On lattice-valued frames: The completely distributive case

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Abstract

We provide an extension of the notion of chain-valued frame introduced by Pultr and Rodabaugh in [Category theoretic aspects of chain-valued frames: parts I and II, Fuzzy Sets and Systems 159 (2008) 501–528 and 529–558] by relaxing the assumption that $L$ be a complete chain. As a result of this investigation we formulate the category $L\text{-Frm}$ of $L$-frames under the weaker assumption that $L$ is a completely distributive lattice. In particular, $L\text{-Frm}$ is complete and cocomplete. Finally we prove that, in a certain sense, the assumption of $L$ being a completely distributive lattice cannot be weakened.

Keywords: $L$-frames; $L\text{-Frm}$; Completely distributive lattice; $L$-topological space; $L\text{-Top}$; Iota functor; Completeness; Cocompleteness

1. Introduction

This paper is motivated by the following sentences from [10, Subsection 4.4.1]:

For simplicity and ease of exposition most sections of Parts I and II have the standing assumption that $L$ be a complete chain (unless indicated elsewhere). Relaxing the conditions of a complete chain, particularly in Section I.4, is a significant question outside the scope of this two-part series and the subject of future investigation.

It is precisely our intention here to provide an extension of the notion of an $L$-frame to a wider setting, namely for completely distributive lattices. The definition we introduce extends that of [8–10] (see also [4, Section 6]) in the sense that, when particularized to the chain-valued case, it coincides with the former one.

In this sense we provide an answer to the question of relaxing the conditions on the underlying lattice stated by Pultr and Rodabaugh in [10]. Indeed, all the results concerning the category-theoretic properties of $L\text{-Frm}$ proved in Section 4 of [9] remain valid for this new setting.

But we do more, we prove that the condition of the underlying lattice being a completely distributive lattice cannot be weakened. This is done by showing that if the system of level mappings of any $L$-topology is an $L$-frame, then the lattice $L$ must be necessarily completely distributive.

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2. Preliminaries

We state here only the information needed to understand the presented results. For more detailed preliminaries concerning $L$-frames, we refer to [9] and [10]. But for the convenience of the reader we recall here some basic facts from the theory of frames (cf. [5]) and continuous lattices (cf. [2]). In what follows $L$ will always denote a complete lattice.

$L$ is said to be a frame if finite meets are distributive over arbitrary joins, i.e., if for each $x \in L$ and every subset $S \subseteq L$ one has

$$x \wedge \bigvee S = \bigvee \{x \wedge \beta : \beta \in S\}.$$ 

A frame morphism is a map $f : L \to M$ between frames which preserves finite meets and arbitrary joins (so it preserves universal bounds). The category of frames and frame morphisms will be denoted by $\text{Frm}$.

An element $p \in L$ is called prime if for each $x, \beta \in L$ with $x \wedge \beta \leq p$ either $x \leq p$ or $\beta \leq p$. Following [2], we denote by $\text{PRIME } L$ the set of all prime elements of $L$ and $\text{Spec } L = \text{PRIME } L \setminus \{1\}$.

A frame $L$ is said to be spatial if it is order-isomorphic to the lattice of open sets of some topological space. Recall that (see e.g. [2, Section V-4]) $L$ is a spatial frame if and only if $L$ is generated by its prime elements, i.e. if

$$x = \bigwedge \{p \in \text{Spec } L : x \leq p\} = \bigwedge (\uparrow x \cap \text{Spec } L) \quad \text{for all } x \in L,$$

where $\uparrow x = \{\beta \in L : x \leq \beta\}$. (We shall occasionally say that the subset Spec $L$ is meet-dense in $L$). Equivalently, $L$ is spatial if and only if for any two distinct $x, \beta \in L$ there is $p \in \text{Spec } L$ such that either $x \leq p$ and $\beta \not\leq p$ or $x \not\leq p$ and $\beta \leq p$.

Given $x, \beta \in L$, we say that $x$ is way below $\beta$, in symbols $x \ll \beta$, if and only if

for all $S \subseteq L$ such that $\beta \leq \bigvee S$ there exist $\gamma_1, \ldots, \gamma_n \in S$ such that $x \leq \bigvee_{i=1}^{n} \gamma_i$.

Recall that $L$ is continuous if and only if the way-below relation is approximating, i.e., if and only if $x = \bigvee \{\beta \in L : \beta \ll x\}$ for each $x \in L$.

2.1. Completely distributive lattices and the binary relation $\ll$

We shall be particularly interested in the opposite relation of the way-below relation in the lattice $L^{\text{op}}$, denoted by $\ll$. Namely, given $x, \beta \in L$ we have $x \ll \beta$ if

for all $S \subseteq L$ such that $\bigwedge S \leq x$ there exist $\gamma_1, \ldots, \gamma_n \in S$ such that $\bigwedge_{i=1}^{n} \gamma_i \leq \beta$.

For each $x \in L$ we write $\uparrow x = \{\beta \in L : x \ll \beta\}$. Then we have that $L^{\text{op}}$ is continuous if and only if it satisfies

$$x = \bigwedge (\{\beta \in L : x \ll \beta\}) = \bigwedge \uparrow x \quad \text{for all } x \in L.$$

In what follows, the following properties of the binary relation $\ll$ will be needed. Let $x, \beta, \gamma, \delta \in L$, then:

1. $x \ll \beta$ implies $x \leq \beta$.
2. $x \leq \beta \ll \gamma \leq \delta$ implies $x \ll \delta$.

We can now state the following result which follows immediately from [2, Theorem I-3.16].

**Theorem 2.1.** Let $L$ be a complete lattice. Then the following statements are equivalent:

1. $L$ is completely distributive.
2. $L$ is a spatial frame and $L^{\text{op}}$ is continuous.

As a consequence, we can prove the following result that will play a key role in the rest of the paper.
Corollary 2.2. A complete lattice $L$ is completely distributive if and only if it satisfies the following two properties:

(i) $\forall x \in L$ and $p \in \text{Spec } L$, we have $p = \bigwedge_{p \in \text{Spec } L} (\uparrow x \cap \text{Spec } L)$.

(ii) $\forall p \in \text{Spec } L$, we have $p = \bigwedge_{p \in \text{Spec } L} (\uparrow p \cap \text{Spec } L)$.

Proof. $\Rightarrow$: As mentioned before, condition (i) is equivalent to $L$ being a spatial frame, so we only need to prove the second condition. Let $p \in \text{Spec } L$, since $L^{op}$ is continuous, it follows that $p = \bigwedge_{p \in \text{Spec } L} (\uparrow p)$ and so

$$p = \bigwedge_{p \in \text{Spec } L} (\uparrow x \cap \text{Spec } L) = \bigwedge_{p \in \text{Spec } L} \bigwedge_{x \leq p} \bigwedge_{q \in \text{Spec } L} q.$$

$\Leftarrow$: We only need to prove that $L^{op}$ is continuous. But this is now obvious since for each $x \in L$ we have

$$x = \bigwedge_{x \leq p \in \text{Spec } L} p = \bigwedge_{x \leq p \in \text{Spec } L} \bigwedge_{p \in \text{Spec } L} \bigwedge_{q \in \text{Spec } L} q.$$

Recall that for a completely distributive lattice $L$, one has $\text{Spec } L = L \setminus \{1\}$ and

$$x < y \iff x = \bigwedge \{ \gamma \in L : x < \gamma \}.$$

Also, since any complete chain is completely distributive, it follows from Corollary 2.2 that $\bigwedge (\uparrow p \cap \text{Spec } L) = p$, for each $p \in \text{Spec } L$. Hence $\uparrow p \cap \text{Spec } L \neq \emptyset$ for each $p \in \text{Spec } L$.

Finally, note that for $x \in L$ and $p \in \text{Spec } L$ one has

$$x \in \text{Spec } L \iff \forall S \subseteq L \text{ such that } \bigwedge S \leq x \text{ there exist } \gamma \in S \text{ such that } \gamma \leq p.$$
Remark 2.4.

(1) Note that the mapping \( t_p : \tau \to \tau_L(p) \) is a frame morphism for each \( p \in \text{Spec } L \) (this is not true in general if \( p \) fails to be prime). Consequently, we can consider the system of frame morphisms

\[
(t_p : \tau \to \tau_L(p)|p \in \text{Spec } L).
\]

(2) If \( L \) satisfies the second condition of Corollary 2.2, i.e., if \( p = \bigwedge (\uparrow p \cap \text{Spec } L) \) for each \( p \in \text{Spec } L \), then we have that \( [f \neq p] = \bigcup_{q \in \uparrow \emptyset \cap \text{Spec } L} [f \neq q] \) for each \( p \in \text{Spec } L \) and \( f \in L^X \). Consequently, for each \( p \in \text{Spec } L \),

\[
t_p = \bigvee_{q \in \uparrow p \cap \text{Spec } L} t_q. \tag{2.4.1}
\]

In particular, the assignment \( p \mapsto t_p \) is antitone.

(3) If \( L \) is a spatial frame then the family \( \{ t_p(f) : f \in \tau, p \in \text{Spec } L \} \) is a subbase of \( \tau_L(p) \). Indeed, for each \( x \in L \) we have \( x = \bigwedge (\uparrow x \cap \text{Spec } L) \) and so

\[
t_x(f) = [f \neq x] = \bigcup_{p \in \uparrow x \cap \text{Spec } L} [f \neq p] = \bigcup_{p \in \uparrow x \cap \text{Spec } L} t_p(f).
\]

Hence (cf. [6, Remark 3.7]),

\[
\tau_L(\tau) = \left( \bigcup_{p \in \text{Spec } L} t_p(\tau) \right). \tag{2.4.2}
\]

(4) Let \( L \) be a spatial frame. Since for each distinct \( f, g \in \tau \) there exists \( x \in X \) such that \( f(x) \neq g(x) \), hence there exists \( p \in \text{Spec } L \) such that either \( f(x) \leq p \) and \( g(x) \neq p \) or \( f(x) \neq p \) and \( g(x) \leq p \) and so \( [f \neq p] \neq [g \neq p] \). It follows that

\[
\text{if } f \neq g \in \tau \text{ then } t_p(f) \neq t_p(g) \text{ for some } p \in \text{Spec } L. \tag{2.4.3}
\]

(5) As a consequence of the previous comments and Corollary 2.2 we have that if \( L \) is a completely distributive lattice, then the system of frame morphisms

\[
(t_p : \tau \to \tau_L(p)|p \in \text{Spec } L)
\]

satisfies conditions (2.4.1), (2.4.2) and (2.4.3).

With the same aim as in [8, Proposition 3.1] we can prove now that in the case of completely distributive lattices the system of frame morphisms \( (t_p : \tau \to \tau_L(p)|p \in \text{Spec } L) \) contains all the information needed to recover the given \( L \)-topology \( \tau \).

Proposition 2.5. Let \( L \) be a completely distributive lattice, \( A \) be a frame and \( (X, \mathcal{O}X) \) be a topological space. Let \( (\varphi_p : A \to \mathcal{O}X|p \in \text{Spec } L) \) be a system of frame morphisms satisfying:

- \( \varphi_p = \bigvee_{q \in \uparrow p \cap \text{Spec } L} \varphi_q \) for each \( p \in \text{Spec } L \). \tag{2.5.1}
- \( \mathcal{O}X = \left( \bigcup_{p \in \text{Spec } L} \varphi_p(A) \right) \). \tag{2.5.2}
- \( \text{if } a \neq b \in A \text{ then } \varphi_p(a) \neq \varphi_p(b) \text{ for some } p \in \text{Spec } L. \tag{2.5.3}

Then there is a frame \( \tau \) and an isomorphism \( h : A \to \tau \) such that the following hold:

(1) \( \tau \) is an \( L \)-topology on \( X \);
(2) \( \tau_L(\tau) = \mathcal{O}X \); and
(3) for each \( p \in \text{Spec } L \), \( t_p \circ h = \varphi_p \).
Proof. Let us define, \( h(a) \in L^X \) for each \( a \in A \) as follows:

\[
  h(a)(x) = \bigwedge \{ p \in \text{Spec} \ L : x \notin \varphi_p(a) \}
\]

and put

\[
  \tau = \{ h(a) : a \in A \}.
\]

Let \( p, q \in \text{Spec} \ L \) be such that \( h(a)(x) \leq p \) and \( p \preceq q \). It follows from Corollary 2.3 that there exists \( p' \in \text{Spec} \ L \) such that

\[
  h(a)(x) = \bigwedge \{ p \in \text{Spec} \ L : x \notin \varphi_p(a) \} \preceq p' \preceq q.
\]

Hence there exists \( p'' \in \text{Spec} \ L \) such that \( x \notin \varphi_{p''}(a) \) and \( p'' \preceq p' \preceq q \). Since condition (2.5.1) in particular implies \( \varphi_q(a) \leq \varphi_{p''}(a) \), it follows that \( x \notin \varphi_q(a) \). We conclude, because of (2.5.1), that \( x \notin \bigcup_{q \in \text{Spec} \ L} \varphi_q(a) = \varphi_p(a) \). Consequently \( h(a)(x) \leq p \) implies \( x \notin \varphi_p(a) \), but the other implication is obvious and so we have the key equivalence

\[
  h(a)(x) \leq p \iff x \notin \varphi_p(a).
\]

We now check that \( h \) is injective: Given \( a \neq b \in A \), by (2.5.3) we have \( \varphi_p(a) \neq \varphi_p(b) \) for some \( p \in \text{Spec} \ L \). We can assume, without loss of generality, that there exists \( x \in \varphi_p(a) \) such that \( x \notin \varphi_p(b) \). Then \( h(b)(x) \leq p \) and \( h(a)(x) \neq p \) which implies \( h(b) \neq h(a) \).

\( h \) is a frame morphism: Let \( a, b \in A, x \in X \) and \( p \in \text{Spec} \ L \). We have

\[
  h(a \land b)(x) \leq p \iff x \notin \varphi_p(a \land b) = \varphi_p(a) \cap \varphi_p(b)
  \iff x \notin \varphi_p(a) \text{ or } x \notin \varphi_p(b)
  \iff h(a)(x) \leq p \text{ or } h(b)(x) \leq p
  \iff h(a)(x) \land h(b)(x) \leq p
\]

and so, since \( L \) is a spatial frame, we have that \( h(a \land b) = h(a) \land h(b) \). On the other hand, given \( \{ a_i \}_{i \in I} \subseteq A, x \in X \) and \( p \in \text{Spec} \ L \) we have

\[
  h \left( \bigvee_{i \in I} a_i \right)(x) \leq p \iff x \notin \varphi_p \left( \bigvee_{i \in I} a_i \right) = \bigvee_{i \in I} \varphi_p(a_i)
  \iff x \notin \varphi_p(a_i) \text{ for all } i \in I
  \iff h(a_i)(x) \leq p \text{ for all } i \in I
  \iff \bigvee_{i \in I} h(a_i)(x) \leq p
\]

and so, once again because \( L \) is a spatial frame, we have that \( h(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} h(a_i) \).

Consequently, the image of \( A \) under \( h \) is a subframe of \( L^X \), i.e. an \( L \)-topology \( \tau \) on \( X \). Equivalently, \( h : A \to \tau \) is an isomorphism.

Further, for each \( p \in \text{Spec} \ L \) and each \( a \in A \) we have \( t_p(h(a)) = [h(a) \neq p] = \varphi_p(a) \). It follows from (2.5.2) that

\[
  t_L(\tau) = \{ \{ t_p(h(a)) : p \in \text{Spec} \ L, a \in A \} \} = \{ \{ \varphi_p(a) : p \in \text{Spec} \ L, a \in A \} \} = \mathcal{O}_X. \quad \square
\]

3. \( L \)-frames

The notion of chain-valued frame, was introduced—as mentioned in the introduction of [9]—to be an abstraction of the distinctive properties of the system of level mappings from an \( L \)-topology \( \tau \) into \( t_L(\tau) \). These conditions, when \( L \) is a complete chain, were taken as axioms (F0), (F1) and (F2) in order to define \( L \)-frames and the associated category \( L\text{-Frm} \) (see [8, 3.3] and [9, 2.2.6]).

Clearly enough, conditions (2.4.2) and (2.4.3) of Remarks 2.4 are precisely (F1) and (F2). It is then natural to check whether condition (2.4.1) above and axiom (F0) coincide when \( L \) is a chain.

A (\( \gamma \in S \) is down-directed (or filtered) if it is non-empty and for any \( \alpha, \beta \in S \) there is some \( \gamma \in S \) such that \( \gamma \leq \alpha \land \beta \). In a complete chain \( L \) any non-empty subset is down-directed and \( \text{Spec} \ L = L \setminus \{ 1 \} \) and
hence we shall write here Spec $L$ instead of $L_\rightarrow$ and $L^\bullet$ as used in [8–10]. Consequently, if $L$ is a complete chain and 
$(\varphi^A_p : A^u \to A^l|p \in \text{Spec } L)$ is a system of frame morphisms, then the axiom (F0) in [8–10] can be equivalently stated 
as follows:

(F0) $\varphi^A_{\bigwedge S} = \bigvee_{s \in S} \varphi^A_s$ for each downdirected $S \subseteq \text{Spec } L$.

Now the following results holds.

**Proposition 3.1.** Let $L$ be a completely distributive lattice and $A \equiv (\varphi^A_p : A^u \to A^l|p \in \text{Spec } L)$ a system of frame morphisms. Then the following are equivalent

1. $\varphi^A_{\bigwedge S} = \bigvee_{s \in S} \varphi^A_s$ for each downdirected $S \subseteq \text{Spec } L$.
2. $\varphi^A_p = \bigvee_{q \in \uparrow\cap \text{Spec } L} \varphi^A_q$ for each $p \in \text{Spec } L$.

**Proof.** (1) $\Rightarrow$ (2): Let $p \in \text{Spec } L$. It follows from [2, Lemma V-1.6.], applied to $L^{op}$, that the subset $\uparrow\cap \text{Spec } L$ is 
downdirected, and from Corollary 2.2(2) that $p = \bigwedge(\uparrow\cap \text{Spec } L)$. Hence

$$\varphi^A_p = \varphi^A_{\bigwedge(\uparrow\cap \text{Spec } L)} = \bigvee_{q \in \uparrow\cap \text{Spec } L} \varphi^A_q.$$

(2) $\Rightarrow$ (1): Let $S \subseteq \text{Spec } L$ be downdirected. We first note that $\bigwedge S \in \text{Spec } L$ (cf. [9, Sublemma 3.1.5.1]). Indeed, 
let $x, y \in L$ such that $x \leq y$. Then there exist $x_s, x_y \in S$ such that $x \neq x_s$ and $y \neq x_y$. Since $S$ is downdirected, 
there exist $x \in S$ such that $s \leq x_s \wedge x_y$, and $x \in \text{Spec } L$ implies that $x \wedge y \neq s$. Hence $x \wedge y \neq \bigwedge S$.

On the other hand, if $p, q \in \text{Spec } L$ with $p \leq q$, then $\uparrow\cap \text{Spec } L \supseteq \uparrow\cap \text{Spec } L$ and so $\varphi^A_p \geq \varphi^A_q$. Hence

$$\varphi^A_{\bigwedge S} \geq \bigvee_{s \in S} \varphi^A_s.$$ 

Finally, for each $q \in \uparrow(\bigwedge S) \cap \text{Spec } L$ there is some $s_q \in S$ such that $s_q \leq q$ and so $\varphi^A_q \leq \varphi^A_{s_q} \leq \bigvee_{s \in S} \varphi^A_s$. We 
conclude that $\varphi^A_{\bigwedge S} = \bigvee(\varphi^A_q : q \in \uparrow(\bigwedge S) \cap \text{Spec } L) \leq \bigvee_{s \in S} \varphi^A_s$. $\square$

In view of Proposition 3.1 and Remarks 2.4, it is natural to consider the following extension of the definition of a 
chain-valued frame to the less restrictive case of completely distributive lattices.

**Definition 3.2 (L-frame).** Let $L$ be a completely distributive lattice. An $L$-frame $A$ is a system

$$(\varphi^A_p : A^u \to A^l|p \in \text{Spec } L)$$

of frame morphisms—$A^u$ is the upper frame and $A^l$ is the lower frame—satisfying each of these conditions:

(F0) $\varphi^A_p = \bigvee_{q \in \uparrow\cap \text{Spec } L} \varphi^A_q$ for every $p \in \text{Spec } L$.

(F1) $A^l = (\bigcup_{p \in \text{Spec } L} \varphi^A(A^u))$. (collectionwise extremely epimorphic)

(F2) If $a \neq b$ then $\varphi^A_p(a) \neq \varphi^A_p(b)$ for some $p \in \text{Spec } L$. (collectionwise monomorphic)

An $L$-frame morphism $h : A \to B$ is an ordered pair of frame morphisms

$$h^u : A^u \to B^u \quad \text{and} \quad h^l : A^l \to B^l$$

satisfying

$$\forall p \in \text{Spec } L, \ h^l \circ \varphi^A_p = \varphi^B_p \circ h^u.$$ 

The resulting category, with composition and identities component-wise in Frm, is denoted by $L$-Frm.
We recall also the upper and lower forgetful functors
\[ U^u : L\text{-Frm} \to \text{Frm}, \quad U^l : L\text{-Frm} \to \text{Frm} \]
defined by sending \((\phi^A_p : A^u \to A^l|p \in \text{Spec } L)\) to \(A^u\) (resp. \(A^l\)), and \((h^u, h^l)\) to \(h^u\) (resp. \(h^l\)).

Note that because of Proposition 3.1, the new notion coincides with that of [8–10] when \(L\) is a complete chain.

It is now easy to check that all the results proved in Section 4 of [9] can now be extended to this new setting. The proofs of the results included here are exactly the same as those of the corresponding results in [9] and hence we omit them. The only changes to be done are to consider the set Spec \(L\) instead of \(L^*\) and the reformulation of axiom (F0) considered here.

We omit here all the categorical background needed which can be found in [9].

Following [9], let \(\overline{\text{Frm}}_0\) and \(\overline{\text{Frm}}_1\) be the categories in which the objects are frame morphisms for which only (F0) (resp. (F0) and (F1)) is (resp. are) satisfied.

Lemma 3.3. (Cf. Pultr and Rodabaugh [9, Lemma 4.3.1]). Let \(D : C \to \overline{\text{Frm}}_0\) be a diagram in \(\overline{\text{Frm}}_0\), and for \(\tau = u, l\), let
\[
(\pi_i^\tau : A^\tau \to U^\tau D(i))_{i \in |C|}, \quad (\gamma_i^\tau : U^\tau D(i) \to A^\tau)_{i \in |C|}
\]
be the limit, resp. colimit in \(\text{Frm}\) of \(U^\tau D\). Then there is exactly one system
\[
A \equiv (\phi_i^A : A^u \to A^l|p \in \text{Spec } L)
\]
in \(\overline{\text{Frm}}_0\) (up to isomorphism) such that
\[
((\pi_i^{u/l}), \pi_i^{u/l}) : A \to D(i))_{i \in |C|}, \quad ((\gamma_i^{u/l}), \gamma_i^{u/l}) : D(i) \to A)_{i \in |C|}
\]
is the limit, resp. colimit of \(D\) in \(\overline{\text{Frm}}_0\). Consequently \(\overline{\text{Frm}}_0\) is complete and cocomplete and each of the forgetful functors
\[ U^u, U^l : \overline{\text{Frm}}_0 \to \text{Frm} \]
preserves all limits and colimits.

Proof. As in [9, Lemma 4.3.1]. \(\square\)

Lemma 3.4. (Cf. Pultr and Rodabaugh [9, Lemma 4.4.1]). \(\overline{\text{Frm}}_1\) is mono-coreflective in \(\overline{\text{Frm}}_0\). Consequently, \(\overline{\text{Frm}}_1\) inherits colimits from \(\overline{\text{Frm}}_0\), and the limits of \(\overline{\text{Frm}}_1\) are the coreflections of limits of \(\overline{\text{Frm}}_0\). Hence \(\overline{\text{Frm}}_1\) is complete and cocomplete.

Proof. As in [9, Lemma 4.4.1]. \(\square\)

Theorem 3.5. (Cf. Pultr and Rodabaugh [9, Theorem 4.5.1]). \(L\text{-Frm}\) is epi-reflective in \(\overline{\text{Frm}}_1\). Consequently, \(L\text{-Frm}\) inherits limits from \(\overline{\text{Frm}}_1\), and the colimits of \(L\text{-Frm}\) are the reflections of colimits of \(\overline{\text{Frm}}_1\). Hence \(L\text{-Frm}\) is complete and cocomplete.

Proof. As in [9, Theorem 4.5.1]. \(\square\)

4. Possible extensions to more general underlying lattices

This section, and in fact the whole paper, is motivated by Section 4.4 in [10], when the authors suggest that relaxing the condition of a complete chain is a significant question, particularly in Section 4 of [9].

In the previous section we have already specified an answer by proving that the condition of a complete chain can be relaxed to a completely distributive lattice and that the completeness and cocompleteness of \(\overline{\text{Frm}}_0\), \(\overline{\text{Frm}}_1\) and \(L\text{-Frm}\) are still satisfied.

In this context the natural question arises whether weakening of complete distributivity is still possible. As an answer to this question we show that complete distributivity is necessary for the property that for every \(L\)-topological space \((X, \tau)\) the system
\[
(t_p : \tau \to t_L(\tau)|p \in \text{Spec } L)
\]
of frame homomorphisms \(t_p\) satisfies (F0) and (F2).
Proposition 4.1. Let $L$ be a complete lattice. If the system of frame morphisms 

$$(t_p : \tau \to t_L(\tau)|p \in \text{Spec } L)$$

satisfies axiom (F0) for each $L$-topological space $(X, \tau)$, then 

$$p = \bigwedge(\uparrow p \cap \text{Spec } L) \text{ for each } p \in \text{Spec } L.$$ 

Proof. Choose $X = L$ and $\tau = \{1, \emptyset, 1_L\} \subseteq L^X$ (where $1_\emptyset(x) = 0$, $\text{Id}(x) = x$ and $1_L(x) = 1$ for each $x \in L$). By hypothesis we have that $t_p = \bigcup_{q \in \uparrow p \cap \text{Spec } L} Lq$ for each $p \in \text{Spec } L$ and so 

$$L \setminus p = [\emptyset \neq p] = t_p(\text{Id}) = \bigcup_{q \in \uparrow p \cap \text{Spec } L} t_q(\text{Id}) = \bigcup_{q \in \uparrow p \cap \text{Spec } L} [\emptyset \neq q] = \bigcup_{q \in \uparrow p \cap \text{Spec } L} L \setminus q$$ 

for each $p \in \text{Spec } L$. It follows that $\alpha \leq p$ if and only if $\alpha \leq q$ for each $q \in \uparrow p \cap \text{Spec } L$ and so $p = \bigwedge(\uparrow p \cap \text{Spec } L)$. \hfill $\Box$

Proposition 4.2. Let $L$ be a frame. If the system of frame morphisms 

$$(t_p : \tau \to t_L(\tau)|p \in \text{Spec } L)$$

satisfies axiom (F2) for each $L$-topological space $(X, \tau)$, then $L$ is spatial.

Proof. Choose $\tau = \{ \emptyset, 1 \} \subseteq L$. Note that $\tau$ is an $L$-topology because $L$ is a frame. Let $\alpha \neq \beta \in L$. By hypothesis there exists $p \in \text{Spec } L$ such that $t_p(\emptyset) \neq t_p(1)$ and so 

$$[\emptyset \neq p] = t_p(\emptyset) \neq t_p(1) = [1 \neq p].$$

Hence either $\alpha \neq p$ and $\beta \leq p$ or $\alpha \leq p$ and $\beta \neq p$. We conclude that $L$ is spatial. \hfill $\Box$

Finally, we conclude from Corollary 2.2 and Propositions 4.1 and 4.2:

Corollary 4.3. Let $L$ be a frame. If the system of frame morphisms 

$$(t_p : \tau \to t_L(\tau)|p \in \text{Spec } L)$$

is an $L$-frame for each $L$-topological space $(X, \tau)$, then $L$ is a completely distributive lattice.

5. Concluding remarks and further work

The purpose of this paper was to discuss the open question stated in [9,10] related to the possibility of relaxing the condition of $L$ being a complete chain in the development of $L$-frames. As noticed by the authors (see e.g. [10, Subsection 4.4.3]), significant parts of the above mentioned papers hold for $L$ being a completely distributive lattice. However, the fact that the definition of $L$-frames required $L$ to be a complete chain was the main obstacle to obtain all the results in this more general setting of completely distributive lattices.

In our paper we extend the axioms of many valued frames to the case when the base is a completely distributive lattice. We prove two things: (1) All the results in [9, Section 4] remain valid in the new setting. (2) The condition of the base being completely distributive lattice cannot weakened.

We would also like to mention that a significant part of [4] is based on the notion of $L$-frame as considered in [9,10], i.e. under the assumption of $L$ being a complete chain. Here again it would be interesting to see if that part of [4] in which $L$ is assumed to be a chain can be extended to this new setting.

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