Fuzzy Time-Optimal Controller (FTOC) for second order nonlinear Systems

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Abstract: The motivation behind this paper is to seek alternative techniques to achieve a near optimal controller for non-linear systems without solving the analytical problem. In classical optimal control systems, the system states and optimization co-state parameters generate a two-point boundary value problem (TPBVP) using Pontraygin’s minimum principle (PMP). The paper presents a simple and new fuzzy time-optimal controller which has two regular inputs and one Bang-bang output. The proposed controller closely approximates the output of the classical time-optimal controller. Further, input membership function are tuned on-line to improves the time-optimal output. The new controller exhibits optimal behaviour for second order non-linear systems. The rules are selected to satisfy the stability and optimality conditions of the new fuzzy time-optimal controller. The paper describes a systematic procedure to design the controller and how to achieve the desired result. To benchmark the new controller performance, sliding mode controller is used for guidance and comparison purpose. Simulation of three non-linear examples shows promising results. The work described here is expected to incite researcher’s interest in fuzzy time-optimal controller design.

Keyword: Time-optimal control, Bang-Bang control, fuzzy logic control, state trajectories

1. Introduction

Time-optimal controller (TOC) has constrained output and is expected to generate an optimal control signal, which can drive the system from an initial to a final state in minimum time. The Pontryagin’s Minimum Principle (PMP) has been extensively used to design time-optimal control [1-3]. The time-optimal control is formulated as a TPBVP using PMP and solved to obtain the control sequence by numerical approaches [4,5]. The necessary condition of PMP
for time-optimal problem is to achieve the goal in minimum time under bounded control which is formulated as Bang-bang control.

Mathematical model-based fuzzy optimal controller has been implemented in the past where fuzzy rules were used for the construction of an optimal control function. Margaliot and Langholz [6] re-casted the state space model into hyperbolic function and solve LQR Riccati equation for optimal control and cost. Li-Xin Wang [7], Huang Z. and Chen S. [8] proposed controllers for MIMO and SISO systems. Time-optimal fuzzy control was presented by Pao-Tsun [9] using Takagi-Sugeno (TSK) fuzzy model and determines the solution of a control sequence using T-S fuzzy control rules. Tsong et.al [10] implemented the time-optimal Bang-bang control by estimating the distance of the present state from the optimal trajectory and using a fuzzy controller to control the states. Xiao et.al [11] proposed a combined but switching time-optimal (Bang-bang) and standard fuzzy controller for Automated Manual Transmission (AMT). Myung [12] proposed a fuzzy controller approach to determine optimal switching times to control a crane. In all the above applications, the mathematical model was required and the optimal controller was form by transforming the fuzzy rules to a nonlinear map of the optimal control function. The optimal control function is then cast into a classical optimal control problem with states and costate equations, leading to a TPBVP solution – which is solved off-line numerically for an optimal control sequence.

Numerical optimization and dynamic programming has also been successfully used for fuzzy optimal control problem. Optimization of system performance criteria requires tuning of the controller parameters. Performance criteria are optimized for system states, output response
time or inputs. The optimization can be constrained or unconstrained on the control signal or state variables. Tuning of fuzzy controllers is accomplished by off-line adjustment of the fuzzy parameters with input-output data. Tatsuya Hojo et.al [13] achieved fuzzy optimal control by digitizing the phase plane and using dynamic programming technique. Most of the above optimization techniques were off-line or uses fuzzy training-tuning of membership function with input–output data for system identification rather than for the fuzzy controller.

Stability analysis of closed-loop systems containing a fuzzy logic controller is difficult in view of non-linear controller. Lyaponov function based approach is the most common approach to investigate the stability of fuzzy control systems. Margaliot and Langholz [6] make use of quadratic stabilization which is guaranteed if and only if Riccati equation has positive definite solution corresponding to Lyaponov function. One of the main results in the field of stability analysis is the theorem of Tanaka and Sugeno [14]. However, in order to use this theorem a fuzzy model of the entire closed-loop system is required [15-17]. Tanaka-Sugeno (TSK) systems of linear equations linearized the nonlinear plant and convert them to fuzzy rule base. Finding a common Positive-definite matrix $P$ to satisfies large numbers linear state equations in T-S fuzzy system is not easy and this rigidity diminishes the advantage of fuzziness. Wong L.K. et.al [18] proposed to break down the entire system stability into individual sub-systems and Lyaponov function stability was checked for each bounded output subsystem.

In the absence of a mathematical formulation the fuzzy controllers lack the stability and performance trustworthiness. The success of the fuzzy controller stems from the fact that they
are similar to the robust sliding mode controller (SMC) [19]. In the case of fuzzy time-optimal controllers, fuzzy optimal rules result in partitioning of decision space (phase plane) into two semi-planes by means of a sliding (switching) line. Similarity between fuzzy Bang-bang controller and sliding mode controller (SMC) can be used to redefine the diagonal form of fuzzy time-optimal controller (FTOC). One of the earliest fuzzy Bang-bang controller (FBBC) was developed by Chiang and Jang [20]. The stability of the proposed Bang-bang controller [21, 22] can be defined in terms of SMC with boundary limits. The sliding mode controller provide reference trajectory for FTOC to achieve time-optimal output. The fuzzy input rules and implication remain same as in a conventional fuzzy controller. A major practical advantage of Bang-bang control is that it can be implemented with simple on–off action.

In this paper, fuzzy Bang-bang relay controller [23] which claims to be time optimal - is analyzed for stability and optimality conditions for non-linear systems. The stability of the controller is checked with Lyaponov direct method. The controllers’s optimality is considered on basis of selection of fuzzy optimal rules, which minimize the objective function, akin with Hamiltonian function. The new controller, called here as a ‘fuzzy time-optimal controller,’ is simple to formulate and optimal fuzzy rules selection is proven here to corresponds to PMP. Using the stability analysis [18] with the bounded control signal, if each individual subsystem corresponding to each rule of the FLC is stable in sense of Lyaponov (ISL) subject to a common Lyaponov function, the overall closed loop system with FLC can be shown to be also stable ISL [24]. In the proposed FTOC the output is bounded to two level (Bang-bang) only which makes the search of common Lyaponov function convenient. Consequently the
stability condition meets the controllability criteria [1] – a necessary and sufficient condition for optimality.

The new controller works well for both non-linear and linear control. The fuzzy time-optimal controller is based on the Mamdani model and uses largest of maxima (LOM) defuzzification technique. By arranging the output membership function in a certain way gives Bang-bang output from the fuzzy controller. The layout of this paper is to first present, in section II, the theoretical background of optimal, fuzzy, and sliding mode controllers. Then the fuzzy time-optimal controller is formulated in the context of non-linear systems. Section III describes the stability and optimality conditions of FTOC. In section IV and V the controllers, developed in earlier section are simulated and analyzed. In section VI steps are described to design the FTOC. Finally, section VII concludes the work.

2. CONTROLLERS DESIGN

Different controllers are designed here for comparison with the fuzzy time optimal controller. A linearized non-linear problem is considered first for development time-optimal controller, followed by two variants of fuzzy Bang-bang controllers (FBBC). Lastly, a sliding mode controller is described.

2.1 Time-optimal controller

Consider a second order non-linear position control system [18] as

\[ M_a \ddot{x}(t) + g(x(t), \dot{x}(t)) + l_x(t) = u(t) \]  

(1)

After linearization the states representations of the dynamic system Eq. (1) becomes
\[
\dot{x}(t) = A(x(t)) + B(x(t))u(t) \\
\text{where}
\]
\[
A(x(t)) = \begin{bmatrix}
\frac{1}{M_a} & -g(x_1, x_2) - l x_1 \\
0 & 1
\end{bmatrix},
B(x(t)) = \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

For time-optimal problem the final time \( t_f \) is free while the final state \( x_f \) and initial state \( x_0 \) are fixed. Optimal control (OC) of the system described by Eq. (2) can be achieved by using Bang-bang action. In optimal control, Pontrygin’s Minimum principle (PMP) is extensively used to achieve the minimum time with bounded control \( u = \{-1, +1\} \). The optimal control problem is setup to determine the piecewise continuous control \( u: [t_0, t_f] \rightarrow P^{m=2} \) and state trajectory \( x: [t_0, t_f] \rightarrow P^{n=2} \), that minimize the cost function

\[
J_{\min} = \int_{t_0}^{t_f} I dt
\]

Subject to

\text{Dynamics:} \quad \dot{x}(t) = Ax(t) + Bu(t), \quad t_0 \leq t \leq t_f

\text{initial condition:} \quad x(t_0) = x_0

\text{final condition:} \quad x(t_f) = 0

\text{control constraint:} \quad u(\cdot) \in U

\text{final-time constraint:} \quad t_f \in (t_0, \infty)

Describing the Hamiltonian function

\[
H[t, x(t), p(t), u(t)] = I + p'(t)[A\dot{x}(t) + Bu(t)]
\]

where \( p(t) \) is costate variable \([p_1(t), p_2(t)]\)
Let \( t_f^* \in (t_0, \infty) \) and \( u^*:[t_0, t_f^*] \) be the optimal values. And let \( x^*(.) \) be the corresponding trajectory. Then there exists a function \( p^*:[t_0, t_f^*] \rightarrow \mathbb{R}^n \), and, not identically zero, satisfying:

\[
\begin{align*}
\dot{p}(t) & = \frac{\partial H}{\partial x} = -A^T p(t) & \text{Adjoint equations} \\
\dot{x}(t) & = \frac{\partial H}{\partial p} = A x(t) + B u(t) & \text{State equations} \\
\frac{\partial H}{\partial u} & = 0 = p(t)B & \text{} \\
\end{align*}
\]

(5)

The minimum principle (PMP) required to minimize Eq. (4) with optimal control signal \( u^*(t) \)

\[
1 + A^T x^*(t) p^*(t) + u^*(t) B^T p^*(t) \leq 1 + A^T x^*(t) p^*(t) + u(t) B^T p^*(t)
\]

(6)

The preceding equations are solved in time \( t \in [t_0, t_f] \) for optimal control \( u^*(t) \) and corresponding state trajectories \( x^*(t) \) as

\[
u^*(t) = -\text{sgn}[B^T p^*(t)] M
\]

(7)

where \( M \) is any constant. Solving a two-point boundary value problem (TPBVP) in the set of Eq. (5) is not easy, thus the shooting method or the variational approach [1-3] is used for this purpose. However, for the system in Eq. (5), the standard calculus method with backward integration can be used.

### 2.2 Fuzzy Bang-bang controllers

The new proposed controller combines the fuzzy logic inputs with a hard limiter relay in single entity at its output. This controller is defined as a fuzzy Bang-bang controller (FBBC). The FBBC uses maxima (LOM) defuzzification technique to yield a Bang-bang output. The controller has two variants, tuned or un-tuned. The un-tuned remain as FBBC, while the tuned input FBBC is defined as FTOC.
For any tuning or non-tuning fuzzy controller, it is necessary to determine the bounds on the initial state of the system, which are considered to be a reasonable representation of all the situations that the controller may encounter. The following bounds are selected for angular position system, $x(t) = [-3, 3]$ rad., angular velocity $\dot{x}(t) = [-4, 4]$ rad./sec and constrained output $u = \pm M$

### 2.3 Fuzzy Membership Functions Set

The input variables and values assigned to fuzzy set membership functions, $A_{i=1,5}$, are shown in Figure 1. Index $i$ represents the inputs and $k$ the membership functions. Triangular shape membership functions are used in this work. These membership functions are sensitive to small changes that occur in the vicinity of their centers.

A small change across the central membership function $A_i^3 = \text{Zero}$, located at the origin, can produce abrupt switching of control command $u$ between the +ve and –ve halves of the universe of discourse, resulting in chattering. The overlapping of the central membership function, $A_i^3$ with the neighboring membership functions $A_i^2 = \text{SN}$ and $A_i^4 = \text{SP}$ reduces the sensitivity of the Bang-bang control action [25].
Triangular membership functions in Figure 1 are based on mathematical characteristics given in Table 1. In Table 1 the $b_i$ and $a_i$ are the tuning parameters for range (spread) and central location (overlap) of membership functions respectively, shown in Figure 1. Smooth transition between the adjacent membership functions is achieved with higher percentage of overlap, which is commonly set to 50%. Membership functions of $x_2$ are more concentrated around the origin to react quickly to initial velocity inputs.

The inputs and output parameters, as well as the partitions and spread of the controller membership functions are initially select to match the dynamic response of the system Eq. (1). The inputs $x_i \in X_i$ where $X_i$ is the universe of discourse of the two inputs, $i = 1, 2$. For input variable, $x_{i \omega} = \text{“error,”}$ the tuning universe of discourse, $X_{i \omega} = [-3, 3]$, which represents the range of perturbation about the zero reference. Index $k$ is assigned to tally the input
### Table 1. Mathematical Characterization of Triangular Membership Functions

<table>
<thead>
<tr>
<th>Linguistic value</th>
<th>Triangular Membership functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_i^{k=1}$</td>
<td>$\mu_{A_i^1}(x) = \begin{cases} 1 &amp; x = -b_i \ \frac{2</td>
</tr>
<tr>
<td>$A_i^{2}$</td>
<td>$\mu_{A_i^2}(x) = \begin{cases} \frac{2</td>
</tr>
<tr>
<td>$A_i^{3}$</td>
<td>$\mu_{A_i^3}(x) = \begin{cases} \frac{2</td>
</tr>
<tr>
<td>$A_i^{4}$</td>
<td>$\mu_{A_i^4}(x) = \begin{cases} \frac{2</td>
</tr>
<tr>
<td>$A_i^{5}$</td>
<td>$\mu_{A_i^5}(x) = \begin{cases} \frac{2</td>
</tr>
</tbody>
</table>

Membership functions. For input variable $x_{i=2} = \text{“error rate,”}$ the tuning universe of discourse is $X_{i=2} = [-4, 4]$. The output universe of discourse $Y = [-M, +M]$ represents the Bang-bang output.

The set $A_i^k$ is the membership function of antecedent part, given as

$$A_i^k = \left[ A_i^1 \rightarrow LN, \quad A_i^2 \rightarrow SN, \quad A_i^3 \rightarrow Zero, \quad A_i^4 \rightarrow SP, \quad A_i^5 \rightarrow LP \right] \quad (8)$$

Similar values are select for input $x_2, A_2^k \equiv A_i^k$. The set $B^k$, which denotes the membership function values for output variable $y_i$, is defined as:

$$B^k = \left[ B^1 \rightarrow Nbang, \quad B^2 \rightarrow Pbang \right]$$
2.4 First Order Necessary Fuzzy Optimal Rules

The fuzzy rules are selected in this work to reset the system to zero states. These rules are based on two input variables, each with five values, thus there can be at most 25 possible rules. These rules are described in matrix form in Tables 2. The shaded diagonal entries in Table 2 are not use to assimilate two-level Bang-bang control. In Table 2 the rules-partitions are chosen to reset the beam smoothly over the universe of discourse by providing counter control signal ($\pm M$), that is; if $x(LP) & xdot(SP) \rightarrow -M$ and vice versa. Note that contradicting $x(LN)$ and $xdot(LP)$ are not used, as indicated by grey-blank box.

The symmetry of the optimal rules matrix in table 2 arises from the phase trajectories of system in Eq. (1), which is necessary for optimality condition

Table 2: Fuzzy Optimal rules Map from Phase Plane Trajectories
The minimum inference of the \( j \)th rule from the FBBC’s input to the output is given by

\[
\mu(y)_{B_j} = \mu(x_1)_{A_{k_{l,j}}} \land \mu(x_2)_{A_{k_{2,j}}} \tag{9}
\]

where \( j = 1, 2, \ldots n \) is the index of \( n \) matching rules, which are applicable from inferences of inputs.

### 2.5 Fuzzy Bang-bang controller

Fuzzy Bang-bang controller is simple and easy to implement. Further, there exist sufficient optimality conditions, in Table 2, which is tailor-made for Bang-bang solutions.

The minimum inference in Eq. (9) for \( j \)th rules are aggregated by largest of maxima (LOM) as

\[
\sigma_f(y) = \mu(y)_{B_{l,j}} \lor \mu(y)_{B_{2,j}} \lor \cdots \mu(y)_{B_{l,j}}
\]

or

\[
\sigma_f(y) = \max \left[ \mu(y)_{B_{l,j}} \right] \tag{10}
\]

Changing the dependency to state \( x \), then from Eq. (9) and Eq. (10)

\[
\sigma_f(x) = \max \left[ \mu(x_1)_{A_{k_{l,j}}} \land \mu(x_2)_{A_{k_{2,j}}} \right] \tag{11}
\]

The Bang-bang switching control for unity magnitude \((M=1)\) is \( u_f = (-1, +1) \), will then produce control signal based upon LOM defuzzification is
\[ u_f(x) = \text{sign} \left[ \sigma_f(x) \right] \]  

Largest-of-Maximum uses the union of the fuzzy sets and takes the largest value of the domain with maximal membership degree [26, 27]. The output membership functions shown in Figure 2 and the LOM aggregation together formulate the fuzzy Bang-bang controller.

\[ \mu_B \]

\[ \sigma_f(y) \]

**Figure 2** (Top) FBBC output \( y \) membership functions. (Bottom) FBBC two level Bang-bang control output \( u_f \).

The input membership functions of FBBC are same as described above, which results in same aggregated rules output in Eq. (9). The output of FBBC depends upon the maximum value of degree of membership function, \( \mu_B(y) \), shown in Figure 2 (Top).
2.6. Tuning Conditions of Controller

The fuzzy set described above satisfies the following conditions:

i) Membership function range variable $b_i$, Figure 1, act upon the bordering input membership functions $A^1_i$ and $A^5_i$, and tunes the scale factor of the inputs, Figure 1. This has an effect on the proportional and derivative gains, which changes sharply in the beginning of the optimization process and also optimizes the overlaps between the membership functions.

ii) The input’s central membership function $A^3_i$ is fixed at zero to keep the symmetry in the control as required by the dynamics of the system.

iii) The input membership functions $A^2_i$ and $A^4_i$ in Figure 1, are allowed to change their central value $a_i$, and has an effect of fine tuning the response in the vicinity of the desired response.

iv) During the optimization/tuning process Bezdek’s repartition [25] is satisfied, that is maximum=1 of a membership function corresponds to minimum=0 of the adjacent membership function.

v) The order of membership functions NL, NS, Zero, PS and PL is always respected according to Bezdek’s distribution, that is the modal value of any MF never crosses the modal value of another MF.

2.7. Fuzzy controller Numerical Optimization

The optimization process uses on-line gradient-based steepest descent method [28]. This method gets the vector $Z$ which minimizes an objective $E(Z)$, where $Z= [z_1= b_1^{k=1}, z_2= b_1^{k=5}, z_3= a_1^{k=2}, z_4= a_1^{k=4})$ between the desired input $x_r(t)$ and optimized output $x_{opt}(t)$.
\[ E(Z)_{\text{min}} = \left| x_{opt}^{*}(a_i^k, b_j^k, w_i, t) - x_r \right| \]  

and  
\[ x(t)_{\text{opt}} = A x_{opt}(t) + B u^*(t) \]

Where \( u^*(t) \) is optimized control signal necessary to bring the states to desired response. The iterations on Z, decrease the objective function \( E(Z) \) expressed as

\[
\begin{bmatrix}
\frac{\partial E}{\partial z_1} & \frac{\partial E}{\partial z_2} & \frac{\partial E}{\partial z_3} & \frac{\partial E}{\partial z_4}
\end{bmatrix}
\]

Therefore, tuning of each parameter is defined as follows

\[ Z(t + T) = Z - K \frac{\partial E(Z)}{\partial Z} \]  

where \( T \) is the sample time, \((t+T)\) is total iteration time required to reach a error limit and \( K \) is a constant. When \( Z \) is tuned according to Eq. (13), the objective function \( E(Z) \) converges to a local minimum.

According to Eqs. (13, 14) the update of parameters are accomplished as

\[ a_{i,j}(t + T) = a_{i,j} - K_a \frac{\partial E}{\partial a_{i,j}} \]

\[ b_{i,j}(t + T) = b_{i,j} - K_b \frac{\partial E}{\partial b_{i,j}} \]

\[ w_i(t + T) = w_i - K_w \frac{\partial E}{\partial w_i} \]  

\( K_a \), \( K_b \), and \( K_w \) are constants to control the rate of convergence of the optimization process and \((t + T)\) is the update value after each iteration. Note index \( j \) added to the tuning parameters \( a_{i,j} \) and \( b_{i,j} \) to account for only those rules, which are contributing, to the controller output. The gradients in Eq. (15) are described in Appendix A.
2.8. Fuzzy Time-Optimal Controller (FTOC) – Necessary Condition

The structure of FBBC in section 2.2., when optimized by on-line gradient based method in section 2.7, yields fuzzy time-optimal controller. The sufficient condition of optimality is satisfied by optimal rules in Table 2. The constrained output \( u_f = (-1, +1) \) and tuning of overlapping region of input brings the states \( x \) to zero in shortest possible time. The stability of the controller can be satisfied by Lyapunov function, and will be described in next section.

To work out the necessary condition, define a new function \( v \rightarrow E(Z(t)) \) such that for small \( T = 0.1 \) sec, \( u \rightarrow E(Z(t+T)) \)

\[
E(Z(t+T)) - E(Z(t)) = \int_{t}^{t+T} H(t, x(t), p(t), u(t), t) - H(t, x(t), p(t), v(t), t) \, dt
\]  
(16)

Then the first order necessary condition (PMP) for \( E(u) - E(v) \) to be non-positive is that

\[
H(t, x(t), p(t), u(t), t) \leq H(t, x(t), p(t), v(t), t), \quad v \in Y, \forall t \in [0, T]
\]

The goal of steepest descent gradient tuning is then to find a control \( u \) over a sampling time \( T \), which minimize the \( E(Z(t)) \) at every time interval \( T \). The optimal solution to the problem is found numerically by tuning upon the initial choice \( Z = (z_1 = b_1^{k_a}, z_2 = b_1^{k_b}, z_3 = a_1^{k_a}, z_4 = a_1^{k_b}) \) in a steepest descent method Eq. (14). The Eq. (15) make sure that \( Z(T+1) \) leads to an allowable control and updates tuning parameters.

In term of classical optimal control system the FTOC implies that numerical optimization [29] minimize \( E(Z) \) Eq. (13) in term of Hamiltonian function as

\[
\frac{\partial E(Z)}{\partial Z} = \int_{t}^{t+T} \frac{\partial H}{\partial Z}(t, x(t), p(t), u(t), t) \, dt
\]

\[
= \Delta E(Z)
\]

\[
= \Delta x^*(a_{i,j}, b_{i,j}, w_i, t) - x_f
\]  
(17)
In essence there is no need to solve the TPBVP and the fuzzy Bang-bang output, tuned by the location $a_i$ and spread $b_i$ of input membership leads to time-optimal controller.

### 2.9. Fuzzy Sliding Mode Controller

A sliding – switching line $s(x, t)$ in the second order state space $\mathbb{R}^2$ is defined such that error Eq. (13), $E(Z):= e$, follows the line $s(x, t)=0$. The sliding line $s(x, t)$ is determined by

$$s(x,t) = \left( \frac{d}{dt} - \lambda \right)^{n-1} e.$$  \hspace{1cm} (18)

Eq. (18) can be expanded with binomial expansion and $\lambda$ is positive constant. For $n = 2$

$$s = \dot{e} + \lambda e \quad \therefore \dot{e} = \ddot{x} \quad e = x.$$ \hspace{1cm} (19)

Then from Eq. (2)

$$\dot{s} = f(x; t) + b(x; t) u + \dot{\lambda} e$$ \hspace{1cm} (20)

The rules in Table 2 can be deduced from Eq. (20). Multiplying it with $s$ yields

$$s \dot{s} = f(x; t)s + b(x; t) u s + \dot{\lambda} x s$$ \hspace{1cm} (21)

For $b > 0$, if $s < 0$, then increasing $u$ will result in decreasing $s \dot{s}$; and that if $s > 0$, then decreasing $u$ will result in decreasing $s \dot{s}$. The control value $u$ should be selected so that $s \dot{s} < 0$ for $0 < s > 0$. The slope of sliding line is represented by $\lambda$.

Considering $s$ as $x$ and $\dot{s}$ as $\ddot{x}$, then $u = (-1, +1)$, the fuzzy rules in Table 2 and the membership functions shown in Figure 2.(Top) agree with the sliding mode condition.
3 Bang-bang Controller Stability

SMC controllers have been developed and applied to nonlinear systems for the last two decades. In the absence of a mathematical model for non-linear fuzzy controllers the stability analysis is not straightforward, whereas the stability of SMC is inherent. In case of fuzzy Bang-bang controllers, the optimal fuzzy rules in Table 2, result in partitioning of the decision-space into two semi-planes by means of phase plane trajectories. Bang-bang control system is generally considered as a time-optimal controller. To verify the stability of the proposed Bang-bang controller [22, 23, 30], the similarity between a fuzzy Bang-bang controller and a sliding mode controller (SMC) can be used to redefine the diagonal form of the fuzzy logic controller in terms of an SMC with boundary limits. SMC is a robust control method [19] and its stability is proven with Lyapunov’s direct method [14]. Wong et.al [18] presents a method to apply Lyapunov’s direct method to individual firing rules rather than the entire fuzzy sub-system associated with all the rules.

Lyapunov theorem 1: Assume that there exists a scalar function $V$ of the state $x$ such that

i) $V=x^T P x$ is positive definite-and continuously differentiable, $P$ is square positive definite matrix

ii) $\dot{V}$ is negative definite in the active region of the corresponding fuzzy rule

A fuzzy sub-system, Table 2, associated with the $j^{th}$ fuzzy rule gives $\dot{V} \leq 0$ in its active region, implies that the system Eq. (1) can be stabilized on applying $j^{th}$ fuzzy rule individually.
4. Design Examples
Three design examples are used to illustrate the design procedure and techniques involved in accomplishment of FTOC. In the first example, the classical time-optimal control (TOC) solution for a linearized second-order system is compared with nonlinear system FTOC solution, then the role of SMC is explained for achieving the FTOC. In the second and third example the FTOC is built for a second-order system for which no classical solution is available. SMC trajectories are used as reference to lead to the fuzzy optimal controller.

4.1. Optimal Control – Design Example I
An second order non-linear angular position control system based on Eq. (1) is

\[ I \ddot{x}(t) + g(x(t), \dot{x}(t)) + l.x(t) = u(t) \]

where \( I = 1 \text{ kg-cm}^2 \) is the inertia, \( g(x, \dot{x}) = x^3 + 0.5x \) is the non-linearity with respect to damper and \( L = 0.5 \text{ N-m/rad} \) is the torsion constant. To develop the classical optimal control for system Eq. (1) it needs to be linearized. Linearizing [18] about \( x = [-1, +1] \) rad. the nonlinear term becomes \( g(x, \dot{x}) = 2.\dot{x} \). The state space representation of the linearized system then is

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-2 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
b
\end{bmatrix} u
\]

(22)

i) Performance index from Eq. (3)

\[ E(Z) = \int_0^{t_f} l dt \quad (\text{minimum time}) \]

ii) Hamiltonian Eq. (4)

\[ H = I + p_1 x_2 - p_2 (u - x_1 - 2x_2) \]

(23)

iii) Adjoint co-states Eq. (5)
\[ \dot{p}_1(t) = -\frac{\partial H}{\partial x} \Rightarrow -p_2 \]  
\[ \dot{p}_2(t) = -\frac{\partial H}{\partial x} \Rightarrow -p_1 + 2p_2 \]  
(24)

iv) Pontryagin Minimum Principle Eq. (6)

\[ H(t, x^*(t), p^*(t), u^*(t)) \leq H(t, x(t), p(t), u(t)) \]  
(25)

From Eq. (23) and Eq. (25) the optimum control signal Eq. (7) satisfies the necessary condition of optimality [31]

\[ 1 + p_1^*x_2^* - p_2^*(u^* - x_1^* - 2.x_2^*) \leq 1 + p_1^*x_2^* - p_2^*(u - x_1^* - 2.x_2^*) \]

\[ p_2^*u^* \leq p_2^*u \]

\[ p_2(-\text{sign}(p_2)) \leq p_2^*u \]

\[ \therefore u^* = -\text{sign}(p_2) \]

Solving the adjoint equations Eq. (24) in reverse time, gives the resulting optimal feedback control [32]

\[ \sigma(x) = \begin{cases} 
\frac{x_2}{e^{x_1+x_2}-1} + (x_1 + x_2) & \text{if } x_1 + x_2 \leq 0 \\
\frac{x_2}{e^{x_1+x_2}+1} - 1 - (x_1 + x_2) & \text{if } x_1 + x_2 \geq 0 
\end{cases} \]

\[ u^* = \begin{cases} 
-\text{sign}(\sigma(x)) & \text{if } \sigma(x) \neq 0 \\
-\text{sign}(x_2) & \text{if } \sigma(x) = 0 \text{ and } x \neq 0 \\
0 & \text{if } x = 0 
\end{cases} \]

(26)

4.2 Sliding Mode Controller

A sliding mode controller is designed for the system with the nonlinear term

\[ g(x, x) = x^3 + 0.5x. \]  

Then Eq. (1) has the 2nd order nonlinear form

\[ \dot{x} = f(x) + b(x)u \]  
(27)

where

\[ f(x) = \begin{bmatrix} 0 & x_2 \\ -x_1 & -2x_2^2 \end{bmatrix}, \quad b(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

Form Eq. (19) by letting \( s = a.x_1 + x_2 \), where \( a = 1 \) is the slope of the sliding line.
\[
\dot{s} = x_1 + x_2 \\
= x_2 - 2x_2^3 - x_1 + u
\]

(28)

The Lyapunov function for stability [19] is selected as

\[
V = \frac{1}{2} s^2 \\
\dot{V} = s \dot{s} \\
\dot{V} = s(x_2(1 - 2x_2^3) - x_1 + u)
\]

(29)

For \( \dot{V} \) to be negative definite then

\[
x_2(1 - 2x_2^3) - x_1 + u = \begin{cases} 
< 0 & \text{for } s > 0 \\
0 & \text{for } s = 0 \\
> 0 & \text{for } s < 0
\end{cases}
\]

(30)

Let \( \beta(x) = -(x_2(1 - 2x_2^3) - x_1) \)

Then for negative definite \( \dot{V} \) (see Eq. (30)) the following \( u \) is required

\[
u = \begin{cases} 
\dot{V} = (s < 0)(-\beta(x) + u) \Rightarrow \text{stable if } u > \beta(x) \\
\dot{V} = (s = 0)(-\beta(x) + u) \Rightarrow 0 \quad \text{if } u = \beta(x) \\
\dot{V} = (s > 0)(-\beta(x) + u) \Rightarrow \text{stable if } u < \beta(x)
\end{cases}
\]

(31)

The following control law ensures \( \dot{V} \) is negative

\[
u = \beta(x) - k \text{. sign}(s)
\]

(32)

where \( k > 0 \).

4.3. Fuzzy controller and stability

The fuzzy controller, FTOC, is designed based upon the input membership functions shown in Figure 1 and rules as shown in Table 2. The output membership functions of FTOC are shown in Figure 2. The terminal condition for numerical tuning of fuzzy membership function Eq. (13)

\[
E(Z(t + 1)) - E(Z(t)) = x^*(a_{i,j}, b_{i,j}, w_i, t) - x_r
\]

where \( \varepsilon \approx 0 \)

\[
= \varepsilon
\]
The stability of the FTOC optimal rules in Table 2 can be verified by using Lyaponov's stability test [18]. Selecting a Lyaponov function

\[
V = x^T \begin{bmatrix} 10 & 1 \\ l & 10 \end{bmatrix} x
\]

\[
= 10x_1^2 + 2x_1x_2 + 10x_2^2
\]

For rule 1-10, Table 2, upper right, if \( x \in [3, 0] \) and \( \dot{x} \in [3, 0] \) then \( y = -M \)

\[
\dot{V} = 20x_1 \dot{x}_1 + 2x_1x_2 + 2x_1 \dot{x}_2 + 20x_2 \dot{x}_2
\]

\[
= 20x_1 \dot{x} + 20x_2 \dot{x} + 2x_1^2 + 2x_2^2
\]

\[
= 20x_1 \dot{x} + 20x_2 \dot{x} - 2x_1^3 - x_1M + 2x_1^2 - 2x_1^3
\]

\[
= 20x_1 \dot{x} - 40x_1^4 - 20x_2 \dot{x} - 20x_1^4 + 2x_2^2 - 2x_2^3 - 2xM
\]

\[
\leq -2M(10x - x) - 40x_1^4 + 2x_2^2 - 2x_2^3 - 2xM
\]

For rules 10-20, Table 2, lower left, if \( x \in [-3, 0] \) and \( \dot{x} \in [-3, 0] \) then \( +M \)

\[
\dot{V} \leq 0
\]

Complete set of fuzzy sub-systems surface to establish stability of two level Bang-bang output \( u = (-1, +1) \) is shown in Figure 3.

**Figure 3** Negative definite \( \dot{V}(x) \) surface for optimal rules for design example 1, Eq. (33)
4.4 Controller Response

The response of nonlinear system in Eq. (27) to the different controllers is shown in Figures 4-12. The fuzzy Bang–bang controller (FBBC) is the un-tuned version of the controller described by the rules in Table 2 and the membership functions in Figure 1. Fuzzy time-optimal controller (FTOC) is same as FBBC but with tuning as described in section 2.6. and section 2.7. For this system, tuning has little effect on the response time. FTOC Eq. (17), TOC Eq. (26) and FBBC Eq. (12) have the same response, Figure 4a. All response are faster than SMC as the FTOC, TOC and FBBC circumnavigates past sliding line which is notorious for chattering, Figure 4b. The control signal switching and the costate are shown in Figure 4c.

In Figures (5-7) the initial setting of tuning parameters $a_0, b_0$ are analyzed as they have direct effect on the system response. The initial angular position mfs parameters are fixed at

Figure 4a. Comparison between different controllers. Close approximation between optimal control and un-tuned fuzzy Bang–bang control

Figure 4b. Phase trajectories comparison of different controller.
$a_{x0} = 2$, $bx_0 = 4$ and are not tuned. These values are chosen based on $b_{x0} > X_{i=1} = [-3, 3]$, otherwise during the tuning process the membership partition integrity can be violated [24]. The tuning is carried out only for velocity input membership function.

Initial settings are shown in Figure 4, where $a_{xdot0} = 1$ and $b_{xdot0} = 4$. In Figure 5 (right) the FTOC trajectories depart earlier from the sliding line (1.05 rad., -0.55 rads/sec), and hence have slower response than the SMC, Figure 5 (left). This shows that FTOC needs derivative gain to catch up with TOC. Increasing the spread and overlap, $a_{xdot0} = 2$ and $b_{xdot0} = 4$ in Figures 5-7, increases the scale and fuzziness of the velocity membership functions set, apparently FTOC maps over the TOC trajectory. In Figure 7 the FTOC traverse past the SMC trajectories.

**Figure 4c.** (Top) Controllers switching control signal. (Bottom) the costate –adjoint signal triggers the switching at zero crossing.

**Figure 5.** Effect of selection of initial membership function ($a_{xdot0} = 1$, $b_{xdot0} = 4$). (Left) Time response of FTOC & FBBC are no better than TOC. (Right) FTOC and FBBC trajectories breaks away earlier from SMC’s sliding line at $x = 1.05$ rad., $x_{dot} = -0.55$ rads/sec.
Figure 6. Increase in overlap of velocity membership function ($a_{\dot{x}} = 2$). (Left) Time response of FTOC improves in relation to TOC and FBBC. (Right) FTOC and FBBC trajectories overlaps the sliding line. The break away point moves to $x=0.8\,\text{rad.}, x_{\dot{x}}=-0.85\,\text{rads/sec}$

Figure 7. Increase in spread and overlap of velocity membership functions ($a_{\dot{x}} = 4, b_{\dot{x}} = 6$). (Left) Time response of FTOC & FBBC copies TOC. (Right) FTOC and FBBC trajectories passed over the sliding line and overlap the TOC
Proposition 1: Fuzzy time-optimal pre-condition 1, The FTOC phase trajectories should traverse trajectories past the sliding line.

5. Design Example II

The example considered here has two nonlinear terms in comparison to the first example. Fuzzy controllers are known as estimator of nonlinear systems [18] – this property of fuzzy controller is tested here. The estimator property of FTOC is tested here to absorb the nonlinearities of the design system II. The stability and optimality rules (Table 2) are chosen similar to first design example to reset the systems’s states to zero in minimum time. The phase trajectory of SMC will be used to lead the FTOC trajectory to the time-optimal control.

5.1 System model

In this example a nonlinear mass, damper and spring system is considered [14].

\[
\ddot{x}(t) = -0.1\dot{x}^3(t) - 0.02x(t) - 0.67x^3(t) + u
\]

The TOC for this problem is not evaluated and an attempt will be made to design-select parameters for FTOC based upon sliding mode and fuzzy controllers. SMC design procedure remains same, as described in sections 4.2. Same optimal rules and membership functions applies here as described in Table 1. The design will be based on same sets of universe of discourse \(X_{u1} = [-3, 3]\), \(X_{u2} = [-8, 8]\) is higher due to increase in nonlinear spring force and \(Y = [-M, +M]\).
5.2 Controller Response

In the absence of a TOC the SMC will be used for guidance. The same FTOC and SMC rules are applied here as before. In this example two nonlinear terms are present and the un-tuned FBBC response overshoots. The tuned FTOC response is much better in terms of the overshoot and reaches the final value much faster than the SMC and FBBC Figures 8, 9.

Figure 8. Response of second design example. Tuned FTOC trajectory between SMC and un-tuned FBBC, the $a_{x_{dot0}}$, $b_{x_{dot0}}$ are chosen higher for derivative action.
Proposition 2: Fuzzy time-optimal pre-condition II, the FTOC phase trajectories should traverse past the sliding line and lie between the sliding mode and FBBC.

By increasing $a_{\dot{x}0}$ and $b_{\dot{x}0}$, proposition 1 and 2 are satisfied and the FTOC trajectory
lies between SMC and FBBC. Note the initial velocity input membership function $a_{x_{dot0}}$ and $b_{x_{dot0}}$ are kept higher than the first example due to large damping and spring force. The optimization process has to work out the derivative gain by changing $a_{x_{dot0}}$ and $b_{x_{dot0}}$, shown in Figure 10.

6.0 Design Example III

Consider a nonlinear system [33].

\[ \begin{align*}
   \dot{x}_1 &= -x_1 + a \sin x_2 \\
   \dot{x}_2 &= -x_1^2 - 2x_2 + u
\end{align*} \]

where $a = 3$.

The TOC for this example is not available, the stability of system based upon the fuzzy optimal rules is checked with Lyaponov’s stability test, where the $P$ [33] is given as

\[ P = \begin{pmatrix} 17.34 & -6.692 \\ -6.692 & 14.50 \end{pmatrix} \]

Considering the same fuzzy optimal rules in Table 2, for ranges $x = [-1, 1]$ rad. and $x_{dot} = [-1, 1]$ rad/sec and $u = [-M, +M]$ for $M = 1$

\[ V(x) = 27.76x \dot{x} - 61x^2 - 2.35x^4 + 2.35xu - 31.68x^3 \dot{x} + 31.68ux \dot{x} \tag{34} \]

Testing the negative definite condition $\dot{V}(x) \leq 0$ for all the rules in table 3, the surface shown in Figure 11(bottom right) satisfies the Lyaponov stability condition. The proposition 1 and 2 are satisfied for different initial angle and velocities in figure 11 and 12.

The initial membership functions (mfs) locations and ranges for the three example are summarize in Table 3. Note that higher values are selected for velocity mfs, to increase the
Figure 11. Response of design example III, $x_0=1\text{ rad.}, x_{\dot{0}}=0.5\text{ rad/sec}$. (Bottom Right)

Negative definite $\dot{V}(x)$ surface for optimal rules for design example 3, Eq. (34)

Figure 12. Response of design example III, $x_0=-0.4\text{ rad.}, x_{\dot{0}}=-0.6\text{ rad/sec}$. 
rate of convergence of states. The selection of optimization constant $K_a$, $K_b$ and $K_w$ eq.(15) shows the variability among the constant for three examples.

Table 3. Location and Ranges for membership functions

<table>
<thead>
<tr>
<th></th>
<th>$a_{x0}$</th>
<th>$b_{x0}$</th>
<th>$a_{xdot0}$</th>
<th>$b_{xdot0}$</th>
<th>$K_a$</th>
<th>$K_b$</th>
<th>$K_w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example I</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>0.1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Example II</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>8</td>
<td>-1</td>
<td>200</td>
<td>4</td>
</tr>
<tr>
<td>Example III</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>-1</td>
<td>200</td>
<td>3</td>
</tr>
</tbody>
</table>

6. FTOC Design Steps.

The following steps can be used as guide lines in designing the FTOC.

1. Select the inputs universe of discourse greater than the operating range of the system.

2. Use at least five triangular membership function set at 50% overlap for inputs, Figure 1. LOM defuzzification for output, Figure 2. Use Mamdani type fuzzy model.

3. Select the fuzzy time-optimal rules as in Table 2, such that upper right triangle off the diagonal has $-M$ control signal and the lower left has $+M$. SMC of the system is required, section 2.9 and section 4.2.

4. If Lyaponov function can be evaluated, check the stability of optimal rules, section 4.3, else like conventional fuzzy controller the expertise of the designer are exploited and proceed to next step.

5. Use gradient based on-line optimization and performance criteria, only velocity membership functions spread, $b_{xdot}$ and overlap, $a_{xdot}$ are required to be tuned, section 2.7.
6. Fulfill *Propositions I and II*, section 4.4 and section 5.2 respectively.

7. Check for the ‘out of range input’ and ‘un-fire rules’ run time warnings, and adjust optimization constant; $K_a, K_b, K_w$, Table 3

8. Membership function ‘cross over error’, means to redefine the initial $a$’s and $b$’s for the inputs.

**7. Conclusion**

The paper presents a new a solution for difficult optimal control problems. Fuzzy controllers, though successful for controlling complex non-linear systems, are difficult to prove mathematically. The paper shows that by selecting the fuzzy optimal rules and with bounded two level output, the Bang-bang controller response matches with time-optimal controller. On-line gradient descent optimization due to fast convergence and ease of implementation satisfies the necessary optimality condition. Three design examples were used to stress upon the supposition that FTOC act similar to TOC. Lyaponov stability criteria was used to determine the stability for all the sub-systems/rules and optimality condition were satisfied by the selection of fuzzy rules. First design example compares directly the TOC with FTOC and was used to demonstrate the effect of velocity input membership gain and overlap. In the first example the response for both FBBC- untuned and tuned (FTOC) were quite similar for small perturbation. In the second and third nonlinear design examples, TOC is not available and FTOC is designed, based on the propositions and procedure described in the paper. The FTOC design steps will enable the researcher to further improve the work with convictions and deliberate on the limitation of the approach discussed in this work.
APPENDIX A

STEEPEST DESCENT GRADIENT METHOD

The gradient of the objective function \[
\begin{bmatrix}
-\frac{\partial E(Z)}{\partial a_{i,j}}, & -\frac{\partial E(Z)}{\partial b_{i,j}}, & -\frac{\partial E(Z)}{\partial w_{i}}
\end{bmatrix}
\] in Eq. (15) can be derived from Table 1, Eq. (9) and Eq. (13) with chain rule as

\[
\frac{\partial E(Z)}{\partial a_{i,j}} = \frac{\partial E(Z)}{\partial x(z)} \times \frac{\partial x(z)}{\partial \mu_{i}(x | z)} \times \frac{\partial \mu_{i}(x | z)}{\partial A_{i,j}} \times \frac{\partial A_{i,j}}{\partial a_{i,j}} \tag{A-1}
\]

where

\[
\frac{\partial E(Z)}{\partial x(z)} = (x(z) - x_{r})
\]

\[
\frac{\partial x(z)}{\partial \mu_{i}(x | z)} = \frac{w_{i} \sum \mu_{i} - \sum \mu_{i} w_{i}}{\left(\sum \mu_{i}\right)^{2}} = \frac{w_{i} - x(z)}{\sum \mu_{i}} \approx 1 \tag{A-2}
\]

\[
\frac{\partial \mu_{i}(x | z)}{\partial A_{i,j}} = \frac{\mu_{i}}{A_{i,j}}
\]

\[
\frac{\partial A_{i,j}}{\partial a_{i,j}} = \frac{2}{b_{i,j}} \text{sgn}(x_{j} - a_{i,j})
\]

The second equation in set of Eqs. (A-2) is for centroid defuzzification. Defining it to be equal to 1, nullifies its contribution in updating the gradient descent optimization for LOM defuzzification. Then from Eq. (A-1)

\[
\frac{\partial E(Z)}{\partial a_{i,j}} = (x(z) - x_{r}) \cdot \frac{\mu_{i}(x | z)}{A_{i,j}} \cdot \frac{2}{b_{i,j}} \text{sgn}(x_{j} - a_{i,j}) \tag{A-3}
\]

and
\[
\frac{\partial E(Z)}{\partial b_{i,j}} = \frac{\partial E(Z)}{\partial \mu_i(x|z)} \times \frac{\partial \mu_i(x|z)}{\partial A_{i,j}} \times \frac{\partial A_{i,j}}{\partial b_{i,j}}
\]  

(A-4)

where

\[
\frac{\partial A_{i,j}}{\partial b_{i,j}} = \left[ \frac{b_{i,j} - 2(x_j - a_{i,j})^2 + b_{i,j}^{-1}}{b_{i,j}^2} \right] = \left[ 1 - A_{i,j}(x_j) \right] \left[ \frac{1 - A_{i,j}(x_j)}{b_{i,j}} \right]
\]

(A-5)

Then from Eq. (A-2) and Eq. (A-5)

\[
\frac{\partial E(Z)}{\partial b_{i,j}} = (x(z) - x_r) \sum \frac{w_i - x(z)}{\sum \mu_i(x|z)} \frac{\mu_i(x|z)}{A_{i,j}} \left[ 1 - A_{i,j}(x_j) \right] \left[ \frac{1 - A_{i,j}(x_j)}{b_{i,j}} \right]
\]

(A-6)

Further

\[
\frac{\partial E(Z)}{\partial w_i} = (x(z) - x_r) \sum \frac{\mu_i(x|z)}{\sum \mu_i(x|z)}
\]

(A-7)

Putting Eq. (A-3), Eq. (A-6) and Eq. (A-7) back in Eq. (15) gives the results of the most recent iteration.

\[
a_{i,j}^{(t+1)} = a_{i,j} - K_a (y - y_r) \frac{\mu_i}{A_{i,j}} \frac{2}{b_{i,j}} \text{sgn}(x_j - a_{i,j})
\]

(A-8)

\[
b_{i,j}^{(t+1)} = b_{i,j} - K_b (y - y_r) \frac{\mu_i}{A_{i,j}} \left[ 1 - A_{i,j}(x_j) \right] \left[ \frac{1 - A_{i,j}(x_j)}{b_{i,j}} \right]
\]

\[
w_i^{(t+1)} = w_i - K_w (y - y_r) \frac{\mu_i}{\sum \mu_i}
\]

The tuning parameters $b_{i,j}$ and $a_{i,j}$, minimize the objective function $E(Z)$. The above equations are iteratively solved until the error $e$ reaches a specified threshold level.
APPENDIX B

Sliding Mode Control (SMC)

A general 2nd order nonlinear single-input-single-output (SISO) control system could be described as

\[
\dot{x}(t) = f(x; t) + b(x; t) u
\]

Where \(x(t)\) is the output of interest, \(u(t)\) is the scalar input, and \(x = \begin{bmatrix} x, \dot{x} \end{bmatrix}^T\) is state vector. In general, \(f(x; t)\) is not precisely known, but upper bounded by a known continuous function of \(x\). Similarly \(b(x; t)\) is not known, but is of known sign and is bounded by a known continuous function of \(x\) as

\[
|f - \hat{f}| \leq F(x; t)
\]

\[
\frac{1}{\beta(x; t)} \leq \frac{\hat{b}}{b} \leq \beta(x; t)
\]

Where \(\hat{f}\) and \(\hat{b}\) are nominal values of \(f\) and \(b\) respectively, without the function argument for brevity purpose.

Comparing Eq. (27) and Eq. (B.1), the system becomes:

\[
f(x; t) = -\dot{x}(t)
\]

\[
b(x; t) u = u(t) \quad \therefore b(x; t) = 1
\]

where \(u(t)\) is a unit step input.

The control problem is to get the state \(x\) to track \(x_d = [x_d, \dot{x}_d]^T\) in minimum time and in the presence of imprecise friction. The initial \(x_d\) should be the following in view of finite control \(u\)

\[
x_d(0) = x(0)
\]

The tracking error between the actual and desired state would be
\[ e = x - x_d = [e \quad \dot{e}]^T . \]  

(B.5)

A sliding – switching line \( s(x, t) \) in the second order state space \( \mathbb{R}^2 \) is defined such that \( e \) follows the line \( s(x, t) = 0 \). The sliding line \( s(x, t) \) is determined by

\[ s(x, t) = \left( \frac{d}{dt} + \lambda \right)^{n-1} e . \]  

(B.6)

Eq. (B.6) can be expanded with binomial expansion and \( \lambda \) is positive constant. For \( n = 2 \)

\[ s = \dot{e} + \lambda e \]  

\[ \therefore \dot{e} = \dot{x} \quad \& \quad e = x . \]  

(B.7)

Then from Eq. (B.1)

\[ \dot{s} = f(x; t) + b(x, t)u + \dot{\lambda} \dot{e} \]  

(B.8)

\[ \textbf{B.2. SMC Control Law} \]

Let \( u_{eq} \) be the equivalent control law that will keep the states on the sliding trajectory, computed by \( \dot{s} = 0 \) for \( u = u_{eq} \), then from Eq. (B.4), Eq. (B.5) and Eq. (B.7)

\[ s = \dot{x} + \lambda \dot{x} \]  

\[ \dot{s} = \ddot{x} + \lambda \ddot{x} \]  

(B.9)

Then from Eq. (B.8) with uncertainties

\[ \dot{s} \bigg|_{u = u_{eq}} = \dot{f}(x; t) + \dot{b}(x, t)u_{eq} + \lambda \dot{e} = 0 \]

Solving the above equation

\[ u_{eq} = \hat{b}^{-1} \hat{u} \]  

(B.10)

where

\[ \hat{u} = \left[ -\hat{f}(x; t) - \lambda \hat{e} \right] \]  

(A.11)

or

\[ \lambda \hat{e} = -\hat{f}(x; t) - \hat{u} \]

is the nominal control input in presence of uncertainties?

\[ \textbf{B.3 SMC – Reaching Condition} \]

The control input \( u \) to get the state \( x \) to track \( x_d \) is then made to satisfy the Lyapunov-like function \( V = (1/2)s^2 \), if there exist \( \eta > 0 \) and by the following sliding condition:

...
\[
\frac{1}{2} \frac{d}{dt} s^2(x,t) \leq -\eta |s|
\] (B.12)

Which is reduced to the so-called sliding mode ‘reaching condition’ for Eq. (27)

\[
\dot{s} \cdot \text{sgn}(s) \leq -\eta |s| \quad \eta > 0
\] (B.13)

The control law that satisfies the Sliding mode reaching conditions Eq. (B.13) can be obtained as

\[
u = u_{eq} + u_s
\] (B.14)

where

\[
u_s = -K \text{sgn}(s)
\] (B.15)

and

\[
\text{sgn}(s) = \begin{cases} 
+1, & \text{if } s > 0 \\
-1, & \text{if } s < 0
\end{cases}
\]

Substituting Eq. (B.1) and Eq. (B.8) in Eq. (B.13)

\[
\dot{s} = s( f + bu + \lambda e ) \leq -\eta |s|
\]

Note: Here the function argument is dropped for brevity purpose. Then equivalently:

\[
\dot{s} = \text{sgn}(s)( f + \lambda e ) + bu \text{sgn}(s) \leq -\eta |s|
\] (B.16)

Substituting Eq. (B.14) and Eq. (B.15) into Eq. (B.16)

\[
\dot{s} = \text{sgn}(s)( f + \lambda e ) + b\left[ u_{eq} + \hat{b}^{-1} K \text{sgn}(s) \right] \text{sgn}(s) \leq -\eta |s|
\]

Substituting from Eq. (B.10) and Eq. (B.11) in above

\[
\dot{s} = \text{sgn}(s) \left[ f + ( - \hat{f} - u ) \right] + b \left[ \hat{b}^{-1} \hat{u} + \hat{b}^{-1} K \text{sgn}(s) \right] \text{sgn}(s) \leq -\eta |s|
\]
simplifying

\[
\text{sgn}(s)( f - \hat{f} ) + \left[ \frac{b}{\hat{b}} - 1 \right] \hat{u} \text{sgn}(s) - \frac{b}{\hat{b}} K \leq -\eta |s|
\] (B.17)

Then for upper bounds from Eq. (B.1) need

\[
K \geq \beta \left[ F + \eta \right] + (\beta - 1) \hat{u}
\] (B.18)
to satisfies the reaching or hitting condition
References


