MIXED MIMETIC SPECTRAL ELEMENT METHOD FOR STOKES FLOW: A POINTWISE DIVERGENCE-FREE SOLUTION

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Abstract. In this paper we apply the recently developed mimetic discretization method \[44\] to the mixed formulation of the Stokes problem in terms of the vorticity-velocity-pressure formulation. The mimetic discretization presented in this paper and in \[44\] is a higher-order method for curvilinear quadrilaterals. It relies on the language of differential $k$-forms, $k$-cochains as its discrete counterpart, and the relations between them in terms of the mimetic operators: reduction, reconstruction and projection. The reconstruction consists of a mimetic spectral element method. The most important result of the mimetic framework is the commutation between differentiation at the continuous level with that on the finite dimensional and discrete level. As a result operators like gradient, curl and divergence are discretized exactly. For Stokes flow, this implies a pointwise divergence-free solution. This is confirmed using a set of test cases on both Cartesian and curvilinear meshes. It will be shown that the method preforms optimally for all admissible boundary conditions.

1. Introduction

We consider Stokes flow, which models a viscous, incompressible fluid flow in which the inertial forces are negligible with respect to the viscous forces, i.e. when the Reynolds number is very small, $Re \ll 1$. Since $Re = UL/\nu$, small Reynolds numbers appear when either considering extremely small length scales, or when dealing with a very viscous liquid, or when one treats slow flows. Despite the simple appearance of Stokes flow model, there exists a large number of numerical methods to simulate Stokes flow. They all reduce to two classes of either circumventing the Ladyshenskaya-Babuška-Brezzi (LBB) condition or satisfying this condition, \[29\]. The first class can roughly be split into two subclasses, one is the group of stabilized methods, see e.g. \[9, 36\] and the references therein, the other the group of least-squares methods, see e.g. \[10, 40\].

The class that satisfies the LBB condition is the group of compatible methods. In compatible methods discrete vector spaces are constructed such that they satisfy the discrete LBB condition. The most famous are the curl conforming Nédélec \[49\] and divergence conforming Raviart-Thomas \[58\] and Brezzi-Douglas-Marini \[16\] spaces. A subclass of compatible methods are mimetic methods. Mimetic methods do not solely search for appropriate vector spaces, but try to mimic structures and symmetries of the continuous problem, see \[11, 15, 44, 46, 51, 63\]. As a consequence of this mimicking, mimetic methods automatically satisfy the compatibility criteria.

In this work we use the language of differential geometry to identify these structures, since it clearly identifies the metric and metric-free part of the differential models. The latter has a discrete counterpart in the language of algebraic topology. In mimetic methods we employ commuting diagrams to indicate the strong analogy between differential geometry and algebraic topology. The most important commuting property employed in this work is the commutation between the projection operator and differentiation in terms of the exterior derivative. This means that also in finite dimensional spaces, operations like gradient, curl and divergence are performed exactly. This implies, among others, and most importantly that incompressible Navier-Stokes...
and Stokes flow are guaranteed to be pointwise divergence-free, because the projection operator commutes with the divergence operator.

The similarities between differential geometry and algebraic topology in physical theories were first described by Tonti, [63]. A mimetic framework relating differential forms and cochains was initiated by Hyman and Scovel, [58], and extended first by Bochev and Hyman, [11], and later by Kreeft, Palha and Gerritsma [44]. A framework, closely related to the mentioned mimetic framework, is the finite element exterior calculus framework by Arnold, Falk and Winther [5, 6]. A more geometric approach is described in the work by Desbrun et al. [22, 23]. An excellent introduction and motivation for the use of differential forms in the description of physics and the use in numerical modeling can be found in the 'Japanese papers' by Bossavit, [12, 13].

We make use of spectral element interpolation functions as basis functions. In the past nodal spectral elements were mostly used in combination with Galerkin (GSEM) [8, 41], and least-squares formulations (LSSEM) [53, 55]. The GSEM satisfies the LBB compatibility condition by lowering the polynomial degree of the pressure by two with respect to the velocity. The GSEM method is only weakly divergence-free, meaning that the divergence of the velocity field only converges to zero with mesh refinement. The LSSEM circumvents the LBB condition in order to be able to use equal order polynomials. The drawback of this method is the poor mass conservation property, [42, 56].

The present study uses mimetic spectral element interpolation or basis functions on curvilinear quadrilaterals of arbitrary order as described in [31, 44]. The mixed mimetic spectral element method (MMSEM) satisfies the LBB condition and gives a pointwise divergence-free solution for all mesh sizes. The mimetic spectral element interpolation functions are tensor product based interpolants. In every direction either a nodal or an edge interpolation function is used. By using tensor products, we are able to interpolate points, lines, surfaces, volumes, hyper-volumes and higher degree n-cube manifolds.

Although mimetic spectral elements are used to simulate Stokes flow and to derive numerical properties, alternative compatible/mimetic functions could be used without much change, e.g. compatible B-splines, as in Buffa [17, 18], Evans [27] and mimetic B-splines as in Hiemstra [35].

This paper is organized as follows: first in Section 2 a brief summary of the most important concepts from differential geometry is given. In Section 3 the Stokes model is formulated in terms of differential forms and a mixed formulation is formulated. Its well-posedness is proven. Section 4 discusses the discretization of the Stokes model. It introduces algebraic topology and a set of mimetic operators relating differential forms to cochains; the reduction operator, \( R \), the reconstruction operator, \( I \), and its composition, the bounded linear projection, \( \pi_h := I \circ R \). As reconstruction functions the mimetic spectral element basis functions are used in this paper. It is shown that the MMSEM satisfies discrete well-posedness. In Section 5 numerical results are discussed that show optimal convergence of all variables on curvilinear quadrilateral meshes. Secondly, the lid-driven cavity problem is shown on a square as well as a triangle domain. The last testcase is the flow around a cylinder moving with a constant velocity.

## 2. Differential geometry

This paper presents the Stokes model in the language of differential forms. Differential forms offer significant benefits in the construction of structure-preserving spatial discretizations. For example, the generalization of differentiation in terms of the exterior derivative resembles the gradient, curl and divergence operators from vector calculus, and the generalized Stokes theorem resembles their corresponding integration theorems, respectively. The coordinate-free action of the exterior derivative and generalized Stokes theorem give rise to commuting properties with respect to mappings between different manifolds. These kind of commuting properties are essential for the structure preserving behavior of the mimetic method.

Only those concepts from differential geometry which play a role in the remainder of this paper will be explained. Much more can be found in [2, 28, 30, 44].

### 2.1. Differential forms

Let \( \Lambda^k(\Omega) \) denote a space of differential forms or \( k \)-forms, on a sufficiently smooth bounded \( n \)-dimensional oriented manifold \( \Omega \subset \mathbb{R}^n \) with boundary \( \partial \Omega \). Every
element \( a \in \Lambda^k(\Omega) \) has a unique representation of the form\(^1\)
\[
(2.1) \quad a = \sum_i f_i(x) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k},
\]
where \( i_1 < \ldots < i_k \) and where \( f_i(x) \) is an \( l \)-times continuously differentiable scalar function, \( f_i(x) \in C^l(\Omega) \). Examples of 0-forms, \( a \in \Lambda^0(\Omega) \), 1-forms, \( b \in \Lambda^1(\Omega) \), and 2-forms, \( c \in \Lambda^0(\Omega) \), in \( \Omega \subset \mathbb{R}^2 \), with scalar functions, \( f, g, h \), respectively, are given below,
\[
a = f(x,y), \quad b = g_1(x,y)dx + g_2(x,y)dy, \quad c = h(x,y)dx \wedge dy.
\]
Differential \( k \)-forms are naturally integrated over \( k \)-dimensional manifolds, i.e. for \( a \in \Lambda^k(\Omega) \) and \( \Omega_k \subset \mathbb{R}^n \), with \( k = \dim(\Omega_k) \),
\[
(2.2) \int_{\Omega_k} a \in \mathbb{R} \iff \langle a, \Omega_k \rangle \in \mathbb{R},
\]
where \( \langle \cdot, \cdot \rangle \) indicates a duality pairing between the differential form and the geometry. Note that the \( n \)-dimensional computational domain is indicated as \( \Omega \), so without subscript. The wedge product, \( \wedge \), of two differential forms \( a \in \Lambda^k(\Omega) \) and \( b \in \Lambda^l(\Omega) \) is a mapping: \( \wedge : \Lambda^k(\Omega) \times \Lambda^l(\Omega) \to \Lambda^{k+l}(\Omega) \), \( k+l \leq n \). The wedge product is a skew-symmetric operator, i.e. \( a \wedge b = (-1)^{kl} b \wedge a \).

The pointwise inner-product of \( \Lambda^k \) and \( \Lambda^l \) forms, \( \langle \cdot , \cdot \rangle : \Lambda^k(\Omega) \times \Lambda^l(\Omega) \to \mathbb{R} \), is constructed using inner products of one-forms, that is based on the inner product on vector spaces, see [28, 30]. Let \( a, b \in \Lambda^k(\Omega) \) and \( \alpha_i, \beta_j \in \Lambda^1(\Omega) \), \( i, j = 1, \ldots, k \), then
\[
(2.3) \quad \langle a, b \rangle = (\alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k) := \det \{ (\alpha_i, \beta_j) \}.
\]

The wedge product and inner product induce the Hodge-* operator, \( * : \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega) \), a metric operator that includes orientation. Let \( a, b \in \Lambda^k(\Omega) \), then
\[
(2.4) \quad a \wedge b := \langle a, b \rangle \sigma,
\]
where \( \sigma \in \Lambda^n(\Omega) \) is a unit volume form, \( \sigma = +1 \). The space of \( k \)-forms on \( \Omega \) can be equipped with an \( L^2 \) inner product, \( \langle \cdot , \cdot \rangle : \Lambda^k(\Omega) \times \Lambda^k(\Omega) \to \mathbb{R} \), given by,
\[
(2.5) \quad \langle a, b \rangle := \int_{\Omega} (a, b) \sigma = \int_{\Omega} a \wedge b.
\]

The differential forms live on manifolds and transform under the action of mappings. Let \( \Phi : \Omega_{\text{ref}} \to \Omega \) be a mapping between two manifolds. Then we can define the pullback operator, \( \Phi^* : \Lambda^k(\Omega) \to \Lambda^k(\Omega_{\text{ref}}) \), expressing the \( k \)-form on the reference manifold, \( \Omega_{\text{ref}} \). The mapping, \( \Phi \), and the pullback, \( \Phi^* \), are related by
\[
(2.6) \quad \int_\Omega a = \int_{\Phi(\Omega_{\text{ref}})} \Phi^* a \iff \langle a, \Phi(\Omega_{\text{ref}}) \rangle = \langle \Phi^* a, \Omega_{\text{ref}} \rangle, \quad \forall a \in \Lambda^k(\Omega).
\]
A special case of the pullback operator is the trace operator. The trace of \( k \)-forms to the boundary, \( \text{tr} : \Lambda^k(\Omega) \to \Lambda^k(\partial\Omega) \), is the pullback of the inclusion of the boundary of a manifold, \( \partial\Omega \mapsto \Omega \), see [44].

An important operator in differential geometry is the exterior derivative, \( d : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega) \). It is induced by the generalized Stokes’ theorem, combining the classical Newton-Leibnitz, Stokes circulation and Gauss divergence theorems. Let \( \Omega_{k+1} \) be a \((k+1)\)-dimensional manifold and \( a \in \Lambda^k(\Omega) \), then
\[
(2.7) \quad \int_{\partial\Omega_{k+1}} a = \int_{\Omega_{k+1}} da \iff \langle a, \partial\Omega_{k+1} \rangle = \langle da, \Omega_{k+1} \rangle,
\]
where \( \partial\Omega_{k+1} \) is a \((k)\)-dimensional manifold being the boundary of \( \Omega_{k+1} \). Due to the duality pairing in (2.7), the exterior derivative is the formal adjoint of the boundary operator \( \partial : \Omega_{k+1} \to \Omega_k \). The exterior derivative is independent of any metric and coordinate system. Applying the exterior derivative twice always leads to the null \((k+2)\)-form, \( d(da) = 0^{(k+2)} \) for all \( a \in \Lambda^k(\Omega) \). This property ensures that the range, \( B^k := d\Lambda^{k-1}(\Omega) \subset \Lambda^k(\Omega) \), of the exterior derivative on \((k-1)\)-forms is contained in the nullspace, \( \mathcal{Z}^k := \{ \forall a \in \Lambda^k(\Omega) \mid da = 0 \} \subset \Lambda^k(\Omega) \), of the exterior derivative.

\(^1\)Occasionally we write for \( a \in \Lambda^k(\Omega), a^{(k)} \), to emphasize the kind of differential form.
derivative on $k$-forms, $\mathcal{B}^k \subseteq \mathcal{Z}^k$. As a consequence, on contractible domains the exterior derivative gives rise to an exact sequence, called De Rham complex \([30]\), and indicated by $(\Lambda, d)$,

$$
\mathbb{R} \hookrightarrow \Lambda^0(\Omega) \xrightarrow{d} \Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n(\Omega) \xrightarrow{d} 0.
$$

In vector calculus a similar sequence exists, where, from left to right for $\mathbb{R}^3$, the $d$'s denote the vector operators grad, curl and div. Every space of differential forms in the complex can be decomposed into the nullspace of $d$, and its complement part, $\Lambda^k(\Omega) = Z^k \oplus Z^k,_{\perp}$. The exterior derivative and wedge product are related according to Leibniz's rule as

$$
d(a \wedge b) = da \wedge b + (-1)^k a \wedge db, \quad \text{for } a \in \Lambda^k(\Omega), \ b \in \Lambda^l(\Omega), \ k + l < n.
$$

Furthermore, the pullback operator and exterior derivative possess the following commuting property,

$$
\Phi^* da = d\Phi^* a, \quad \forall a \in \Lambda^k(\Omega),
$$

as illustrated in the following commuting diagram,

$$
\Lambda^k(\Omega) \xrightarrow{d} \Lambda^{k+1}(\Omega) \quad \downarrow \Phi^* \quad \downarrow \Phi^*
$$

$$
\Lambda^k(\Omega_{ref}) \xrightarrow{d} \Lambda^{k+1}(\Omega_{ref}).
$$

The inner product gives rise to the formal adjoint of the exterior product, the codifferential operator, $d^* : \Lambda^k(\Omega) \rightarrow \Lambda^{k-1}(\Omega)$,

$$
(d a, b)_\Omega = (a, d^* b)_\Omega, \quad \forall a \in \Lambda^{k-1}(\Omega), \ b \in \Lambda^k(\Omega).
$$

In case of non-zero trace, and by combining (2.5), (2.7) and (2.9), we get

$$
(a, d^* b)_\Omega = ( da, b)_\Omega - \int_{\partial \Omega} \operatorname{tr} a \wedge \operatorname{tr} \ast b.
$$

Also the codifferential operator, is nilpotent, $d^*(d^* a) = 0$, i.e., its range is contained in its nullspace, $\mathcal{B}^* \subseteq \mathcal{Z}^*$. where $\mathcal{B}^* := d^* \Lambda^{k+1}(\Omega) \subset \Lambda^k(\Omega)$ and $\mathcal{Z}^* := \{ \forall a \in \Lambda^k(\Omega) \mid d^* a = 0 \} \subset \Lambda^k(\Omega)$. On contractible manifolds this gives rise to the following exact sequence,

$$
0 \xrightarrow{d^*} \Lambda^0(\Omega) \xrightarrow{d^*} \Lambda^1(\Omega) \xrightarrow{d^*} \cdots \xrightarrow{d^*} \Lambda^n(\Omega) \xrightarrow{d^*} \mathbb{R}.
$$

Again, in vector notation from right to left the $d^*$'s denote again the grad, curl and div operators in $\mathbb{R}^3$. However, whereas the exterior derivative is a metric-free operator, the codifferential operator is metric-dependent. The Hodge-Laplace operator, $\Delta : \Lambda^k(\Omega) \rightarrow \Lambda^k(\Omega)$, is constructed as a composition of the exterior derivative and the codifferential operator,

$$
-\Delta a := (d^* d + dd^*) a, \quad \forall a \in \Lambda^k(\Omega).
$$

2.2. Hilbert spaces. Function spaces play an important role in the analysis of numerical methods. Of importance in this paper are the Hilbert spaces. On an oriented Riemannian manifold, we can define Hilbert spaces for differential forms. Let all $f_i(x)$ in \([2.1]\) be functions in $L^2(\Omega)$, then $a$ in \([2.1]\) is a $k$-form in the Hilbert space $L^2\Lambda^k(\Omega)$. The norm corresponding to the space $L^2\Lambda^k(\Omega)$ is $\|a\|_{L^2\Lambda^k} = \sqrt{(a, a)_\Omega}$ or simply $\|a\|$. Although extension to higher Sobolev spaces are possible, we focus here on the Hilbert space corresponding to the exterior derivative. The Hilbert space $H^k\Lambda(\Omega)$ is defined by

$$
H^k\Lambda(\Omega) = \{ a \in L^2\Lambda^k(\Omega) \mid da \in L^2\Lambda^{k+1}(\Omega) \},
$$

and the norm corresponding to $H^k\Lambda(\Omega)$ is defined as

$$
\|a\|^2_{H^k\Lambda} := \|a\|^2_{L^2\Lambda^k} + \|da\|^2_{L^2\Lambda^{k+1}}.
$$

The $H^k\Lambda$-semi-norm of $a$ is the $L^2$-norm of the exterior derivative of $a$, $|a|_{H^k\Lambda} = \|da\|_{L^2\Lambda^{k+1}}$. Semi-norms in higher degree Sobolev spaces, $H^m\Lambda^k$, measure the $m$th derivative of the functions $f_i(x)$ of the $k$-form, $a$. The full norm, $\|a\|_{H^m\Lambda^k}$, can be obtained by taken the square root of the
sum over all $H^j \Lambda^k$-semi-norms, $0 \leq j \leq m$. The $L^2$-de Rham complex, $(H \Lambda, d)$, or the Hilbert version of the de Rham complex, is the exact sequence of maps and spaces given by

$$\mathbb{R} \hookrightarrow H^{0}(\Omega) \xrightarrow{d} H^{1}(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H^{m}(\Omega) \xrightarrow{d} 0.$$  

In vector operations the $L^2$-de Rham complex becomes for $\Omega \subset \mathbb{R}^3$,

$$H^{1}(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^{2}(\Omega),$$

and for $\Omega \subset \mathbb{R}^2$, either

$$H^{1}(\Omega) \xrightarrow{\text{grad}} H(\text{rot}, \Omega) \xrightarrow{\text{rot}} L^{2}(\Omega), \quad \text{or} \quad H^{1}(\Omega) \xrightarrow{\text{curl}} H(\text{curl}, \Omega) \xrightarrow{\text{div}} L^{2}(\Omega).$$

The two are related by the Hodge-$\star$ operator (2.4), see [50],

$$(2.19) \quad H^{0}(\Omega) \xrightarrow{\star} H^{1}(\Omega) \xrightarrow{\star} L^{2}(\Omega) \xrightarrow{\star} H^{1}(\Omega) \xrightarrow{\star} H^{0}(\Omega) \xrightarrow{\star} L^{2}(\Omega) \xrightarrow{\star} H^{1}(\Omega).$$

Similar double Hilbert complex can be constructed in $\mathbb{R}^3$. An important inequality in stability analysis, relating the $L^2 \Lambda^k$-norm and the $H \Lambda^k$-norm, is the Poincaré inequality.

**Lemma 1 (Poincaré inequality).** [6] Consider the de Rham complex $(\Lambda, d)$, then the exterior derivative is a bounded bijection from $Z^k \perp$ to $B^k$, and hence, by Banach’s bounded inverse theorem, there exists a constant $c_P$ such that

$$\| a \|_{H \Lambda^k} \leq c_P \| d a \|_{L^2 \Lambda^k}, \quad \forall a \in Z^k \perp.$$

### 3. Mixed formulation for the Stokes problem

#### 3.1. Stokes problem in vector notation.**

Let $\Omega \subset \mathbb{R}^n$ be a bounded $n$-dimensional domain with boundary $\partial \Omega$. On this domain we consider the Stokes problem, consisting of a momentum equation and the incompressibility constraint, resulting from the conservation of mass. In vector notation Stokes problem is given by

$$(3.1a) \quad -\nu \Delta \vec{u} + \text{grad} \ p = \vec{f}, \quad \text{on} \ \Omega,$$

$$(3.1b) \quad \text{div} \ \vec{u} = 0, \quad \text{on} \ \Omega,$$

with $\vec{u}$ the velocity vector, $p$ the pressure, $\vec{f}$ the forcing term and $\nu$ the kinematic viscosity. For the vorticity-velocity-pressure formulation, the Laplace operator is split using the vector identity, $-\Delta \vec{u} = \text{curl} \ curl \ \vec{u} - \text{grad} \ \text{div} \ \vec{u}$, and by introducing vorticity as auxiliary variable, $\omega = \text{curl} \ \vec{u}$. In terms of a system of first-order partial differential equations, the Stokes problem becomes

$$(3.2a) \quad \text{curl} \ \omega + \text{grad} \ p = \vec{f}, \quad \text{on} \ \Omega,$$

$$(3.2b) \quad \omega - \text{curl} \ \vec{u} = 0, \quad \text{on} \ \Omega,$$

$$(3.2c) \quad \text{div} \ \vec{u} = 0, \quad \text{on} \ \Omega.$$

In case of velocity boundary conditions the pressure is only determined up to a constant. So as a post process step either the pressure in a point in $\Omega$ can be set, or a zero average pressure can be imposed; i.e.

$$(3.3) \quad \int_{\Omega} p \ d\Omega = 0.$$

3.2. Stokes problem in terms of differential forms. The kind of form a variable has is directly related to the kind of manifold this variable can be integrated over. For example, from a physics point of view velocity is naturally integrated along a line (stream line), a 1-manifold, indicating that velocity is a 1-form. However, from a more mathematical / finite-volume point of view, velocity in incompressible (Navier)-Stokes equations is often associated to a flux through a surface, indicating that velocity should be an \((n-1)\)-form \((n=\text{dim}(\Omega))\). The two are directly related by the Hodge duality, \(u^{(n-1)} = \ast \tilde{u}^{(1)}\), see \((2.19)\). The Hodge-\(\ast\) not only changes the corresponding type of integral domain, but also its orientation (along a line = inner, through a surface = outer).

Note that the Hodge-\(\ast\) is often combined with a constitutive relation. In that case the two variables have clearly a different meaning. In incompressible flow models, mass density plays the role of material property, so we actually have \((\rho u)^{(n-1)} = \ast \rho \tilde{u}^{(1)}\). Since mass density is assumed to be equal to one in incompressible (Navier)-Stokes, this difference is less obvious.

As for the velocity, also for pressure and vorticity there exists an inner and outer oriented version. The inner oriented variables are pressure, \(\tilde{p} \in \Lambda^0(\Omega)\), associated to point values, vorticity and \(\tilde{\omega} \in \Lambda^2(\Omega)\), associated to circulation in a surface. Alternatively, there exists the set of outer-oriented variables, being the pressure, \(p \in \Lambda^n(\Omega)\), measured in a volume, and vorticity, \(\omega \in \Lambda^{n-2}(\Omega)\), corresponding to circulation around a line (both in case of \(\Omega \subset \mathbb{R}^n\), where \(n = 3\)).

Both sets, \((\tilde{p}^{(0)}, \tilde{\omega}^{(1)}, \omega^{(2)})\) and \((\omega^{(n-2)}, u^{(n-1)}, p^{(n)})\) are used in literature to derive mixed formulations and numerical schemes. For the former see \cite{1} \cite{14} and for the latter see \cite{7} \cite{25}.

To obtain a pointwise divergence-free solution, the set of outer-oriented variables are used in this paper, \((\omega, u, p) \in \{\Lambda^{n-2}(\Omega) \times \Lambda^{n-1}(\Omega) \times \Lambda^n(\Omega)\}\), with forcing term \(f \in \Lambda^{n-1}(\Omega)\). Then the Stokes problem in terms of differential forms becomes,

\begin{align}
(3.4a) & \quad -\nu \Delta u + \ast d^* p = f, \quad \text{on } \Omega, \\
(3.4b) & \quad d u = 0, \quad \text{on } \Omega,
\end{align}

where \(\Delta\) is the Hodge-Laplacian defined \((2.14)\). Vorticity was introduced as auxiliary variable to cast this system into a system of first-order equations. Substitution of \((2.14)\) and the incompressibility constraint \((3.4b)\), gives the vorticity-velocity-pressure formulation in terms of differential forms,

\begin{align}
(3.5a) & \quad \omega - d^* u = 0, \quad \text{on } \Omega, \\
(3.5b) & \quad \nu d\omega + d^* p = f, \quad \text{on } \Omega, \\
(3.5c) & \quad d u = 0, \quad \text{on } \Omega.
\end{align}

The actions of the exterior derivatives and codifferentials in this system are illustrated below for a two-dimensional domain.

Example 1 (2D Stokes problem). Let \(\Omega \subset \mathbb{R}^2\) and let the two-dimensional de Rham complex be equivalent to the second complex in \((2.18)\). Then velocity is expressed as

\[ u = -v(x, y)dx + u(x, y)dy. \]

Applying the exterior derivative gives us a 2-form, the divergence of velocity,

\[ d u = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dx \wedge dy. \]

Vorticity is a 0-form, \(\omega = \omega(x, y) \in \Lambda^0(\Omega)\), and the curl of vorticity gives,

\[ d \omega = \frac{\partial \omega}{\partial x} dx + \frac{\partial \omega}{\partial y} dy. \]

The gradient of pressure, \(p = p(x, y)dx \wedge dy \in \Lambda^2(\Omega)\), is the action of the codifferential,

\[ d^* p = -\frac{\partial p}{\partial y} dx + \frac{\partial p}{\partial x} dy. \]

\footnote{With \(\ast\) we indicate a variable contained in the lower complex.}
Then the momentum equation follows,
\[
- \left( \frac{\partial \omega}{\partial x} + \frac{\partial p}{\partial y} \right) dx + \left( \frac{\partial \omega}{\partial y} + \frac{\partial p}{\partial x} \right) dy = -f(y)(x,y)dx + f_x(x,y)dy.
\]
In a similar way the vorticity-velocity relation can be obtained.

3.3. Mixed formulation, boundary conditions and well-posedness. We know how to discretize exactly the metric-free exterior derivative \( d \) (see Lemma [3] Section 4.2), but it is less obvious how to treat the codifferential operator \( d^* \). Fortunately, the two are directly related using \( L^2 \)-inner products as seen in (2.12). Therefore the derivation of the mixed formulation of the Stokes problem consists of two steps: 1). Multiply equations (3.5a)-(3.5c) by the test functions \( L \).

\[
\Lambda \in \{\Lambda^{n-2} \times \Lambda^{n-1} \times \Lambda^n\}
\]

obvious how to treat the codifferential operator \( d^* \) and \( \Lambda \), such that \( \forall \tau \in \Lambda^{n-2}(\Omega) \),

\[
\Omega = \Lambda^{n-2}(\Omega) \times \Lambda^{n-1}(\Omega) \times \Lambda^n(\Omega),
\]

such that

\[
(3.6a) \quad (\tau, \omega)_\Omega - (d\tau, u)_\Omega + \int_{\partial \Omega} \tau \wedge \tau \ast u = 0,
\]

\[
(3.6b) \quad (v, d\omega)_\Omega + (dv, p)_\Omega - \int_{\partial \Omega} v \wedge \tau \ast p = (v, f)_\Omega,
\]

\[
(3.6c) \quad (q, du)_\Omega = 0.
\]

System (3.5) needs to be supplemented with boundary conditions on \( \partial \Omega \). Subdivide the boundary into several parts, \( \partial \Omega = \bigcup_i \Gamma_i \), where \( \Gamma_i \cap \Gamma_j = \emptyset \) for \( i \neq j \). Each part of the boundary can have one of the following four boundary conditions: 1. prescribed velocity (such as no-slip), 2. tangential velocity - pressure, 3. tangential vorticity - normal velocity, and 4. tangential vorticity - pressure boundary conditions. Specific values for analysis purposes are given in Table 1.

For analysis purposes we choose values of the boundary conditions such that the boundary integrals vanishes. To show well-posedness of the mixed formulation (3.6), we define the following bounded bilinear form \( B : [H^{n-2} \times H^{n-1} \times L^2 \Lambda^n] \times [H^{n-2} \times H^{n-1} \times L^2 \Lambda^n] \rightarrow \mathbb{R} \) for the homogeneous Stokes problem,

\[
B(\omega, u, p; \tau, v, q) = (\tau, \omega)_\Omega - (d\tau, u)_\Omega + (v, d\omega)_\Omega + (dv, p)_\Omega - (q, du)_\Omega.
\]

<table>
<thead>
<tr>
<th>Name</th>
<th>Exterior Calculus</th>
<th>Vector Calculus</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal velocity</td>
<td>( \text{tr } u \Rightarrow \text{tr } v = 0 )</td>
<td>( \vec{u} \cdot \vec{n} \Rightarrow \vec{v} \cdot \vec{n} = 0 )</td>
<td>essential</td>
</tr>
<tr>
<td>tangential velocity</td>
<td>( \text{tr } \ast u )</td>
<td>( \vec{u} \times \vec{n} )</td>
<td>natural</td>
</tr>
<tr>
<td>Tangential velocity</td>
<td>( \text{tr } \ast u )</td>
<td>( \vec{u} \times \vec{n} )</td>
<td>natural</td>
</tr>
<tr>
<td>pressure</td>
<td>( \text{tr } \ast p )</td>
<td>( p )</td>
<td>natural</td>
</tr>
<tr>
<td>Tangential vorticity</td>
<td>( \text{tr } \omega \Rightarrow \text{tr } \tau = 0 )</td>
<td>( \vec{\omega} \times \vec{n} \Rightarrow \vec{\tau} \times \vec{n} = 0 )</td>
<td>essential</td>
</tr>
<tr>
<td>normal velocity</td>
<td>( \text{tr } u \Rightarrow \text{tr } v = 0 )</td>
<td>( \vec{u} \cdot \vec{n} \Rightarrow \vec{v} \cdot \vec{n} = 0 )</td>
<td>essential</td>
</tr>
<tr>
<td>Tangential vorticity</td>
<td>( \text{tr } \omega \Rightarrow \text{tr } \tau = 0 )</td>
<td>( \vec{\omega} \times \vec{n} \Rightarrow \vec{\tau} \times \vec{n} = 0 )</td>
<td>essential</td>
</tr>
<tr>
<td>pressure</td>
<td>( \text{tr } \ast p )</td>
<td>( p )</td>
<td>natural</td>
</tr>
</tbody>
</table>

Table 1. Admissible boundary conditions for Stokes flow in vorticity-velocity-pressure formulation.
The following theorem shows that system (3.6) is well-posed by proving that the bilinear form (3.7) satisfies the inf-sup condition.

**Theorem 1 (Well-posedness).** Let $(\Lambda, d)$ be a closed de Rham complex. There exists a constant $\beta > 0$, depending only on the constant $c_P$ in the Poincaré inequality, such that for any $(\omega, u, p) \in H^2\tau, v, q) \in H^n \times H^n \times L^2\Lambda^n$ with

\[
B(\omega, u, p; \tau, v, q) \geq \beta (\|\omega\|_{\Lambda^n}^2 + \|u\|_{\Lambda^n}^2 + \|d^* p\|_{L^2\Lambda^n}) (\|\tau\|_{\Lambda^n}^2 + \|v\|_{\Lambda^n}^2 + \|d^* q\|_{L^2\Lambda^n}).
\]

**Proof.** Extensive proof can be found in [10] [43]. Only the key steps are given here. Since $u \in \mathcal{Z}^{n-1}$, $u = d\psi$ for some $\psi \in \mathcal{Z}^{n-2}$. By Poincaré’s inequality we get $\|\psi\|_{\Lambda^{n-2}} \leq c_P \|u\|_{L^2\Lambda^{n-1}}$, where $c_P \geq 1$. Let

\[
(3.8) \quad \tau = \omega - \frac{1}{c_P} \psi \in H^2, \quad v = d\omega + u + d^* p \in H^2, \quad q = p \in \Lambda^n.
\]

This gives the following two bounds,

\[
(3.9) \quad \|\tau\|_{H^2}^2 + \|v\|_{H^2}^2 + \|q\|_{L^2\Lambda^n}^2 \leq C (\|\omega\|_{\Lambda^n}^2 + \|u\|_{\Lambda^n}^2 + \|p\|_{L^2\Lambda^n}^2),
\]

\[
(3.10) \quad B(\omega, u, p; \tau, v, q) \geq \frac{1}{c_P} (\|\omega\|_{\Lambda^n}^2 + \|u\|_{\Lambda^n}^2 + \|d^* p\|_{L^2\Lambda^n}^2).
\]

The theorem easily follows from the bounds (3.9) and (3.10). It shows that $\beta$ only depends on the Poincaré constant $c_P$. \qed

**Corollary 1.** [7] [25] Problem (3.8) is well-posed according to Theorem 1. That is, for any $f \in L^2\Lambda^{n-1}(\Omega)$, there exists a unique solution $(\omega, u, p) \in H^2 \times H^2 \times L^2\Lambda^n$ satisfying (3.6). Moreover, this solution satisfies:

\[
(3.11) \quad \|\omega\|_{H^2}^2 + \|u\|_{H^2}^2 + \|p\|_{L^2\Lambda^n}^2 \leq c \|f\|_{L^2\Lambda^{n-1}},
\]

where $c$ is a constant depending only on the Poincaré constant $c_P$.

4. Discretization of Stokes problem

The mimetic discretization of Stokes model consists of three parts. First, the discrete structure is described in terms of chains and cochains from algebraic topology, the discrete counterpart of differential geometry. This discrete structure mimics a lot of properties of differential geometry. Secondly, mimetic operators are introduced that relate the continuous formulation in terms of differential forms to the discrete representation based on cochains. Thirdly, mimetic spectral basis functions are described which satisfy the structure defined in the algebraic topology and mimetic operators sections. Finally, well-posedness of the discrete numerical formulation is proven.

4.1. Algebraic Topology. In many numerical methods, especially in finite difference and finite element methods, the discrete coefficients are point values, i.e. zero-dimensional sub-manifolds. In the structure of algebraic topology, the discrete coefficients represent values on $k$-dimensional sub-manifolds, ranging from points to $n$-dimensional volumes, so $0 \leq k \leq n$. These $k$-dimensional sub-manifolds are called $k$-cells, $\tau(k)$. See [44] [48] [49] how they are formally defined. The two most popular classes of $k$-cells in literature to describe the topology of a manifold are either in terms of simplices, see for instance [40] [62] [65], or in terms of cubes, see [45] [64] and Figure 1 for an example of $k$-cubes in $\mathbb{R}^3$. From a topological point of view both descriptions are equivalent, see [20]. Despite this equivalence between simplicial complexes and cubical complexes, the reconstruction maps to be discussed in Section 4.2 differ significantly. For mimetic methods based on simplices see [16] [17] [22] [57], whereas for mimetic methods based on singular cubes see [18] [19] [39] [40].

Here we list the terminology to setup a homology theory in terms of $n$-cubes as given by [45]. Consider a unit $k$-cube given by $I^k = I \times I \times \cdots \times I$ ($k$ factors, $k \geq 0$), where $I = [-1, 1]$ is a one-dimensional closed interval. By definition $I^0$ is a space consisting of a single point. Then a $k$-cube in an $n$-dimensional manifold $\Omega$ is a continuous map $\tau(k) : I^k \rightarrow \Omega$, $0 \leq k \leq n$. 
The boundary of a $k$-cube is a set of $(k-1)$-cubes and are given by the boundary operator, $\partial$. The boundary operator is constructed using face maps. We define the $(k-1)$-cubes, $A_i\tau(k-1)$, $B_i\tau(k-1)$ : $I^{k-1} \to \Omega$, by the face maps

$$A_i\tau(k-1)(x_1, x_2, \ldots, x_{k-1}) = \tau(k)(x_1, \ldots, x_i, x_{i-1}, x_{i+1}, \ldots, x_{k-1}),$$

$$B_i\tau(k-1)(x_1, x_2, \ldots, x_{k-1}) = \tau(k)(x_1, \ldots, x_i, x_{i-1}, x_{i+1}, \ldots, x_{k-1}).$$

$A_i\tau(k-1)$ is called the front $i$-face and $B_i\tau(k-1)$ is called the back $i$-face of $\tau(k)$. The boundary $\partial$ of a $k$-cube $\tau(k)$ is given by

$$\partial \tau(k) := \sum_{i=1}^{k} (-1)^i \left[ A_i \tau(k-1) - B_i \tau(k-1) \right].$$

This definition describes the boundary which we already encountered in (2.7). The boundary of a $k$-cell $\tau(k)$ is again consists of a set of $(k-1)$-cells, as illustrated in Figure 2. From this we can define a cell complex.

**Definition 1 (Cell complex).** A cell complex, $D$, in a compact manifold $\Omega$ is a finite collection of cells such that:

1. The set of $n$-cells in $D$ covers the manifold $\Omega$.
2. Every face of a cell in $D$ is contained in $D$.
3. The intersection of any two $k$-cells, $\tau(k)$ and $\sigma(k)$ in $D$ either share a common face, is empty, or $\tau(k) = \sigma(k)$.

We call a cell complex an oriented cell complex, once we add to each $k$-cell a default orientation according to the definition of $k$-cubes. See [14] for more on orientation of $k$-cells. Figure 3 depicts an example of a cell complex in a compact manifold $\Omega \subset \mathbb{R}^3$. The ordered collection of all $k$-cells in $D$ generate a basis for the space of $k$-chains, $C_k(D)$. Then a $k$-chain, $c(k) \in C_k(D)$, is a formal linear combination of $k$-cells, $\tau(k), i \in D$,

$$c(k) = \sum_i c_i \tau(k), i.$$
In the description of geometry, we restrict ourselves to chains with coefficients in $\mathbb{Z}/3 = \{-1, 0, 1\}$. The meaning of these coefficients is: 1 if the cell is in the chain with the same orientation as its orientation in the cell complex, -1 if the cell is in the chain with the opposite orientation to the orientation in the cell complex and 0 if the cell is not part of the chain. The orientation of the cell is implied by the orientation of the line segment $I$ and the map $\tau(k)$.

We can now extend the boundary operator applied to a $k$-cell, (4.1), to the boundary of a $k$-chain. The boundary operator, $\partial$, is an homomorphism, $\partial : C_k(D) \to C_{k-1}(D)$, defined by

$$\partial c(k) = \partial \sum_i c^i \tau(k),i := \sum_i c^i \partial \tau(k),i,$$

where the action of the boundary operator on the $k$-cubes is given in (4.1). The boundary of a $k$-cell $\tau(k)$ will then be a $(k-1)$-chain formed by the faces of $\tau(k)$. The coefficients of this $(k-1)$-chain associated to each of the faces is given by the orientations.

$$\partial \tau(k),j = \sum_i e^i_j \tau(k-1),i,$$

with

$$\begin{cases} e^i_j = 1, & \text{if } \tau(k-1),i \text{ has the same orientation of } \tau(k),j \\ e^i_j = -1, & \text{if } \tau(k-1),i \text{ has the opposite orientation of } \tau(k),j \\ e^i_j = 0, & \text{if } \tau(k-1),i \text{ is not a face of } \tau(k),j \end{cases}$$

And the boundary of a 0-cell is $\emptyset$. Hence,

$$\partial c(k) = \partial \sum_j c^j \tau(k),j = \sum_j c^j \partial \tau(k),j = \sum_i \sum_j c^j e^i_j \tau(k-1),i.$$  

Recalling that the space of $k$-chains is a linear vector space it follows that the boundary operator can be represented as a matrix acting on the column vector of the $k$-chain. The coefficients $e^i_j$ are the coefficients of an incidence matrix $E_{(k-1,k)}$ that represents the boundary operator. Like the exterior derivative, applying the boundary operator twice on a $k$-chain gives the null $(k-2)$-chain, $\partial \partial c(k) = 0_{(k-2)}$ for all $c(k) \in C_k(D)$, see Figure 4. This was expected, since the exterior derivative and boundary operator are related according to Stokes theorem, (2.7). This property is reflected in the incidence matrices, since they are matrix representations of the topological

![Figure 3. Example of a cell complex. Left: a three dimensional compact manifold. Right: the $k$-cells that constitute the cell complex.](image)
boundary operators. Therefore $E_{(k-2,k-1)}E_{(k-1,k)} = 0$, where for Figure 4 we have

$$E_{(1,2)} = \begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
\end{bmatrix}, \quad E_{(2,3)} = \begin{bmatrix}
-1 \\
1 \\
1 \\
-1 \\
-1 \\
1 \\
\end{bmatrix}.$$

The set of $k$-chains and boundary operators gives rise to an exact sequence, the chain complex $(C_k(D), \partial)$,

$$\ldots \leftarrow \partial C_{k-1}(D) \leftarrow \partial C_k(D) \leftarrow \partial C_{k+1}(D) \leftarrow \partial \ldots . \tag{4.5}$$

Dual to the space of $k$-chains, $C_k(D)$, is the space of $k$-cochains, $C^k(D)$, defined as the set of all linear functionals, $c^{(k)} : C_k(D) \to \mathbb{R}$. The duality is expressed using the duality pairing $\langle \tau^{(k),i}, c^{(k)} \rangle := c^{(k)}(\tau^{(k),i})$. Note the resemblance between this duality pairing and the integration of differential forms, see (2.2). This similarity forms the basis of the mimetic framework.

Let $\{\tau^{(k),i}\}$ form a basis of $C_k(D)$, then there is a dual basis $\{\tau^{(k),i}\}$ of $C^k(D)$, such that $\tau^{(k),i}(\tau^{(k),j}) = \delta_{ij}$ and all $k$-cochains can be represented as linear combinations of the basis elements,

$$c^{(k)} = \sum_i c_i \tau^{(k),i}. \tag{4.6}$$

The cochains are the discrete analogue of differential forms. With the duality relation between chains and cochains, we can define the formal adjoint of the boundary operator which constitutes an exact sequence on the spaces of $k$-cochains in the cell complex. This formal adjoint is called the \textit{coboundary operator}, $\delta : C^k(D) \to C^{k+1}(D)$, and is defined as

$$\langle \delta c^{(k)}, c^{(k+1)} \rangle := \langle c^{(k)}, \partial c^{(k+1)} \rangle, \quad \forall c^{(k)} \in C^k(D) \text{ and } \forall c^{(k+1)} \in C_{k+1}(D). \tag{4.7}$$

Also the coboundary operator satisfies $\delta^2 = 0$ for all $c^{(k)} \in C^k(D)$, see Figure 5 and gives rise to an exact sequence, called the \textit{cochain complex} $(C^k(D), \delta)$,

$$\ldots \delta \to C^{k-1}(D) \delta \to C^k(D) \delta \to C^{k+1}(D) \delta \to \ldots . \tag{4.8}$$

Also the coboundary operator has a matrix representation. As a result of the duality pairing in (4.7), the matrix representation of the coboundary operator is the transpose of the incidence matrix of the boundary operator, $E^{(k,k-1)} := (E_{(k-1,k)})^T$. And again, $E^{(k+1,k)}E^{(k,k-1)} = 0$. Note that expression (4.7) is nothing but a discrete Stokes’ theorem.
4.2. **Mimetic Operators.** The discretization of the flow variables involves a projection operator, \( \pi_h \), from the complete space \( \Lambda^k(\Omega) \) to a subspace \( \Lambda^k_h(\Omega; C_k) \subset \Lambda^k(\Omega) \). In this subspace we are able to express differential forms in terms of \( k \)-cochains defined on \( k \)-chains, and corresponding \( k \)-form interpolation functions (often called basis-functions). As often, also in this case the subspace is a polynomial space. The projection operation actually consists of two steps, a reduction operator, \( R \), that integrates the \( k \)-forms on \( k \)-chains to get \( k \)-cochains, and a reconstruction operator, \( I \), to reconstruct \( k \)-forms from \( k \)-cochains using the appropriate basis-functions. These mimetic operators were already introduced before in [11, 38]. A composition of the two operators gives the projection operator \( \pi_h = I \circ R \) as is illustrated below.

\[
\Lambda^k(\Omega) \xrightarrow{\pi_h} \Lambda^k_h(\Omega; C_k) \xrightarrow{R} C^k(D)
\]

These three operators together set up the mimetic framework. An extensive discussion on mimetic operators can be found in [44]. Here the most important properties are listed.

**Definition 2.** The reduction operator \( R : \Lambda^k(\Omega) \rightarrow C^k(D) \) is a homomorphism that maps differential forms to cochains. This non-injective and surjective linear map is also called the De Rham map and is defined by integration as

\[
\langle Ra^{(k)}, \tau \rangle_{(k)} := \int_{\tau_{(k)}} a^{(k)}, \quad \forall \tau \in C_k(D).
\]

Then for all \( c_{(k)} \in C_k(D) \), the reduction of the \( k \)-form, \( a^{(k)} \in \Lambda^k(\Omega) \), to the \( k \)-cochain, \( a^{(k)} \in C^k(D) \), is given by

\[
a^{(k)}(c_{(k)}) := \langle Ra^{(k)}, c_{(k)} \rangle \overset{\text{(4.9)}}{=} \sum_i c^i \langle Ra^{(k)}, \tau_i \rangle_{i} \overset{\text{(4.9)}}{=} \sum_i c^i \int_{\tau_{i}} a^{(k)} = \int_{c_{(k)}} a^{(k)}.
\]

It is the integration of a \( k \)-form over all \( k \)-cells in a \( k \)-chain that results in a \( k \)-cochain. A special case of reduction is integration of an \( n \)-form \( a \in \Lambda^n(\Omega) \) over \( \Omega \), then

\[
\int_{\Omega} a^{(n)} := \langle Ra^{(n)}, \sigma \rangle_{(n)}
\]

where the chain \( \sigma_{(n)} = \sum_i \tau_{(n),i} \) (so all \( c^i = +1 \)) covers the entire computational domain \( \Omega \). The reduction map commutes with respect to differentiation.

**Lemma 2.** The reduction map has a commuting property with respect to continuous and discrete differentiation,

\[
Rd = \delta R \quad \text{on} \quad \Lambda^k(\Omega).
\]
This commutation can be illustrated as
\begin{align*}
\Lambda^k & \quad \xrightarrow{d} \quad \Lambda^{k+1} \\
\downarrow \mathcal{R} & \quad \downarrow \mathcal{R} \\
C^k & \quad \delta \quad \xrightarrow{} \quad C^{k+1}
\end{align*}

Proof. This property can be proven using Stokes' theorem (2.7) and the duality property of (4.7),

\begin{equation}
\langle Rda(k), c(k) \rangle \quad \text{(4.12)} \quad = \int_{c(k)} da(k) \quad \text{(2.7)} \quad = \int_{\partial c(k)} a(k) \quad \text{(4.7)} \quad = \langle \delta Ra(k), c(k) \rangle.
\end{equation}

The operator acting in the opposite direction to the reduction operator is the reconstruction operator, \( I \). The reconstruction operator \( I : C^k(D) \rightarrow \Lambda^k_h(\Omega; C_k) \), also called the Whitney map, is an isomorphism that maps \( k \)-cochains onto finite dimensional \( k \)-forms. The reconstructed differential forms belong to the space \( \Lambda^k_h(\Omega; C_k) \), which is a proper subset of the complete \( k \)-form space \( \Lambda^k(\Omega) \). While the reduction step is clearly defined in Definition 2, in the choice of interpolation forms there exists some freedom.

**Definition 3.** Although the choice of a reconstruction method allows for some freedom, \( I \) must satisfy the following properties:

- Reconstruction \( I \) must be the right inverse of \( \mathcal{R} \), so it returns identity (consistency property),

\begin{equation}
\mathcal{R} I = Id \quad \text{on} \quad C^k(D).
\end{equation}

- Like \( \mathcal{R} \), also the reconstruction operator \( I \) has to possess a commuting property with respect to differentiation. A properly chosen reconstruction operator \( I \) must satisfy a commuting property with respect to the exterior derivative and coboundary operator,

\begin{equation}
d I = I \delta \quad \text{on} \quad C^k(D).
\end{equation}

This commutation can be illustrated as
\begin{align*}
\Lambda^k_h & \quad \xrightarrow{d} \quad \Lambda^{k+1}_h \\
\uparrow I & \quad \uparrow I \\
C^k & \quad \delta \quad \xrightarrow{} \quad C^{k+1}
\end{align*}

**Remark 1.** Moreover, we want it to be an approximate left inverse of \( \mathcal{R} \), so the result is close to identity (approximation property)

\begin{equation}
\mathcal{I} \mathcal{R} = Id + \mathcal{O}(h^p) \quad \text{in} \quad \Lambda^k(\Omega).
\end{equation}

where \( \mathcal{O}(h^p) \) indicates a truncation error in terms of a measure of the grid size, \( h \), and a polynomial order \( p \).

**Definition 4.** The composition \( \mathcal{I} \circ \mathcal{R} \) will denote the projection operator, \( \pi_h := \mathcal{I} \mathcal{R} : \Lambda^k(\Omega) \rightarrow \Lambda^k_h(\Omega; C_k) \), allowing for an approximate continuous representation of a \( k \)-form \( a \in \Lambda^k(\Omega) \),

\begin{equation}
\pi_h a = \pi_h a = \mathcal{I} \mathcal{R} a, \quad a_h \in \Lambda^k_h(\Omega; C_k) \subset \Lambda^k(\Omega).
\end{equation}

where \( \mathcal{I} \mathcal{R} a \) is expressed as a combination of \( k \)-cochains and interpolating \( k \)-forms.

A proof that \( \pi_h \) is indeed a projection operator is given in [44]. For a projection operator \( \pi_h \) to be a useful operator, we require it to be a bounded projection operator.

**Definition 5 (Bounded projection).** The projection operator \( \pi_h \) is a bounded projection if for \( C < \infty \),

\begin{equation}
\| \pi_h \|_{\mathcal{L}(\Lambda^k, \Lambda^k_h)} := \sup_{a \in \Lambda^k} \frac{\| \pi_h a \|_{\Lambda^k_h}}{\| a \|_{\Lambda^k}} \leq C \quad \iff \quad \| \pi_h a \|_{\Lambda^k_h} \leq C \| a \|_{\Lambda^k}.
\end{equation}
Lemma 3. There exists a commuting property for the projection and the exterior derivative, such that

\begin{equation}
\text{d}\pi_h = \pi_h\text{d} \quad \text{on } \Lambda^k(\Omega).
\end{equation}

This can be illustrated as

\[
\begin{array}{ccc}
\Lambda^k & \xrightarrow{\text{d}} & \Lambda^{k+1} \\
\downarrow{\pi_h} & & \downarrow{\pi_h} \\
\Lambda^k_h & \xrightarrow{\text{d}} & \Lambda^{k+1}_h.
\end{array}
\]

Proof. This is a direct consequence of the definitions of the reduction (4.11), reconstruction (4.14) and projection operators (4.16).

\[d\pi_h a = \text{d}IR a = \text{I} \delta R a = \text{IR} d a = \pi_h d a, \quad \forall a \in \Lambda^k(\Omega).\]

Note that it is the intermediate step \(I\delta R a\) that is used in practice for the discretization, see Examples 3 and 4, Section 4.5. Lemma 3 is the most important result in this paper. As a direct consequence we obtain the pointwise divergence-free solution, as illustrated in the following example.

Example 2. Consider the relation \(d u = g\), with \(u \in \Lambda^{n-1}(\Omega)\) and \(g \in \mathcal{B}^n(\Omega)\). In vector notation the \(d\) represents the \(\text{div}\) operator. Now let \(d u_h = g_h\) be the discretization of our continuous problem, with \(u_h \in \Lambda^{n-1}_h(\Omega)\) and \(g_h \in \mathcal{B}^n_h(\Omega)\). Then by using (4.18) we get

\[du_h - g_h = d\pi_h u - \pi_h g = \pi_h (du - g) = 0.\]

It follows that our discretization is exact. In case \(g = 0\), we have a pointwise divergence-free solution of \(u_h \in \Lambda^{n-1}_h(\Omega)\).

The projection does not commute with codifferential operator. This is the main reason why to rewrite the codifferentials into exterior derivatives and boundary integrals, by means of integration by parts.

We do not restrict ourselves to affine mappings only, as is required in many other compatible finite elements, like Nédelec and Raviart-Thomas elements and their generalizations [5, 49, 58], but also allow curvilinear maps. This allows for better approximations in complex domains with curved boundaries, without the need for refinement. The following lemma shows the commuting property between projection and pullback. An extensive proof is given in [44].

Lemma 4. [44] For all \(a \in \Lambda^k(\Omega)\), there exists a commuting property between the projection operator \(\pi_h\) and the pullback \(\Phi^*\), such that

\begin{equation}
\Phi^*\pi_h a = \pi_h \Phi^* a \quad \text{on } \Lambda^k(\Omega).
\end{equation}

This commutation can be illustrated as

\[
\begin{array}{ccc}
\Lambda^k(\Omega) & \xrightarrow{\Phi^*} & \Lambda^k(\Omega_{\text{ref}}) \\
\downarrow{\pi_h} & & \downarrow{\pi_h} \\
\Lambda^k_h(\Omega, C_k) & \xrightarrow{\Phi^*} & \Lambda^k_h(\Omega_{\text{ref}}, C_k)
\end{array}
\]

Next we will obtain the discrete Poincaré inequality, which is an essential step in proving numerical stability. Because the complexes \((\Lambda, \text{d})\) and \((\Lambda_h, \text{d})\) are each others supercomplex and subcomplex, respectively. The discrete Poincaré inequality is therefore directly related to the Poincaré inequality in Lemma 1, the linear projection in Theorem 4 and its boundedness, see Definition 5.

Lemma 5 (Discrete Poincaré inequality). [5] Let \((\Lambda, \text{d})\) be a bounded closed complex, \((\Lambda_h, \text{d})\) a subcomplex, and \(\pi_h\) a bounded linear projection. Then

\begin{equation}
\|v\|_{HA^k} \leq c_P \|\pi_h\| \|dv\|_{L^2A^k}, \quad v \in \mathcal{Z}_h^{k,\perp}.
\end{equation}
nodes are their boundaries. Corresponding to this set of nodes (0-chains) there exists a projection \(N\) consists of \(\exists\) so it is enough to show that \(\|v\|_{H^1} \leq \|\pi_h z\|_{H^1}\). Now, \(v - \pi_h z \in \Lambda_h^k\) and \(d (v - \pi_h z) = 0\), so \(v - \pi_h z \in \mathcal{Z}_h^k\). Therefore

\[
\|v\|_{H^1} = (v, \pi_h z)_{H^1} + (v, v - \pi_h z)_{H^1} = (v, \pi_h z)_{H^1} \leq \|v\|_{H^1} \|\pi_h z\|_{H^1}.
\]

The result follows from the operator norm of the projection, Definition 3.

The proof of Lemma 5 makes use of the commuting property in Lemma 3. Using the discrete Poincaré inequality for \(\Lambda^k\) and the norm of the projection operator \(\pi\), such that for any \((\tau, v, q) \in \Lambda_h^k \times \Lambda_h^{k-1} \times \Lambda_h^n\), there exists \((\omega, u, p) \in \Lambda_h^{k-2} \times \Lambda_h^{k-1} \times \Lambda_h^n\) with

\[
B(\omega, u, p; \tau, v, q) \geq \beta_h (\|\omega\|_{H^1} + \|u\|_{H^1} + \|p\|_{L^2}) (\|\tau\|_{H^1} + \|v\|_{H^1} + \|q\|_{L^2}).
\]

Proof: This is just Theorem 1 applied to the complex \((\Lambda_h, d)\), combined with the fact that the constant in the Poincaré inequality for \(\Lambda_h^k\) is \(c_P\|\pi_h\|\) by Lemma 5.

Theorem 2 (Discrete well-posedness). Let \((\Lambda_h, d)\) be a subcomplex of \((\Lambda, d)\), admitting uniformly bounded linear projections. Then there exists a constant \(\beta_h > 0\), depending on \(c_P\) and the norm of the projection operator \(\pi_h\), such that for any \((\tau, v, q) \in \Lambda_h^k \times \Lambda_h^{k-1} \times \Lambda_h^n\), there exists \((\omega, u, p) \in \Lambda_h^{k-2} \times \Lambda_h^{k-1} \times \Lambda_h^n\) with

\[
(4.21) \quad B(\omega, u, p; \tau, v, q) \geq \beta_h (\|\omega\|_{H^1} + \|u\|_{H^1} + \|p\|_{L^2}) (\|\tau\|_{H^1} + \|v\|_{H^1} + \|q\|_{L^2}).
\]

4.3. Mimetic spectral element basis-functions. Now that a mimetic framework is formulated using differential geometry, algebraic topology and the relations between those - the mimetic operators - we derive projections that satisfy the properties of the mimetic operators. The projected differential forms used in this paper are polynomials, based on the idea of spectral element methods [19]. Spectral element methods have many desirable features such as arbitrary polynomial representation, favourable conditioning, element wise local support, and optimal stability and approximation properties. However, the definition of the reconstruction operator requires a new set of spectral element interpolation functions. The mimetic spectral elements were derived independently by [31] [39], and more extensively discussed in [41]. The most important properties of the mimetic spectral element method are presented here.

In spectral element methods the domain \(\Omega\) is decomposed into \(M\) non-overlapping, in this case curvilinear quadrilateral, closed sub-domains \(\Omega_m\):

\[
\Omega = \bigcup_{m=1}^{M} \Omega_m, \quad \Omega_m \cap \Omega_l = \partial \Omega_m \cap \partial \Omega_l, \quad m \neq l,
\]

where in each sub-domain a Gauss-Lobatto grid is constructed, see Figures 8 and 12 in the next section.

The collection of Gauss-Lobatto grids in all elements \(\Omega_m\) constitutes the cell complex \(D\). For each spectral element there exists a sub cell complex, \(D_m\). Note that \(D_m \cap D_l, \quad m \neq l\), is not an empty set in case they are neighboring elements, but contains all \(k\)-cells, \(k < n\), of the common boundary, see Definition 4.

Each sub-domain is mapped from the reference element, \(\Omega_{\text{ref}} = [-1, 1]^n\), using the mapping \(\Phi_m : \Omega_{\text{ref}} \to \Omega_m\). Then all flow variables defined on \(\Omega_m\) are pulled back onto this reference element using the following pullback operation, \(\Phi_m^* : \Lambda_h^k(\Omega_m) \to \Lambda_h^k(\Omega_{\text{ref}})\). In two dimensions the reference element is given by \(\Omega_{\text{ref}} := \{(\xi, \eta) \mid -1 \leq \xi, \eta \leq 1\}\).

The basis-functions that interpolate the cochains on the quadrilateral elements are constructed using tensor products. It is therefore sufficient to derive interpolation functions in one dimension and use tensor products afterwards to construct \(n\)-dimensional basis functions. Because the projection operator and the pullback operator commute (4.19), the interpolation functions are discussed for the reference element only.

Consider a 0-form \(v \in \Lambda^0(\Omega_{\text{ref}})\) on \(\Omega_{\text{ref}} := \xi \in [-1, 1]\), on which a cell complex \(D\) is defined that consists of \(N + 1\) nodes, \(\xi_i\), where \(-1 \leq \xi_0 < \ldots < \xi_N \leq 1\), and \(N\) edges, \([\xi_{i-1}, \xi_i]\), of which the nodes are their boundaries. Corresponding to this set of nodes (0-chains) there exists a projection
using $N$th order Lagrange polynomials, $l_i(\xi)$, to approximate a 0-form, as

$$\pi_h a = \sum_{i=0}^{N} a(\xi_i) l_i(\xi).$$

(4.23)

Lagrange polynomials have the property that they interpolate nodal values and are therefore suitable to reconstruct the cochain $a^{(0)} = \mathcal{R} a(\xi)$ containing the set $a_i = a(\xi_i)$ for $i = 0, \ldots, N$. So Lagrange polynomials can be used to reconstruct a 0-form from a 0-cochain. Lagrange polynomials are in fact 0-forms themselves, $l_i(\xi) \in \Lambda^0_h(\Omega_{\text{ref}}; C_0)$. Lagrange polynomials are constructed such that their value is one in the corresponding point and zero in all other grid points,

$$\mathcal{R} l_i(\xi) = l_i(\xi_p) = \begin{cases} 
1 & \text{if } i = p \\
0 & \text{if } i \neq p.
\end{cases}$$

(4.24)

This satisfies (4.13), where in this case $I = l_i(\xi)$. Gerritsma [31] and Robidoux [59] derived a similar projection for 1-forms, consisting of 1-cochains and 1-form polynomials, that is called the edge polynomial, $e_i(\xi) \in \Lambda^1_h(\Omega_{\text{ref}})$.

**Lemma 6.** Following Definitions 3 and 5, apply the exterior derivative to $\pi_h a(\xi)$, it gives the 1-form $\pi_h u(\xi) = d\pi_h a(\xi) = \mathcal{I} \delta \mathcal{R} a(\xi)$ given by

$$\pi_h u(\xi) = \sum_{i=1}^{N} u_i e_i(\xi),$$

with 1-cochain $u^{(1)}$, where

$$u_i = \langle \mathcal{R} u, \tau_{(1),i} \rangle = \int_{\tau_{(1),i}} u(\xi) = \int_{\tau_{(1),i}} d a(\xi) = \int_{\partial \tau_{(1),i}} a(\xi),$$

$$= a(\xi_i) - a(\xi_{i-1}) = a_i - a_{i-1},$$

(4.26)

with the edge polynomial defined by

$$e_i(\xi) = -\sum_{k=0}^{i-1} d l_k(\xi) = \sum_{k=0}^{N} d l_k(\xi) = \frac{1}{2} \sum_{k=0}^{N} d l_k(\xi) - \frac{1}{2} \sum_{k=0}^{i-1} d l_k(\xi).$$

(4.27)

*Proof. See [31, 44, 59].

The value corresponding to line segment (1-cell) $\tau_{(1),i}$ is given by $u_i = a_i - a_{i-1}$ and so $u^{(1)} = \delta a^{(0)}$ is the discrete derivative operator in 1D. This operation is purely topological, no metric is involved. It satisfies (4.14), since $d\mathcal{I} a^{(0)} = \mathcal{I} \delta a^{(0)}$. Note that we have $d e_i(\xi) = \sum d \circ d l_i(\xi) = 0$. The 1-form edge polynomial can also be written as below, separating the edge function into its polynomial and its basis,

$$e_i(\xi) = e_i(\xi) d\xi, \quad \text{with} \quad e_i(\xi) = -\sum_{k=0}^{i-1} \frac{d l_k}{d\xi},$$

Similar to (4.24), the edge functions are constructed such that when integrating $e_i(\xi)$ over a line segment it gives one for the corresponding element and zero for any other line segment, so

$$\mathcal{R} e_i(\xi) = \int_{\xi_{p-1}}^{\xi_p} e_i(\xi) = \begin{cases} 
1 & \text{if } i = p \\
0 & \text{if } i \neq p.
\end{cases}$$

(4.28)

This also satisfies (4.13), where in this case $I = e_i(\xi)$. The fourth-order Lagrange and third-order edge polynomials, corresponding to a Gauss-Lobatto grid with $N = 4$, are shown in Figures 6 and 7.
4.4. **Bounded linear projections.** As mentioned in Definition 5, boundedness of the projection is a requirement, and is therefore shown for the projections introduced above.

The mimetic framework uses Lagrange, \( l_i(\xi) \in H\Lambda^0(\Omega_{\text{ref}}) \), and edge functions, \( e_i(\xi) \in L^2\Lambda^1(\Omega_{\text{ref}}) \), for the reconstruction, \( I \), where the latter is constructed using the former; i.e., from \( \pi_h a = \sum_{i=0}^N a_i l_i(\xi) \in \Lambda^0_h(\Omega_{\text{ref}}; C_0) \), we define \( \pi_h b = \Lambda^1_h(\Omega_{\text{ref}}; C_1) \) such that

\[
\pi_h b = \pi_h da = \sum_{i=1}^N b_i e_i(\xi), \quad \text{with} \quad b_i = a_i - a_{i-1}.
\]

Because we consider tensor products to construct higher-dimensional interpolation, it is sufficient to show that the projection operator is bounded in one dimension. A similar approach was used in [18]. Due to the way the edge functions are constructed, there exists a commuting diagram property between projection and exterior derivative,

\[
\begin{array}{cccc}
\mathbb{R} & \longrightarrow & H\Lambda^0 & \longrightarrow & L^2\Lambda^1 & \longrightarrow & 0 \\
\downarrow & & \pi_h & & \downarrow & & \\
\mathbb{R} & \longrightarrow & \Lambda^0_h & \longrightarrow & \Lambda^1_h & \longrightarrow & 0,
\end{array}
\]

which gives, for \( a \in H\Lambda^0(\Omega_{\text{ref}}) \), the one form

\[
(4.29) \quad d\pi_h a = \pi_h da, \quad \text{in} \; \Lambda^1_h(\Omega_{\text{ref}}) \subset L^2\Lambda^1(\Omega_{\text{ref}}).
\]

Lagrange interpolation by itself does not guarantee a convergent approximation [26], but it requires a suitably chosen set of points, \(-1 \leq \xi_0 < \xi_1 < \ldots < \xi_N \leq 1\). Here, the Gauss-Lobatto distribution is proposed, because of its superior convergence behaviour [19]. The a priori error estimate for this of interpolant in the \( H\Lambda^0 \)-norm is given by,

\[
(4.30) \quad \|a - \pi_h a\|_{H\Lambda^0} \leq C \frac{h^{l-1}}{p^{m-1}} |a|_{H^m\Lambda^0}, \quad l = \min(p + 1, m).
\]

Equation (4.30) also implies that the projection of zero-forms is stable in the \( H\Lambda^0(\Omega_{\text{ref}}) \), as is shown in the following proposition.

**Proposition 1.** For \( a \in H\Lambda^0(\Omega_{\text{ref}}) \) and the projection \( \pi_h : H\Lambda^0 \rightarrow \Lambda^0_h \), there exists the following two stability estimates in \( H\Lambda^0 \)-norm and \( H\Lambda^0 \)-semi-norm:

\[
(4.31) \quad \|\pi_h a\|_{H\Lambda^0} \leq C \|a\|_{H\Lambda^0} \quad \text{and} \quad |\pi_h a|_{H\Lambda^0} \leq C |a|_{H\Lambda^0}.
\]

**Proof.** See [44].
Now that we have a bounded linear projection of zero forms in one dimension, we can also proof boundedness of the projection of one-forms.

**Proposition 2.** Let \(a \in H\Lambda^0\) and \(b = da \in L^2\Lambda^1\), then there exists a bounded linear projection \(\pi_h : L^2\Lambda^1 \to \Lambda^1_h\), such that

\[
\|\pi_h b\|_{L^2\Lambda^1} \leq C\|b\|_{L^2\Lambda^1}.
\]

**Proof.** The proof is based on the result of the previous proposition and the commutation between projection and derivative, Lemma 3

\[
\|\pi_h b\|_{L^2\Lambda^1} = |\pi_h da|_{L^2\Lambda^1} = |d\pi_h a|_{L^2\Lambda^1} = |\pi_h a|_{H^1\Lambda^0} \leq C|a|_{H^1\Lambda^0} = C|da|_{L^2\Lambda^1} = C\|b\|_{L^2\Lambda^1}.
\]

\(\square\)

Propositions 1 and 2 show that the projection \(\pi_h\) is a bounded linear projection, based on Lagrange functions and edge functions. This is essential in obtaining a discrete Poincaré inequality, Lemma 3 and more important discrete well-posedness, Theorem 2. As for zero forms using Lagrange interpolation, we can also give an estimate for the interpolation error of one forms, interpolated using edge functions.

**Proposition 3.** Let \(a \in H\Lambda^0\) and \(b = da \in L^2\Lambda^1\), the interpolation error \(b - \pi_h b \in L^2\Lambda^1\) is given by

\[
\|b - \pi_h b\|_{L^2\Lambda^1} \leq C\frac{h^{l-1}}{p^{m-1}}|b|_{H^{m-1}\Lambda^1},
\]

with \(l = \min(p + 1, m)\).

**Proof.** See [44]. \(\square\)

For the variables vorticity, velocity and pressure in the VVP Stokes formulation, the \(h\)-convergence rates of the interpolation errors become,

\[
\|\omega - \pi_h \omega\|_{H^{n-2}\Lambda^0} = O(h^{N+1}), \quad \|u - \pi_h u\|_{L^2\Lambda^{n-1}} = O(h^N), \quad \|p - \pi_h p\|_{L^2\Lambda^n} = O(h^N),
\]

with \(N\) defined as in Section 4.3. Because of (3.5c) and (4.18), we have \(\|u - \pi_h u\|_{H^{n-1}\Lambda^0} = \|u - \pi_h u\|_{L^2\Lambda^{n-1}}\).

4.5. **Pointwise divergence-free discretization.** One of the most interesting properties of the mimetic method presented in this paper, is that within our weak formulation, the divergence-free constraint is satisfied pointwise. This result follows from the three commuting properties with the exterior derivative, (4.11), (4.14) and (4.18), as was shown in Lemma 3. The corresponding commuting diagrams are repeated in the diagram below for the two dimensional case.

Note that by curl we refer to the two-dimensional variant, applied to a scalar, i.e. \(\text{curl} \omega = (\partial \omega/\partial y, -\partial \omega/\partial x)^T\), see also Example 1 and is also called the normal gradient operator, \(\text{grad}^\perp\), see [50].

In the following two examples we demonstrate the action of the exterior derivative on vorticity, \(\omega_h \in \Lambda^1_0(\Omega_{ref}; C_0)\), and on the velocity flux, \(u_h \in \Lambda^1_0(\Omega_{ref}; C_1)\). Two dimensional reconstruction is based on tensor product construction of the one dimensional reconstruction function introduced above.
Example 3 (Curl operator). Consider $z_h \in \Lambda^1_h$ where $z_h = d \omega_h$. Then $\omega_h$ is expanded in the reference coordinates $(\xi, \eta)$ as

$$\omega_h(\xi, \eta) = \sum_{i=0}^{N} \sum_{j=0}^{N} \omega_{i,j} l_i(\xi) l_j(\eta).$$

Apply the exterior derivative in the same way as in Lemma 6, it gives

$$z_h(\xi, \eta) = d \omega_h(\xi, \eta) = \sum_{i=0}^{N} \sum_{j=0}^{N} (\omega_{i,j} - \omega_{i-1,j}) e_i(\xi) l_j(\eta) + \sum_{i=0}^{N} \sum_{j=0}^{N} (\omega_{i,j} - \omega_{i,j-1}) l_i(\xi) e_j(\eta),$$

(4.36a)

$$= - \sum_{i=1}^{N} \sum_{j=0}^{N} z^\eta_{i,j} e_i(\xi) l_j(\eta) + \sum_{i=0}^{N} \sum_{j=0}^{N} z^\xi_{i,j} l_i(\xi) e_j(\eta),$$

where $z^\xi_{i,j} = \omega_{i,j} - \omega_{i-1,j}$, and $z^\eta_{i,j} = \omega_{i,j} - \omega_{i,j-1}$ can be compactly written as $z = \delta \omega$, with $\omega \in C^0(D)$ and $z \in C^1(D)$, or in matrix notation as $z = E^{(1,0)} \omega$. This relation is exact, coordinate free and invariant under transformations.

Example 4 (Divergence operator). Let $u_h \in \Lambda^1_h$ be the velocity flux defined as

$$u_h(\xi, \eta) = - \sum_{i=1}^{N} \sum_{j=0}^{N} v_{i,j} e_i(\xi) l_j(\eta) + \sum_{i=0}^{N} \sum_{j=0}^{N} u_{i,j} l_i(\xi) e_j(\eta).$$

Then the change of mass, $m_h \in \Lambda^2_h$, is equal to the exterior derivative of $u_h$,

$$m_h(\xi, \eta) = d u_h(\xi, \eta) = \sum_{i=1}^{N} \sum_{j=1}^{N} (u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1}) e_i(\xi) e_j(\eta).$$

(4.38)

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} m_{i,j} e_i(\xi) e_j(\eta),$$

where $m_{i,j} = u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1}$ can be compactly written as $m = \delta u$, with $u \in C^1(D)$ and $m \in C^2(D)$, or in matrix notation as $m = E^{(2,1)} u$. Note that if the mass production is zero, as in our model problem (4.4b), the incompressibility constraint is already satisfied at discrete/cochain level. Interpolation then results in a pointwise divergence-free solution.

5. Numerical Results

Now that all parts of the mixed mimetic method are treated, we can test the performance of the numerical scheme using a set of three test problems. The first one consists of an analytic solution on a unit square, where optimal $h$-convergence and exponential $p$-convergence rates are shown for both Cartesian and curvilinear meshes for all combinations of boundary conditions. The second is a lid-driven cavity flow, where results are compared with a reference solution. Finally, flow around a cylinder moving is considered.

5.1. Manufactured solution. The first test case demonstrates the rate of $h$- and $p$-rate of convergence of the mixed mimetic spectral element method applied to the Stokes model. The model problem is defined on the unit square $\Omega = [0, 1]^2$, with $\nu = 1$ and with the righthandside $f \in \Lambda^1(\Omega)$ given by

$$f = -f_x(x,y)dx + f_y(x,y)dy,$$

$$= - (\pi \sin(\pi x) \cos(\pi y) + 8\pi^2 \cos(2\pi x) \sin(2\pi y)) \, dx$$

$$+ (\pi \cos(\pi x) \sin(\pi y) - 8\pi^2 \sin(2\pi x) \cos(2\pi y)) \, dy.$$
This righthandside results in an exact solution for the vorticity \( \omega \in \Lambda^0(\Omega) \), velocity \( u \in \Lambda^1(\Omega) \), and pressure \( p \in \Lambda^2(\Omega) \) components of the Stokes problem, given by

\[
\begin{align*}
\omega &= \omega(x, y) = -4\pi \sin(2\pi x) \sin(2\pi y), \\
u &= -v(x, y)dx + u(x, y)dy \\
p &= p(x, y) dx \land dy = (\sin(\pi x) \sin(\pi y)) dx \land dy.
\end{align*}
\]

This testcase was discussed before in [32, 55]. Calculations were performed on both a Cartesian as well as a curvilinear mesh as shown in Figure 8. The mapping, \((x, y) = \Phi(\xi, \eta)\), used for the curved grid is given by

\[
\begin{align*}
x(\xi, \eta) &= \xi + \frac{1}{5} \sin(\pi \xi) \sin(\pi \eta), \\
y(\xi, \eta) &= \eta + \frac{1}{5} \sin(\pi \xi) \sin(\pi \eta).
\end{align*}
\]

**Figure 8.** Examples of a Cartesian and a curvilinear mesh used in the convergence analysis. The meshes shown consist of \(4 \times 4\) spectral elements, with for each element, \(N = 4\). The element boundaries are indicated in red.

**Figure 9** shows the \(h\)-convergence and Figure 10 shows the \(p\)-convergence of the vorticity \( \omega \in \Lambda^0(\Omega) \), velocity \( u \in \Lambda^1(\Omega) \) and pressure \( p \in \Lambda^2(\Omega) \). For both figures, the results of the top row are obtained on Cartesian meshes and the lower results are obtained on curvilinear meshes. The errors for the vorticity and velocity are both measured in the \(L^2\) and \(H^1\)-norm, i.e. \(\| \omega - \omega_h \|_{L^2\Lambda^0}, \| \omega - \omega_h \|_{H^1\Lambda^0}, \| u - u_h \|_{L^2\Lambda^1}, \| u - u_h \|_{H^1\Lambda^1}\), respectively. Because the divergence-free constraint is satisfied pointwise, the norm \(\| d(u - u_h) \|_{L^2\Lambda^2}\) is zero or machine precision, see Figure 11 and so the \(H^1\)-norm is equal to the \(L^2\)-norm of the velocity, i.e., \(\| u - u_h \|_{H^1\Lambda^1} = \| u - u_h \|_{L^2\Lambda^1}\). This does not hold for the vorticity, since \(d\omega \in \Lambda^1(\Omega)\) is again a function of sine and cosine functions. However, the norm \(\| d(\omega - \omega_h) \|_{L^2\Lambda^1}\) converges much faster than \(\| \omega - \omega_h \|_{L^2\Lambda^0}\), and so one can say that the norm \(\| \omega - \omega_h \|_{H^1\Lambda^0}\) approaches \(\| \omega - \omega_h \|_{L^2\Lambda^0}\) on grid refinement, as can be seen in Figures 9 and 10. More on the superconvergence of \(\| d(\omega - \omega_h) \|_{L^2\Lambda^1}\) can be found in [43].

In Figure 9 convergence rates are added which show that \(h\)-convergence rates are equal to the \(h\)-convergence rates of the interpolation error (4.34), on both Cartesian as well as curvilinear meshes. **Figure 10** shows that exponential convergence rates are obtained on both types of meshes.

Important to remark is that the results obtained are independent of the kind of boundary conditions used. This is shown in Table 2. This is an important result, because especially optimal convergence for the normal velocity - tangential velocity boundary condition is non-trivial in compatible methods, [4]. The standard elements in compatible methods, the Raviart-Thomas elements, show only sub-optimal convergence for velocity boundary conditions. They lose \(\tfrac{3}{2}\) order of convergence with respect to the results obtained in Figure 9 and Table 2 [4].
5.2. Lid-driven cavity Stokes. For many years, the lid-driven cavity flow was considered as one of the classical benchmark cases for the assessment of numerical methods and the verification of
incompressible (Navier)-Stokes codes. The lid-driven cavity test case deals with a flow in a unit-square box with three solid boundaries and moving lid as the top boundary, moving with constant velocity equal to one to the right. Because of the discontinuities of the velocity in the two upper corners, the solution becomes singular at these corners, where both vorticity and pressure become infinite. Especially these singularities make the lid-driven cavity problem a challenging test case.

For this test case a non-uniform $6 \times 6$ Cartesian spectral element mesh is used. Each spectral element consists of a Gauss-Lobatto grid for $N = 6$, see Figure 12. The solutions of the vorticity, velocity, pressure and stream function are shown in Figure 12. Also shown in Figure 12 is a plot of the divergence of velocity. It confirms a pointwise divergence-free solution up to machine precision. The results are in perfect agreement with those in [61].

Because in the mixed mimetic spectral element method no velocity unknowns are located at the upper corners, no special treatment is needed for the corner singularities, in contrast to many nodal finite-difference, finite-element and spectral element methods, [27, 52, 54]. This is due to the finite-volume like structure of the method, as explained in the section of algebraic topology.

In Figure 13 the centerline velocities are plotted. Three different configurations are used, based on the same cell complex consisting of $9 \times 9$ 2-cells:

- left: $9 \times 9$ spectral elements with $N = 1$, resulting in piecewise constant approximations along the centrelines,
- middle: $3 \times 3$ spectral elements with $N = 3$, resulting in piecewise quadratic approximations along the centrelines,
- right: global spectral element with $N = 9$, resulting in 8th order polynomial approximations along the centrelines.
Despite the low resolution, all approximations lay on top of those in [61].

The singularities can be made even more severe by sharpening the corners, as happens for a lid-driven cavity problem in a triangle. Figure [14] shows the vorticity and velocity distributions. On top of the velocity plot, stream function contours are plotted. The solutions are constructed on a 9 spectral element mesh with \( N = 9 \). A close up of the stream function contours is shown in the left figure of Figure [14]. The stream function contours nicely show the first three Moffatt eddies [47].

5.3. Flow over a cylinder. The last test case considers the flow around a cylinder moving with constant velocity to the left, as defined in [20]. This test case is mostly considered in the context of least-squares finite and spectral element methods, due to their moderate performance in case of large contraction regions, [20,21,56], mainly in terms of conservation of mass. The mixed formulation treated in this paper is equivalent to the optimal least-squares formulation for vorticity-velocity-pressure formulation of Stokes flow [10,43]. This is revealed in the results below.

The cylinder moves with unit velocity along the centerline of a narrow channel. The computational domain is defined as a rectangular box minus the cylinder, as shown in Figure [15]. Also visible in this figure are the 12 spectral elements in which the computational domain is divided. A transfinite mapping, [33], is used to define the curved elements around the cylinder. Velocity boundary conditions of \((u,v) = (1,0)\) are prescribed on the outer boundary and no-slip, \((u,v) = (0,0)\), is prescribed along the boundary of the cylinder. Solution of the vorticity, velocity and pressure, together with stream lines are shown in Figure [15].

Next consider a control volume \(\Omega_c\) consisting of the 6 elements in the domain \(-1.5 \leq x \leq 0, 0.75 \leq y \leq 0.75\). The control volume is chosen such that the ratio in size between inflow and outflow boundary is maximal. In this control volume conservation of mass should hold.

Conservation of mass is expressed, by means of generalized Stokes theorem [2,7], in terms of a
boundary integral as
\begin{equation}
\int_{\Omega_h} \text{div}_\varepsilon \bar{u} \, d\varepsilon = \int_{\partial\Omega_h} u_h = 0, \quad u_h \in Z_h^1.
\end{equation}

From Section 4.5 and the results of the previous test cases we know that the solution of the velocity is divergence-free throughout the domain, independent of the chosen control volume. In Figure 16 a comparison is made for the horizontal velocity component \( u \) at the smallest crosssection above the cylinder, i.e. \( x = 0, \ 0.5 \leq y \leq 0.75 \), between the recently developed LSSCM. [42], and our MMSEM method for \( N = 3, 6, 12 \). Both methods use a similar grid of 12 spectral elements. As can be seen from this figure, the MMSEM method performs already very well for \( N = 3 \), i.e. quadratic polynomial, where the LSSCM still fails for \( N = 6 \), i.e. sixth order polynomial. This is a direct consequence of the pointwise divergence-free discretization.
Figure 15. Solution of velocity, vorticity and pressure for flow around a moving cylinder, on a 12 element, $N = 6$ mesh.

Figure 16. Horizontal velocity at smallest crosssection above the cylinder, on a 12 element mesh, for $N = 3, 6, 12$.

6. Conclusions and future aspects

In this paper we presented the mixed mimetic spectral element method, applied to the vorticity-velocity-pressure formulation of Stokes model. At the heart lies the recently developed higher-order mimetic discretization for quadrilaterals. [44]. The gradient, curl and divergence conforming methods results in a pointwise divergence-free discretization of the Stokes problem, as was confirmed by a set of benchmark problems. These results also showed optimal convergence, independent of the type of boundary conditions. This is a significant improvement with respect to the classical divergence conforming Raviart-Thomas elements [4]. More on convergence behavior and error estimates is presented [45]. In the near future we plan to extend the method with $hp$-refinement based on conforming mortar element method.

References


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