface to get the solution.

In the context of interpolation, our method has several advantages in addition to those mentioned at the beginning of this section. It applies directly to the point samples without need of a prior subdivision or triangulation of the surface. The quality of the result can be expressed in terms of the curvature of the surface and of the sampling density. If the surface is locally planar, the interpolant has linear precision. If, additionally, the gradient $\nabla \Phi(A_i)$ of $\Phi$ at the sample points $A_i, i = 1, ..., n$, is known, we define an interpolant that reproduces exactly a quadratic function – again if the surface is locally planar. We tested the interpolants in different settings. Examples are shown in section 5.

### 1.2 Paper outline

After the introduction, we proceed in section 2 with the definition of some basic concepts, and we recall some known results that are needed in the sequel. In section 3, we define the $T$-neighbors of a surface point $X$ with respect to a set of sample points from the surface. We show that all $T$-neighbors of $X$ lie in a small neighborhood around $X$ if the surface is well sampled. In section 4, we define the $T$-coordinate system on the surface with respect to the sample. We show the main properties of the $T$-coordinates; in particular, we show that the coordinate functions have compact supports and are continuously differentiable almost everywhere on the surface. Although a point on a surface cannot, in general, be expressed as a convex combination of other points on the surface, we show in subsection 4.4, that the local coordinate property is approximately satisfied, with an error that depends on the local curvature of the surface and on the sampling density. In section 5, we describe the applications of $T$-coordinates to scattered data interpolation on a surface, and we show some experimental results. Perspectives and a conclusion are given in the last section.

## 2. BASIC NOTATIONS AND RESULTS

In this section, we give the definition of the main ingredients of our framework and recall some basic results. Notably, we give a short introduction to natural neighbor coordinates for the general case of points in $\mathbb{R}^d$, we define power diagrams and show their relationship to sections of Voronoi diagrams, and we recall results on sampled surfaces.

### 2.1 Voronoi diagrams and natural neighbors

Let $A = \{A_1, ..., A_n\}$ be a set of points in $\mathbb{R}^d$. Without loss of generality, we can assume that no $d+2$ points lie on the same sphere. The Voronoi cell of $A_i$ is $V(A_i) = \{X \in \mathbb{R}^d : \|X - A_i\| \leq \|X - A_j\| \ \forall j = 1, ..., n\}$ where $\|X - Y\|$ denotes the Euclidean distance between points $X, Y \in \mathbb{R}^d$. The collection of Voronoi cells is called the Voronoi diagram of $A$ or $Vor(A)$. Let $A' \subseteq A$ be a subset of points of $A$ whose Voronoi cells have a non-empty intersection. The convex hull $conv(A')$ is called a Delaunay face, and the collection of Delaunay faces is called the Delaunay triangulation of $A$, denoted $Del(A)$. See figure 1.

**Figure 1:** (a) a Voronoi diagram, (b) the dual Delaunay triangulation.
Given a point $X \notin A$, we define $Vor^+ = Vor(A \cup \{X\})$, $Del^+ = Del(A \cup \{X\})$ and $V^+(X)$ and $V^+(A_i)$ to be the Voronoi cells of $X$ and $A_i$, in $Vor^+$. Under the general position assumption, the $d$-dimensional faces of $Del(A)$ are simplexes, hence the name triangulation.

Figure 2: $X$ has five natural neighbors $A_0, \ldots, A_4$.

Let $V(X, A_i) = V^+(X) \cap V(A_i)$, the part of the Voronoi cell $V^+(X)$ that has been 'stolen' from $V(A_i)$ at the insertion of $X$. If $V(X, A_i) \neq \emptyset$, $A_i$ is a natural neighbor of $X$. Let $\nu_i(X)$ be the volume of $V(X, A_i)$ and $\nu(X)$ be the volume of $V^+(X)$. Figure 2 shows an example. Observe that if $X$ belongs to $\text{conv}(A_i)$, the convex hull of $A$, $\nu(X)$ is bounded.

**Definition 2.1.** The natural neighbor coordinates associated to $A$ of a point $X \in \text{conv}(A)$ are the functions $\sigma_i(X) = \frac{\nu_i(X)}{\nu(X)}$, $i=1, \ldots, n$.

As already mentioned, the natural neighbor coordinates satisfy the three conditions of Definition 1.1. Moreover, the support $\Delta_i$ of the natural neighbor coordinate $\sigma_i$, i.e. the set $\{X \in U[p_i(X)] \neq \emptyset\}$, is the interior of the union of the spheres circumscribing the Delaunay simplexes adjacent to $A_i$.

**Definition 2.2.** The natural neighbors of a point $X$ with respect to $A$ are the points $A_i \in A$ with $\sigma_i(X) \neq 0$, $i=1, \ldots, n$.

Notice that, the natural neighbors of $X$ are exactly the vertices other than $X$ of the simplices $Del^+$ adjacent to $X$.

### 2.2 Natural neighbor coordinates in power diagrams and sections of Voronoi diagrams

In this section, we recall the definition of a power diagram, as well as of natural neighbor coordinates defined in power diagrams. We also point out the relationship between $k$-dimensional sections of Voronoi diagrams in $\mathbb{R}^d$ and $k$-dimensional power diagrams.

Let $B = \{B_1, B_2, \ldots, B_n\}$ be a set of weighted points (or sites) in $\mathbb{R}^d$, $B_i = (p_i, w_i)$, in $\mathbb{R}^d \times \mathbb{R}$. A weighted point $B_i$ can be considered as a sphere with center $p_i$ and radius $\sqrt{w_i}$. Notice, that $w_i$ might be negative and the associated sphere imaginary. The power distance $d_i(p)$ of a point $p \in \mathbb{R}^d$ with respect to $B_i$ is defined as $d_i(p) := ||p - p_i||^2 - w_i$. A weighted point $(p_i, w_i)$ (or the corresponding sphere) is orthogonal to another weighted point $(p_2, w_2)$ (or sphere) if $d_i(p_2) = w_2$. Without real loss of generality, we can assume that the weighted points are in general position. This means that no $d + 2$ weighted points are orthogonal to the same sphere. The power cell of $B_i$ is $P(B_i) = \{p \in \mathbb{R}^d : d_i(p) \leq d_j(p) \quad \forall j = 1, \ldots, n\}$. The collection of power cells is called the power diagram of $B$ or $Pov(B)$. The dual of the power diagram is called the regular triangulation or $Reg(B)$. The vertices of $Reg(B)$ belong to $\{p_1, \ldots, p_n\}$ and, under the general position assumption, the $d$-dimensional faces of $Reg(B)$ are simplexes. We call orthosphere of a $d$-simplex of $Reg(B)$ the sphere that is orthogonal to the $d + 1$ weighted points associated to the vertices of the face. Its center is a vertex of $Pov(B)$.

**Lemma 2.5.** Let $X = (p_X, 0)$ be a point such that $P^+(X) \neq \emptyset$ for all $p_X \in \mathbb{C}$. The natural neighbor coordinate $\lambda_i(X)$ of $X$ with respect to $B$ is $C^0$ continuous over $\mathbb{C}$ and $C^1$ continuous except at a finite set of points.

**Observation 2.6.** $Vor'(A)$ is the power diagram of the points $A_i'$ that are the projection of the sample points $A_i \in A$ onto $H$ weighted with $w_i = -||A_i - A_i'||^2$.

**Proof.** Because $A_i'$ is the orthogonal projection of $A_i$ on $H$, we know that $\forall X \in H : ||X - A_i||^2 = ||X - A_i'||^2 + ||A_i - A_i'||^2$. Therefore, $V'(A_i) = \{X \in H : ||X - A_i||^2 \leq ||X - A_i'||^2 \quad \forall j = 1, \ldots, n\} = \{X \in H : ||X - A_i||^2 + ||A_i - A_i'||^2 \leq ||X - A_j'||^2 + ||A_i - A_j'||^2 \quad \forall j = 1, \ldots, n\} = P(A_i, w_i)$. 

### 2.3 Sampled surfaces: definitions and results

In this section, we consider the case where the sample points are taken from a smooth surface $S$, i.e. a twice-differentiable surface. We assume the first and the second derivative of $S$ at $X$ to be continuous and the third derivatives to exist for all $X \in S$. We give a definition of what we call the sampling density of $S$ and we recall several results about the local behavior of the surface samples, notably from [4] and [7].
2.3.1 Voronoi diagram on a surface

We first define the Voronoi diagram of a set of points restricted to a surface, following previous work by Chew [13] and Edelsbrunner and Shah [16].

**Definition 2.7. (Restricted Voronoi diagram, Chew)**
The Voronoi diagram of \( A \) restricted to \( S \) is the (curved) cell complex obtained by intersecting each face of Vor(\( A \)) with \( S \). We denote it by Vor\( _S \)(\( A \)).

We denote by \( V(\{A_i\}) \) the Voronoi cell of Vor\( _S \)(\( A \)) consisting of the points of \( S \) that are closer to \( A_i \) (for the Euclidean distance) than to any \( A_j, j \neq i \). A vertex of \( V(\{A_i\}) \) is the intersection of an edge of \( V(\{A_i\}) \) with \( S \). Hence it is the center of a ball passing through \( d \) points of \( A \) and not enclosing other points of \( A \).

**Definition 2.8. (Restricted Delaunay triangulation, Chew)**
The Delaunay triangulation of \( A \) restricted to \( S \) is the subcomplex of Del(\( A \)) consisting of the faces of Del(\( A \)) whose dual Voronoi edges intersect \( S \). We denote it by Del\( _S \)(\( A \)).

The notion of surface natural neighbors follows directly:

**Definition 2.9. (S-neighbor of \( X \))**
The surface natural neighbors or \( S \)-neighbors of a point \( X \in S \) are the vertices of the faces of Del\( _S \)(\( A \cup \{X\} \)) that are incident to \( X \).

2.3.2 Medial axis and local feature size

Let \( F \subset \mathbb{R}^d \) be a compact object, \( S \) its boundary. We call \( n_S \) the outward unit normal to \( S \) at \( X \). The following definitions allow to characterize a sampling of a surface. See [27] and [4] for basic results on properties of the medial axis.

**Definition 2.10. (Amenta & Bern)**

1. **(Medial axis)** The medial axis of a surface \( S \) in \( \mathbb{R}^d \) is the closure of the set of points with more than one closest point on \( S \).
2. **(Local feature size)** The local feature size \( \text{lfs}(\{X\}) \) of a point \( X \in S \) is the least distance from \( X \) to the medial axis of \( S \).
3. **(\( \epsilon \)-sample)** \( \epsilon \) set of sample points \( A \) of \( S \) is said to be an \( \epsilon \)-sample of \( S \) if every point \( X \in S \) has a sample point at distance at most \( \epsilon \) \( \text{lfs}(X) \). In the sequel, we only consider \( \epsilon \)-samples for \( \epsilon < \frac{1}{2} \).

2.3.3 Properties of well sampled surfaces

In this section, we mainly recall some results of Amenta and Bern [4].

The first lemma states that the local feature size is Lipschitz.

**Lemma 2.11. [4, lemma 1]**
For any two points \( X, Y \in S \), \( \text{lfs}(X) \leq \text{lfs}(Y) + \|X - Y\| \).

It follows that, if \( A \) is a \( \epsilon \)-sample, the maximum distance between \( X \notin A \) and the closest sample point \( A_i \in A \) is \( \frac{1}{\epsilon} \text{lfs}(A_i) \).

Considering two close surface points, Amenta and Bern show that the angle between the line segment connecting the two points and the normal at the points is large, whereas the angle between two normals is small.

**Lemma 2.12. [4, lemma 2]**
For any two points \( X \) and \( Y \) on \( S \) with \( \|X - Y\| \leq \rho \text{lfs}(X) \), the smaller angle between the line segment \( [XY] \) and the normal to \( S \) at \( X \) is at least \( \frac{\pi}{2} - \arcsin\left(\frac{\rho}{2\sqrt{\text{lfs}(X)}}\right) \).

**Lemma 2.13. [4, lemma 3]**
For any two points \( X \) and \( Y \) on \( S \) with \( \|X - Y\| \leq \rho \text{lfs}(X) \), \( \rho \leq \frac{1}{\sqrt{2}} \), the angle between the normals to \( S \) at \( X \) and at \( Y \) is at most \( \frac{\pi}{\sqrt{2} \text{lfs}(X)} \).

Since the Voronoi diagram of an \( \epsilon \)-sample consists of long and skinny cells, the normal direction at a point \( X \in S \) can be estimated from the Voronoi diagram of Vor(\( A \cup \{X\} \)). To be more specific, we recall the definition of a pole.

**Definition 2.14.** The pole \( p_X \) of \( X \) is the Voronoi vertex of Vor(\( A \cup \{X\} \)) which is furthest from \( X \).

As stated in the next lemma, the line passing through \( X \) and its pole provides a good approximation of the (non-oriented) direction of \( n_X \), the normal to \( S \) at \( X \). Consequently, the plane \( T_X \) that contains \( X \) and is orthogonal to \( n_X \) approximates well the tangent plane to \( S \) at \( X \).

**Lemma 2.15. [4, lemma 5]** The smaller angle between \( n_X \) and the line passing through \( X \) and its pole is at most \( 2 \arcsin\left(\frac{1}{\sqrt{2} \text{lfs}(X)}\right) \).

In the sequel, we will need the following lemma that states that the tangent plane of a point \( X \in S \) cannot be parallel to the bisector of two sample points that are at distance at most \( \rho \text{lfs}(X) \) from \( X \), for \( \rho \leq \frac{1}{\sqrt{2}} \).

**Lemma 2.16.** The angle between the tangent plane \( T_X \) to \( S \) at \( X \in S \) and the bisector of two sample points \( A_i \) and \( A_j \) that are at distance at most \( \rho \text{lfs}(X) \) from \( X \), \( \rho \leq \frac{1}{\sqrt{2}} \), is at least \( \frac{\pi}{2} - \arcsin\left(\frac{1}{2 \sqrt{\text{lfs}(X)}}\right) \).

**Proof.** See figure 3 for notations. We derive a lower bound for the angle \( \alpha \) between the vector \( A_iA_j \) and the surface normal \( n_i \) of \( A_i \) and an upper bound for the angle \( \beta \) between the normals at \( A_i \) and \( X \). The angle between \( n_i \) and \( A_iA_j \) is at least \( \alpha - \beta \).

In order to bound \( \alpha \), we apply lemma 2.12. The distance between the sample points is bounded by \( \|A_i - A_j\| \leq \|A_i - X\| + \|A_j - X\| \leq 2\rho \text{lfs}(X) \). Using lemma 2.11, we get \( \text{lfs}(X) \leq \text{lfs}(A_i) + \rho \text{lfs}(X) \) and \( \text{lfs}(X) \leq \frac{2\rho}{\sqrt{2} \text{lfs}(A_i)} \). Hence, \( \|A_i - A_j\| \leq \frac{2\rho}{\sqrt{2} \text{lfs}(A_i)} \). It follows from Lemma 2.13 that \( \alpha \) is at least \( \frac{\pi}{2} - \arcsin\left(\frac{\rho}{\sqrt{2} \text{lfs}(X)}\right) \). The angle \( \beta \) between the normals at \( A_i \) and \( X \) is at most \( \frac{\pi}{\sqrt{2} \text{lfs}(A_i)} \) if \( \rho < \frac{1}{\sqrt{2}} \) (lemma 2.13). Concluding, we get \( \alpha - \beta \geq \frac{\pi}{2} - \arcsin\left(\frac{\rho}{\sqrt{2} \text{lfs}(X)}\right) - \frac{\rho}{\sqrt{2} \text{lfs}(A_i)} \), provided that \( \rho < \frac{1}{\sqrt{2}} \).

Figure 3: Bounding the angle between the normal \( n_2 \) and the segment \([A_i, A_j]\).
3. SURFACE NEIGHBORS

With this section begins the core part of the paper in which we define a local neighborhood of a point \( X \) of a smooth surface \( S \) with respect to an \( \epsilon \)-sample \( A \) of \( S \). In order not to compute geodesic Voronoi diagrams on \( S \), we approximate \( S \) locally by the tangent plane \( T_X \) of \( S \) at \( X \). We determine the natural neighbors of \( X \) in the Voronoi diagram \( \text{Vor}(A) \) restricted to \( T_X \), and we call them \( T \)-neighbors of \( X \). In the remainder of this section, we formally define the \( T \)-neighbors of a point \( X \in S \) and prove that they are close to \( X \).

3.1 Definition of \( T \)-neighbors

In a first time, we assume that for each point \( X \in S \), the normal \( n_x \) to \( S \) at \( X \) is known, and therefore, the tangent plane \( T_X \) at \( X \).

Let \( \text{Vor}'(A) \) be the intersection of \( \text{Vor}(A) \) with the tangent plane \( T_X \). The Delaunay triangulation restricted to \( T_X \) that consists of the faces of \( \text{Del}(A) \) whose dual Voronoi edges intersect \( T_X \) is called \( \text{Del}'(A) \). Alternatively, \( \text{Vor}'(A) \) is the \((d-1)\)-dimensional power diagram of the points \( A_i' \) that are the projection of the sample points \( A_i \in A \) onto \( T_X \) weighted with \( w_i = -||A_i - A_i'||^2 \). Let \( \text{Reg}'(A) \) be the regular triangulation dual to \( \text{Vor}'(A) \). Since two cells of \( \text{Vor}'(A) \) are adjacent iff their corresponding cells in \( \text{Vor}(A) \) are adjacent and intersect \( T_X \), \( \text{Reg}'(A) \) is the projection of \( \text{Del}(A) \) onto \( T_X \).

**Definition 3.1. \( (T \)-neighbor of \( X \))** The \( T \)-neighbors associated to \( A \) of a point \( X \in S \) are the sample points \( A_i \in A \) such that their projection \( A_i' \) is a normal neighbor of \( X \) in \( \text{Vor}'(A) \).

Attention must be paid to two details which ensure that these concepts are well-defined: First, the definitions of section 2.2 assume general position of the point sites. The case that two sites have the same position and the same weight is excluded. In our context, this occurs if the biseector of two \( T \)-neighbors \( A_k \) and \( A_j \) of \( X \) coincides with \( T_X \) : \( A_k \) and \( A_j \) are projected at the same position, and they have the same weight because they are at the same distance to the tangent plane but on opposite sides. However, without real loss of generality, we can assume that no biseector of two sample points is tangent to \( S \).

Notice, still, that we can easily show that the angle between \( T_X \) and the biseector of \( A_k \) and \( A_j \) is strictly positive, if \( \epsilon \leq \frac{1}{2} \). In the sequel of this section, we show that \( A_k \) and \( A_j \) are at distance at most \( \frac{2}{\cos(\alpha)} \text{Ifs}(X) \) to \( X \) (lemma 3.3), thus, lemma 2.16 applies with \( \rho = \frac{2}{\cos(\alpha)} \).

Second, we need the following lemma to show that \( X \) lies in the convex hull of its natural neighbors in \( \text{Vor}'(A) \). It is equivalent to show that \( V'(X) \) is bounded.

**Lemma 3.2.** \( X \) belongs to the convex hull of the projection of its \( T \)-neighbors on \( T_X \).

**Proof.** For a contradiction, assume that \( V'(X) \) is unbounded. Then \( V'(X) \) contains a point at infinity \( p_{\text{inf}} \). Since \( X \) is, among the points of \( A \cup \{X\} \), the closest to \( p_{\text{inf}} \), the halfspace \( H^+ \) limited by the hyperplane \( H \) passing through \( X \) and normal to \( X p_{\text{inf}} \) and containing \( T_X \) does not contain any point of \( A \).

If \( S \) is a compact surface without boundary, its medial axis must intersect \( H^+ \). Indeed, the maximal ball passing through \( X \) and lying in the region limited by \( S \) is centered on \( H^+ \). This center is a point of the medial axis of \( S \). Let \( M \) be the portion of the medial axis that is contained in \( H^+ \). Because the medial axis must intersect \( H^+ \), there exists a point of \( S \cap H^+ \) which is closer to \( M \) than to any point of \( A \). This contradicts the fact that \( A \) is an \( \epsilon \)-sample with \( \epsilon < 1 \).

In this section, we show that \( X \) is not known (this may be the case when the surface is only known at a finite set of points), we can approximate the tangent plane \( T_X \) by the plane \( T_X \) that passes through \( X \) and is orthogonal to the vector joining \( X \) to its pole. We can define \( T \)-neighbors in very much the same way as \( T \)-neighbors : the only difference is that \( T_X \) is replaced by \( T_X \).

3.2 Locality of the \( T \)-neighbors

We now derive a bound on the distance between a point \( X \in S \) and its \( T \)-neighbors with respect to the local feature size \( \text{Ifs}(X) \).

**Lemma 3.3.** Let \( A \) be an \( \epsilon \)-sample of \( S \) and \( X \in S \).

(a) The \( T \)-neighbors of \( X \) are all contained in a ball of radius \( \frac{1}{2} \text{Ifs}(X) \) centered at \( X \).

(b) The \( T \)-neighbors of \( X \) are contained in a ball of radius \( \frac{2}{\cos(3\text{arsin}(\frac{\epsilon}{2\text{Ifs}(X)}) \text{Ifs}(X))} \) centered at \( X \).

(c) In both cases, the \( T \)-neighbors of \( X \) are contained in a ball of radius \( 2\epsilon(1+O(\epsilon)) \text{Ifs}(X) \).

**Proof.** Let \( v \) be a vertex of \( V'(X) = V^+(X) \cap T_X \). We derive an upper bound on the distance between \( X \) and \( v \). Because \( V'(X) \) is bounded by the bisectors of \( X \) and its \( T \)-neighbors, the distance between \( X \) and its \( T \)-neighbors is at most twice the distance between \( X \) and \( v \).

Let \( B_1 \) and \( B_2 \) be the two balls of radius \( \text{Ifs}(X) \) that are tangent to \( S \) at \( X \). Assume without loss of generality that \( B(m_1) \) and \( v \) are on opposite sides of \( S \). Let \( \alpha \) be the angle \( \angle v m_1 X \) where \( m_1 \) is the center of \( B_1 \). We find the same angle \( \alpha = \angle v q X q' \) where \( q' \) is the orthogonal projection of \( X \) onto the line segment \( v m_1 \). See figure 5.
Since \( v \) and \( m_1 \) lie on different sides of \( S \), the line segment \( v m_1 \) must intersect \( S \). Let \( q \) be such an intersection point. We call \( B(v) \) the ball with center \( v \) and radius \( \| X - v \| \). Since \( v \in V^+(X) \), \( B(v) \) is empty of sample points. Because \( B_1 \) is also empty of sample points, \( X \) is the point of \( A \cap \{ X \} \) that is closest to \( q \). It follows that 
\[
\| X - q \| \leq \frac{\sqrt{d}}{3 \| x \|} \text{lfs}(X).
\]
On the other hand, \( \| X - q \| \geq \| X - q' \| = \sin \alpha \text{lfs}(X) \). Hence, \( \alpha \leq \frac{\pi}{4} \). From the triangle \( v X q' \), we know that 
\[
\| X - v \| = \frac{\| X - q \|}{\cos \gamma} \leq \frac{\| X - q \|}{\cos \alpha} = \frac{\| X - q \|}{\sqrt{\frac{3}{4} \| X \|}} \text{lfs}(X) \approx \frac{\sqrt{d}}{3 \| x \|} \text{lfs}(X).
\]

![Figure 5: For the proof of lemma 3.3(a)](image)

(b) We consider a vertex \( v \in \tilde{V}(X) = V^+(X) \cap \tilde{T}X \). Let \( \beta \) be the angle between \( T(X) \) and \( \tilde{T}X \). From lemma 2.15, we know that \( \beta \leq 2 \arccos \left( \frac{\| v \|}{\| X \|} \right) \). See figure 6 for notations. We define \( q \) as before, and we obtain the same bounds 
\[
\| X - q \| \leq \frac{\sqrt{d}}{3 \| x \|} \text{lfs}(X)
\]
and \( \alpha \leq \arcsin \left( \frac{\| v \|}{\| X \|} \right) \).

Let \( \gamma = \angle m_1 v X = \frac{\| v \|}{\| X \|} - \alpha - \beta \geq \frac{\| v \|}{\| X \|} - 3 \arcsin \left( \frac{\| v \|}{\| X \|} \right) \) and \( \sin \gamma \geq \cos \left( 3 \arcsin \left( \frac{\| v \|}{\| X \|} \right) \right) \).

From triangle \( v X q' \), we know that 
\[
\| X - v \| = \frac{\| q' - q \|}{\sin \gamma} \leq \frac{\epsilon}{\sin \gamma} \text{lfs}(X) \approx \epsilon \text{lfs}(X) \text{.}
\]

![Figure 6: For the proof of lemma 3.3(b)](image)

3.3 Local surface patches

The following lemma states that \( X \) and all its \( T \)-neighbors are contained in a small topological ball, if \( \epsilon < 0.3 \). It is a direct consequence of lemma 3.3.

**Lemma 3.4.** Let \( B(X, r) \) be a ball centered at \( X \) with radius 
\[
r = \frac{\sqrt{d}}{3 \| x \|} \text{lfs}(X) \text{.}
\]
If \( \epsilon < \frac{\sqrt{d}}{3 \| x \|} - 1 \), \( B(X, r) \cap S \) is a topological \((d - 1)\)-ball that contains all the \( T \)-neighbors of \( X \).

**Proof.** By lemma 3.3, all the \( T \)-neighbors of \( X \) are contained in a ball \( B(X, r) \) centered at \( X \) of radius 
\[
r = \frac{\sqrt{d}}{3 \| x \|} \text{lfs}(X) \text{.}
\]
With \([7]\), proposition 14, \( S \cap B(X, R) \) is a topological \((d - 1)\)-ball if 
\[
r < \text{lfs}(X) \text{, which is true for } \epsilon < \frac{\sqrt{d}}{3 \| x \|} - 1 \approx 0.3 \text{.}
\]

4. SURFACE COORDINATE SYSTEM

We can now define a surface coordinate system associated to a given sample set \( A \). Similarly to the \( T \)-neighbors, it is defined in the tangent plane of each surface point and consequently called \( T \)-coordinate system. In the rest of the section, we show some properties of \( T \)-coordinate systems: we show that the support of the \( T \)-coordinates is local in subsection 4.2, study their continuity properties in subsection 4.3, and prove that the local coordinate property is approximately satisfied, with an error that depends on the local curvature of the surface and on the sampling density in subsection 4.4. The results for \( T \)-coordinates, the generalization of \( T \)-coordinates to estimated tangent planes, are not explicitly given because, with lemma 3.3(b), they can be easily deduced. Furthermore, lemma 3.3(c) shows that the asymptotic results are the same.

4.1 Definition and basic properties

The same definitions as in section 3 apply.

**Definition 4.1.** \((T \text{-coordinate associated to } A)\text{ The } T \text{-coordinate } \tau_i(X) \text{ of a point } X \text{ of } S \text{ is the natural neighbor coordinate } \lambda_i(X) \text{ of } X \text{ associated to } A_i \text{ in the power diagram } V \text{or}(A), \text{ } i = 1, \ldots, n \text{.}

By construction, the \( T \)-coordinates \( \tau_i \) fulfill properties (ii) and (iii) of a system of coordinates over \( S \) associated to \( A \) as they are listed in definition 1.1. The local coordinate property (i) is satisfied for the projected sample points \( A_i \). With respect to \( A \), the local coordinate property is only true if the surface is locally planar so that all \( T \)-neighbors of \( X \) lie in the tangent plane \( T_X \).

4.2 Locally bounded support

Let \( \Delta \) denote the support of \( \tau \), i.e. the subset of the points \( X \in S \) such that \( \tau(X) \neq 0 \). In order to show the locality of \( \Delta \) on \( S \), we apply the bound on the distance between a point \( X \in S \) and its \( T \)-neighbors.

**Lemma 4.2.** The support \( \Delta \) of \( \tau \) is contained in a ball of radius 
\[
\frac{2}{3 \| x \|} \text{lfs}(A_i) \text{ centered at } A_i \text{.}
\]

**Proof.** Applying lemma 3.3 and lemma 2.11, we obtain 
\[
\| X - A_i \| \leq \delta \text{lfs}(X) \leq \frac{2}{3 \| x \|} \text{lfs}(A_i) \text{ with } \delta = \frac{2}{\sqrt{d - 1}} \text{, which is at most } \frac{2}{3 \| x \|} \text{.}
\]

4.3 Continuity of the coordinate function

In this section, we study the continuity of the function \( \tau_i \) when \( X \) moves on \( S \). Let us first state the lemma:

**Lemma 4.3.** The \( T \)-coordinate \( \tau_i, i = 1, \ldots, n \), is continuous everywhere on \( S \).

**Proof.** Assume that \( S \) is parameterized by \((u, v)\), so that \( X(u, v) = (x(u, v), y(u, v), z(u, v)) \). The coordinate function \( \tau_i(X) \) is, by definition, equal to the natural neighbor coordinate \( \lambda_i(X(u, v)) \) in the power diagram \( V \text{or}(A \cup X) \). When \( X \) moves on \( S \), the projected sample points change their position and their weight continuously. Indeed, since \( S \) is smooth, the projection onto the tangent
plane is a continuous mapping. The weight of $A_i$ is described by the following function:

$$w_i(u, v) = -||A_i - A_i'(u, v)||^2 = - (A_i - A_i'(u, v) \cdot \mathbf{n}_{X(u, v)})^2.$$ 

By assumption, $X(u, v)$ is twice differentiable. The normal $\mathbf{n}_{X(u, v)}$ is $C^1$ continuous.

The same is true for the position of $A_i'(u, v)$ since $A_i'(u, v) = A_i - ||A_i - A_i'(u, v)|| \mathbf{n}_{X(u, v)}$. Recall also from lemma 3.2, that $X$ is always in the convex hull of the projected sample points. Consequently, the $T$-coordinates are continuous on all of $\mathcal{S}$ because the natural neighbor coordinates are continuous as described in section 2.2. □

**Lemma 4.4.** The $T$-coordinate $\tau_i$, $i = 1, ..., n$, is continuously differentiable everywhere on $\mathcal{S}$ except at the sample points and at the points $X \in \mathcal{S}$ such that $T_X$ contains a $(d-2)$-dimensional Voronoi face of the Voronoi cell $V(X)^+$ in $\text{Vor}(\mathcal{A} \cup \{X\})$.

**Proof.** Consider the power diagram $\text{Vor}^+(\mathcal{A} \cup \{X\})$. By lemma 2.5, $\tau_i$ is $C^1$ continuous except at a finite number of points. Still, we want to characterize those points with respect to $\text{Vor}^+(\mathcal{A})$.

For convenience, we adopt the terminology associated to $\mathbb{R}^d$ and call a $(d-2)$-dimensional face an edge. The natural neighbor coordinate $\tau_i$ is not continuously differentiable at a point $X$ such that an edge of $V^+(X)$ has the same supporting line as an edge of $V^+(\mathcal{A})$ say $V^+(A_i') \cap V^+(A_j')$ (see also the proof of lemma 9.1 in the appendix). This means that a point $\mathbf{v}$ of this edge has equal power with respect to $X$, $A_i'$, and $A_j'$. But, by definition of $\text{Vor}^+(\mathcal{A} \cup \{X\})$, this means also that $\mathbf{v}$ is at equal distance from $A_i$, $A_j$, and $X$. Consequently, the Voronoi edge of $V^+(X)$ is an edge of $\text{Vor}(\mathcal{A} \cup \{X\})$. This edge is therefore contained in $T_X$. □

### 4.4 The local coordinate property

In this section, we bound the error committed with respect to the local coordinate property, that is $e(X) = X - \sum_i \tau_i(X)A_i$. This corresponds to bounding the distance of a $T$-neighbor of a point $X \in \mathcal{S}$ to the tangent plane $T_X$, since $X = \sum_i \tau_i(X)A_i'$, $\sum_i \tau_i(X)(A_i' + ||A_i - A_i'|| \mathbf{n}_{X})$ or $X = \sum_i \tau_i(X)(A_i - ||A_i - A_i'|| \mathbf{n}_{X})$ depending on the sign of $A_i'A_i, \mathbf{n}_{X}$.

**Proposition 4.5.**

$$X = \sum_i \tau_i(X)A_i + O(e^2)\text{Ilsf}(X) \mathbf{n}_{X}.$$ 

**Proof.** It is enough to prove that the distance between a $T$-neighbor $A_i \in \mathcal{A}$ of $X \in \mathcal{S}$ and its projection $A_i'$ on the tangent plane $T_X$, is bounded by

$$||A_i - A_i'|| \leq \frac{2e^2}{1 - 2e} \text{Ilsf}(X) = O(e^2) \text{Ilsf}(X).$$

It is easy to see that $||A_i - A_i'|| = ||A_i - X|| \sin \theta$ where $\theta = \angle A_i X A_i'$. Since $A_i$ does not belong to the two balls of radius $\text{Ilsf}(X)$ centered at $X$, $\sin \theta \leq \frac{1}{2(1+2e)}$. With lemma 3.3, we have $||A_i - X|| \leq \frac{2e^2}{1 - 2e} \text{Ilsf}(X)$ which implies the inequality above and the proposition. □

1 We denote the scalar product of two vectors by $\mathbf{a} \cdot \mathbf{b}$.

2 $\mathbf{n}_{X(u, v)} = \mathbf{X}(u, v) \times \mathbf{X}_u(u, v)$ where $\mathbf{X}_u = \frac{\partial \mathbf{X}}{\partial u}$ ($\mathbf{X}_v = \frac{\partial \mathbf{X}}{\partial v}$) is the first derivative of $X(u, v)$ with respect to $u$ (with respect to $v$).

## 5. APPLICATION: INTERPOLATION

As mentioned at the beginning of the paper, in this section, we use the $T$-coordinates in the context of interpolation. In particular, as described in the introduction, section 1.1.2, we want to interpolate a function which is defined on a surface but only known at some points. Also, we deform a polyhedral surface at some of its vertices in the direction of their normal, and we interpolate the displacement of the other vertices.

### 5.1 Interpolating a function on $S$

In this section, the previously defined $T$-coordinate system is used to define an interpolant for functions defined on smooth surfaces. More exactly, we want to approximate $\Phi : \mathcal{S} \to \mathbb{R}$ where $\mathcal{S} \subset \mathbb{R}^d$ is defined as before. We assume that an $\epsilon$-sample of $\mathcal{A}$ with function values $(A_i, z_i) : A_i \in \mathcal{A}, z_i = \Phi(A_i)$ is given. The interpolation is carried out for a point $X \in \mathcal{S}$.

#### 5.1.1 Linear precision

For a point $X \in \mathcal{S}$, we compute its $T$-coordinates within its tangent plane $T_X$. The interpolation of $\Phi(X)$ is given as the linear combination of the function values of the $T$-neighbors weighted by the coordinates:

$$I_0^0(X) = \sum_i \tau_i(X) z_i.$$ 

Assume that $\mathcal{S}$ is locally flat and identical to a plane $H$. At any point $X$ of $\mathcal{S}$ where the $T$-neighbors of $X$ are all in $H$, the interpolant has linear precision. Indeed, if $z_i = a + bA_i$ for all $T$-neighbors of $X$, we have

$$I_0^0(X) = \sum_i \tau_i(X)(a + bA_i) = a + bx$$

by the local coordinate property. If $\mathcal{S}$ is not flat around $X$, we saw in section 4.4 how the error can be bounded with respect to $\text{Ilsf}(X)$.

Let $\Phi(A_i) = a + bA_i$ be a linear function. We have $I_0^0(X) = \sum_i \tau_i(X)(a + bA_i) = a + bx + b \epsilon(X)\mathbf{n}_X(\mathbf{e}, \mathbf{X})$ where $e(X) = \sum_i \tau_i(X)(A_iA_i' \mathbf{n}_X^2) = O(\epsilon^2)\text{Ilsf}(X)$. Hence,

$$I_0^0(X) = \Phi(X) + O(\epsilon^2).$$

#### 5.1.2 Quadratic precision

We can also define an interpolant that reproduces a quadratic function $\Phi(X) = a + b'X + X^2QX$. As additional input, we assume the gradients $g_i = \nabla \Phi(A_i))$ of $\Phi$ at the sample points $A_i$, $i = 1, ..., n$, to be known. The interpolant is defined as follows:

$$I_1^1(X) = \sum_i \tau_i(X)(z_i + \frac{1}{2}g_i(X - A_i)).$$

Assume again that $\mathcal{S}$ is locally flat and identical to a plane $H$. At any point $X$ of $\mathcal{S}$ where the $T$-neighbors of $X$ are all in $H$, the interpolant has quadratic precision. Indeed, if $z_i = \Phi(A_i) = a + b'A_i + A_i^2QX$, $g_i = b + 2QX$, and applying the local coordinate property, we obtain:

$$I_1^1(X) = a + b'X + X^2QX.$$ 

If $\mathcal{S}$ is not flat around $X$, we have:

$$I_1^1(X) = a + b'X + (X + e(X)\mathbf{n}_X)^TQX = \Phi(X) + O(\epsilon^2).$$

where $e(X)$ is defined as in the previous case. Thus, the error of the interpolant is $||\Phi(X) - I_1^1(X)|| = e(X)\mathbf{n}_X^2QX$. Notice, that
when interpolating a linear function, the error committed by \( I^1 \) is half the error committed by \( I^0 \).

In this context, we also implemented the so-called \( Z^1 \) interpolant proposed by Sibson [26] which we adapted to our setting by replacing the natural neighbor coordinates by the \( T \)-coordinates. Sibson’s original version recaptures only spherical quadrics due to the fact that the gradients are estimated from the data. Otherwise, it has quadratic precision and is \( C^1 \) continuous. Its correctness relies on the local coordinate property of the natural neighbor coordinates. Consequently, in our case, a small error is introduced again depending on the local feature size.

### 5.1.3 Experimental results

We interpolated the following three functions on the sphere:

- \( f_1(x, y, z) = 4x^2 + y^4 + 6z^4 \),
- \( f_2(x, y, z) = e^{-2(x-1)^2+y^2+z^2} + 0.5e^{-4(x^2+(y-0.7)^2+(z-0.7)^2)} - 0.25e^{-4(x^2+(y+0.7)^2+(z+0.7)^2)} \),
- \( f_3(x, y, z) = 1 + x^8 + 2y^3 + e^{y^2} + 10xyz \).

To visualize the result, we deformed the sphere at each point along its normal by the amount of the corresponding interpolation result. See figure 7, 8 and 9. The leftmost pictures show the exact result, i.e. the function is evaluated at the 4000 points of the sphere model and the sphere is deformed correspondingly. The right pictures show the result of the interpolation on the 4000 sphere points given function values at 50 and 250 of the 4000 points. The functions \( f_2 \) and \( f_3 \) are test functions from [20] and [3]. In table 1, we show the corresponding error statistics. For each interpolant, mean and max errors are given with respect to the absolute difference between the actual and the interpolated function value on the 4000 evaluation points.

### 5.2 Surface deformation

This section describes experimental results from the application of the linear interpolant defined in the last section. In order to demonstrate the locality and the smoothness of the \( T \)-coordinate, we interpolate a deformation that is applied to some points of a polyhedral surface. Let \( \mathcal{S} \) be the polyhedral surface and \( \mathcal{A} \) its vertex set. On each vertex \( A_i \in \mathcal{A} \), we approximate the normal to \( \mathcal{S} \) by the average of the normals of the facets adjacent to \( A_i \) in \( \mathcal{S} \).

The setting is the following: we consider a small subset \( \mathcal{A}' \subset \mathcal{A} \) of the vertices of the polyhedral surface, and we assign some real value \( z_i \) to each sample point \( A_i \in \mathcal{A}' \). The value \( z_i \) corresponds to a deformation of \( A_i \) along its normal. Then we calculate the interpolant \( I^0(A_j) \) for all other points \( A_j \in \mathcal{A} \setminus \mathcal{A}' \) using the \( T \)-coordinates associated to \( \mathcal{A}' \). The resulting polyhedral surface is obtained by replacing each vertex with its deformed image. The combinatorial structure of \( \mathcal{S} \) remains the same. This is exactly the experimental setting of Brown [11] generalized to point sets from general surfaces.

Figure 11 shows the result of a deformation applied to the bunny. Three points are displaced along their normals, while the other sample points in \( \mathcal{A}' \) remain unchanged. In the bottom images of figure 11, these sample points are depicted on the surface. One can see that the deformation stays local around the displaced points.

### 5.3 Implementation

The implementation is based on the Computational Geometry Algorithms Library CGAL [12]. It makes use of the two-dimensional regular triangulation, the three-dimensional Delaunay triangulation as well as the polyhedral surface class provided by CGAL. The running times for computing the coordinates for a set of points is about three times the running time for computing its 3D Delaunay triangulation. Further details on the implementation can be found in a technical report [8] that will appear soon.

---

**Table 1: The error of the function interpolation on the sphere**

<table>
<thead>
<tr>
<th>pts</th>
<th>( m )</th>
<th>mean ( I^0 )</th>
<th>max ( I^0 )</th>
<th>mean ( I^1 )</th>
<th>max ( I^1 )</th>
<th>mean ( Z^0 )</th>
<th>max ( Z^0 )</th>
<th>mean ( Z^1 )</th>
<th>max ( Z^1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>( f_1 )</td>
<td>0.313</td>
<td>1.58</td>
<td>0.328</td>
<td>1.57</td>
<td>0.262</td>
<td>1.30</td>
<td>0.47</td>
<td>1.04</td>
</tr>
<tr>
<td>250</td>
<td>( f_1 )</td>
<td>0.075</td>
<td>0.64</td>
<td>0.067</td>
<td>0.48</td>
<td>0.056</td>
<td>0.47</td>
<td>0.04</td>
<td>0.05</td>
</tr>
<tr>
<td>50</td>
<td>( f_2 )</td>
<td>0.037</td>
<td>0.29</td>
<td>0.023</td>
<td>0.15</td>
<td>0.019</td>
<td>0.13</td>
<td></td>
<td></td>
</tr>
<tr>
<td>250</td>
<td>( f_2 )</td>
<td>0.008</td>
<td>0.06</td>
<td>0.002</td>
<td>0.02</td>
<td>0.002</td>
<td>0.019</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>( f_3 )</td>
<td>0.526</td>
<td>3.13</td>
<td>0.698</td>
<td>3.39</td>
<td>0.614</td>
<td>3.31</td>
<td></td>
<td></td>
</tr>
<tr>
<td>250</td>
<td>( f_3 )</td>
<td>0.126</td>
<td>1.17</td>
<td>0.137</td>
<td>1.44</td>
<td>0.116</td>
<td>1.41</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 7:** \( f_1 \) (a) evaluation, (b) 50, (c) 250 sample points

**Figure 8:** \( f_2 \) (a) evaluation, (b) 50, (c) 250 sample points

**Figure 9:** \( f_3 \) (a) evaluation, (b) 50, (c) 250 sample points
6. CONCLUSION AND FUTURE WORK

The \( T \)-neighborhood and the \( T \)-coordinate system are associated to a set of sample points from a surface equipped with normal vectors. We show that the neighbors as well as the coordinate system are local on the surface, if the surface is well sampled. The coordinate system is continuous everywhere on the surface. It is continuously differentiable except at a finite number of surface points. Furthermore, we show the application of the coordinate system to different problems related to interpolation. In the near future, we plan several extensions of our method: In order to simplify the presentation in this framework, we restricted ourselves to \((d-1)\)-manifolds, yet, the same definitions apply to all smooth \( k \)-manifolds, \( k < d \), of \( \mathbb{R}^d \). For the same reason, we did not generalize the definitions to manifolds with boundary. Another straightforward extension allows to enlarge the definition domain of the coordinate system beyond the surface to all of \( \mathbb{R}^d \) except the medial axis of the surface. In addition, we will concentrate on the question of efficiency: due to the proved locality of the \( T \)-neighbors, efficient filtering methods will considerably speed up computation time. See [14] for comparable work on this question. Last but not least, we plan to develop further applications e.g. around texture mapping, surface reconstruction, and point set surfaces as they are recently emerging in computer graphics, see for example [1]. Also, the interpolation schemes can easily be extended to vector field interpolation, thus, to interpolation of functions that are defined from the surface to a higher dimensional space.

7. ACKNOWLEDGMENTS

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8. REFERENCES


boundary direction. In [24], Piper shows that the natural neighbor coordinate function is differentiable when it is restricted to a straight line emanating from a data site, except at the data sites themselves. In Voronoi diagrams this is sufficient to show the $C^1$ continuity on the Delaunay circles because a line emanating from a data site can never be tangent to one of its Delaunay balls — it is always a legal cross boundary direction. In power diagrams this is not true in general, and we need to show another $C^1$ continuous cross boundary direction. Other than that, Piper’s proof extends easily to natural neighbor coordinates.

Figure 12: The volume of $V(B_i, X)$ changes $C^1$ continuously as $X$ moves on $C_i$

Instead, we choose $X$ to move on a sphere $C_i(r)$ centered on $p_i$ with radius $r$. Except on the two points of each power sphere that maximize and minimize the distance to $p_i$, there exists a radius $r$ such that $C_i(r)$ traverses the power sphere. For the other points, Piper’s proof applies. The radical axis of $X$ and $B_i$ is tangent to the sphere of radius $\frac{1}{2}r + \frac{w_i}{2}w_X$ centered on $p_i$ — we call this sphere $C_i(r)$. Let $H_X$ be the the radial hyperplane of $X$ and $B_i$, or in other words, the hyperplane tangent to $C_i(r)$ at the intersection point $v$ of $C_i(r)$ with the line through $p_X$ and $p_i$. Let $H_X^{+}$ be the open halfspace defined by $H_X$ which does not contain $p_i$. If $X$ is in the support of $\lambda_i$, $H_X$ intersects $V(B_i)$. The volume $w_i$ is exactly $w_i(X) = V(B_i) \cap H_X$. Using the arguments of Farin [17], it is now easy to see that $w_i$ changes in a quadratic fashion as $X$ moves along $C_i(r)$, thus, it is continuously differentiable. An exception to this behavior occurs if $H_X$ becomes one of the faces of $V(B_i)$. The function is still continuous but no longer differentiable. This is the case if $X$ lies on the line connecting $p_i$ to one of its neighbors $p_j$ and if the power $\Gamma_j(p) = \frac{1}{2}r + \frac{w_j}{2}w_X - w_i$. Notice, that this particularly the case if $X = B_j$. See figure 12 for an illustration.

This is sufficient to show the differentiability of $\lambda_i$ at all points of $C$ except at one point on each line connecting $p_i$ to on of its neighbors on $\text{Reg}(B)$. Notice, that if the weight of $X$, of $B_i$ and of its neighbor $B_j$ is zero, the $C^1$ discontinuity occurs exactly if $X = p_j$.

In order to proof the $C^0$ continuity, the same argument used by Farin [17] for natural coordinates in Voronoi diagrams applies: the area function $w_i(X) = \mu_d(P(X, B_i))$ is continuous everywhere except at $B_i$. However, the normalized coordinate function $\lambda_i$ is continuous because $\lim_{X \to B_i} \lambda_i(X) = \lim_{X \to B_i} w_i(X) = 0$, as follows from the definition of $w_j$, for all $j \neq i$. Since $\sum_{i=1}^{n} \lambda_i(X) = 1$, we deduce $\lim_{X \to B_i} \lambda_i(X) = 1$, which shows that $\lambda_i$ is continuous everywhere in $C$.