Approximation of the Signorini Problem with Friction by a Mixed Finite Element Method

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INTRODUCTION

In many technical and physical situations one meets problems when one deformable body comes into contact with another. A mathematical formulation of these problems leads to the use of variational inequalities. A detailed analysis of contact problems between two elastic bodies has been given in [5], assuming that no friction occurs on the contact surface (for related results see also [9]). It is clear that the frictionless problem gives an approximation of real situation only, so that an involvement of the friction is desirable.

A mathematical formulation of the contact problem between an elastic body and a perfectly rigid support (known as the Signorini problem), involving the friction governed by Coulomb's law, has been introduced in [2]. The existence of its solution for simple geometrical situations has been proven in [8] for the first time. From the existence proof an algorithm (method of successive approximations) for the numerical approximation of the problem follows. Unfortunately, the convergence of the algorithm is an open problem up to this time.

The present paper deals with the approximation of one iterative step, which is defined by the Signorini problem with prescribed normal forces on the contact surface. A mixed formulation of this problem is derived, making use of the duality approach. This formulation allows us to approximate independently the displacement field in the body and the normal and
tangential forces on contact surface. A finite element analysis of the mixed formulation is presented and error estimates for the physical quantities mentioned above are derived.

A partial analysis of another iterative algorithm for the numerical solution of the Signorini problem with friction (see [10]), will be given in a forthcoming paper.

1. VARIATIONAL FORMULATION OF THE SIGNORINI PROBLEM WITH FRICTION

We start with definitions of some functional spaces, which will be necessary in what follows.

Let \( Q \subseteq \mathbb{R}^2 \) be a bounded domain, the Lipschitz boundary \( \Gamma \) of which is decomposed as \( \Gamma = \Gamma_u \cup \Gamma_k \), where \( \Gamma_u \) and \( \Gamma_k \) are disjoint, non-empty and open in \( \Gamma \). Let

\[
V = \{ v \in H^1(Q) \mid v = 0 \text{ on } \Gamma_u \},
\]

\[
V = V \times V,
\]

where \( H^k(Q) \), \( k \geq 0 \) integer, denotes the Sobolev space of functions, derivatives of which up to the order \( k \) are square integrable in \( Q \). We set \( H^0(Q) = L^2(Q) \) and denote by \( ( , )_0 \) the scalar product in \( L^2(Q) \). \( V \) is a Hilbert space, equipped with the norm

\[
|||v||| = (v_i, v_i)_0 + (\varepsilon_{ij}(v), \varepsilon_{ij}(v))_0^{1/2},
\]  

(1.1)

where \( \varepsilon_{ij}(v) = \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i) \), \( i, j = 1, 2 \), \( v = (v_1, v_2) \in V \). By virtue of Korn's inequality ([7]), (1.1) is equivalent to the usual norm in \( H^1(Q) = H^1(Q) \times H^1(Q) \), which will be denoted by \( |||v||| \) (or \( ||v||_k \) for \( k \geq 0 \)).

Let \( \gamma: V \to L^2(\Gamma) \) be the trace operator and

\[
H^{1/2}(\Gamma_k) = \gamma(V), \quad H^{1/2}(\Gamma_k) = H^{1/2}(\Gamma_k) \times H^{1/2}(\Gamma_k).
\]

It is known that \( H^{1/2}(\Gamma_k) \) is a Hilbert space with the norm

\[
||\Phi||_{1/2, \Gamma_k} = \inf_{\psi \in V, \gamma\psi = \Phi} |||v|||.
\]

(1.2)

From (1.2) it is readily seen that

\[
||\Phi||_{1/2, \Gamma_k} = |||u|||,
\]

We use the summation convention: a repeated index implies the summation over the range 1, 2.
where \( u \in V \) is the unique solution of the elliptic boundary value problem

\[
(e_{ij}(u), e_{ij}(v))_0 + (u_i, v_i)_0 = 0 \quad \forall \, v \in (H^1_0(Q))^2
\]

\[
u = \varphi \quad \text{on } \Gamma.
\]

Let \( n = (n_1, n_2) \) and \( t = (-n_2, n_1) \) denote the unit outward normal and tangential vector, respectively, at \( x \in \Gamma \). By \( v_n \) and \( v_t \) we denote the normal and tangential component of \( v \in V \), i.e.,

\[
v_n = \gamma v \cdot n, \quad v_t = \gamma v \cdot t.
\]

Let \( \delta: V \to L^2(\Gamma_k) = L^2(\Gamma_k) \times L^2(\Gamma_k) \),

\[
L^2(\Gamma_k) = \{ v \in L^2(\Gamma) \mid v = 0 \text{ on } \Gamma_u \},
\]

be the mapping, defined by \( \delta v = (v_n, v_t), \, v \in V \) and

\[
W = W \times W = \delta(V).
\]

If \( Q \) is a domain with a sufficiently smooth boundary \( \Gamma \), then \( W = H^{1/2}(\Gamma_k) \). Although this is not our case, \( W \) and \( H^{1/2}(\Gamma_k) \) are isomorphic. Indeed, let \( \varphi \in H^{1/2}(\Gamma_k) \). Then the element

\[
\mu = (\mu_1, \mu_2) = (\varphi \cdot n, \varphi \cdot t)
\]

belongs to \( W \). We shall write \( \mu = \beta \varphi, \, \beta: H^{1/2}(\Gamma_k) \to W \). Hence, the norm in \( W \) can be introduced as

\[
\| \mu \|_W = \| \beta^{-1} \mu \|_{1/2, \Gamma_k},
\]

where \( \beta^{-1} \) is the mapping inverse to \( \beta \).

Let \( S(Q) \) denote the space of symmetric tensor functions

\[
S(Q) = \{ \sigma = (\sigma_{ij})_{i,j=1}^2 \in (L^2(Q))^4, \sigma_{ij} = \sigma_{ji} \text{ a.e. in } Q \},
\]

and

\[
H(\text{div}, Q) = \{ \sigma \in S(Q) \mid \text{div } \sigma = (\partial \sigma_{ij}/\partial x_j, \partial \sigma_{2j}/\partial x_j) \in (L^2(Q))^2 \}.
\]

\( S(Q) \) and \( H(\text{div}, Q) \) are Hilbert spaces with the scalar products

\[
(\sigma, \tau)_S = (\sigma_{ij}, \tau_{ij})_0
\]

and

\[
(\sigma, \tau)_{H(\text{div}, Q)} = (\sigma_{ij}, \tau_{ij})_0 + (\partial \sigma_{ij}/\partial x_j, \partial \sigma_{ik}/\partial x_k)_0,
\]

respectively.
Let $H^{-1/2}(\Gamma_k)$ and $W'$ be dual spaces to $H^{1/2}(\Gamma_k)$ and $W$, respectively. Dual spaces, corresponding to $H^{1/2}(\Gamma_k)$ and $W$, will be denoted by $H^{-1/2}(\Gamma_k) = H^{-1/2}(\Gamma_k) + H^{-1/2}(\Gamma_k)$ and $W' = W' + W''$, respectively. If $\phi^* = (\phi_1^*, \phi_2^*) \in H^{-1/2}(\Gamma_k)$, $\phi = (\phi_1, \phi_2) \in H^{1/2}(\Gamma_k)$.

$$\langle \phi^*, \phi \rangle = \langle \phi_1^*, \phi_1 \rangle + \langle \phi_2^*, \phi_2 \rangle$$

denotes the value of $\phi^*$ at $\phi$. Analogously, for $\mu^* = (\mu_1^*, \mu_2^*) \in W'$, $\mu = (\mu_1, \mu_2) \in W$, we write

$$\langle \mu^*, \mu \rangle = \langle \mu_1^*, \mu_1 \rangle + \langle \mu_2^*, \mu_2 \rangle.$$

It is well known that for $\tau \in H(\text{div}, Q)$, $v \in V$ the following Green's formulas hold ([11]).

(i) There exists a unique mapping $T = (T_1, T_2) \in \mathcal{L}(H(\text{div}, Q), H^{-1/2}(\Gamma_k))$ such that

$$(\tau_{ij}, \varepsilon_{ij}(v))_0 + (\tau_{ij,j}, v_j)_0 = \langle T(\tau), \gamma v \rangle \quad \forall \tau \in H(\text{div}, Q) \forall v \in V,$$

where $\tau_{ij,j} = \partial \tau_{ij}/\partial x_j$. If $\tau \in S(Q) \cap (H^1(Q))^4$, then $T(\tau) = (\tau_{1j}n_j, \tau_{2j}n_j)$ and $\langle , \rangle$ is the scalar product in $L^2(\Gamma_k)$.

(ii) There exists a unique mapping $\tilde{T} = (T_n, T_t) \in \mathcal{L}(H(\text{div}, Q), W')$ such that

$$(\tau_{ij}, \varepsilon_{ij}(v))_0 + (\tau_{ij,j}, v_j)_0$$

$$= \langle \tilde{T}(\tau), \delta v \rangle$$
$$= \langle T_n(\tau), v_n \rangle + \langle T_t(\tau), v_t \rangle \quad \forall \tau \in H(\text{div}, Q) \forall v \in V.$$
By \( \| \varphi^* \|_{-1/2, \Gamma_K} \) we denote the usual dual norm of \( \varphi^* \), namely,

\[
\| \varphi^* \|_{-1/2, \Gamma_K} = \sup_{\varphi \in H^{1/2}(\Gamma_K)} \frac{\langle \varphi^*, \varphi \rangle}{\| \varphi \|_{1/2, \Gamma_K}}.
\]

We show some other expressions of \( \| \varphi^* \|_{-1/2, \Gamma_K} \), which appear to be useful in Section 4.

**Lemma 1.2.** It holds that

\[
\| \varphi^* \|_{-1/2, \Gamma_K} = \| \sigma \|_{H^{1/2}(\Gamma_K)} = \| u(\varphi^*) \| \quad \forall \varphi^* \in H^{-1/2}(\Gamma_K),
\]

where \( u(\varphi^*) \) is the solution of (1.5) and \( \sigma = \varepsilon(u(\varphi^*)) \).

**Proof.** Let \( \tau \in H(\text{div}, \Omega) \) be such that \( T(\tau) = \varphi^* \). Then for every \( \varphi \in H^{1/2}(\Gamma_K) \) we have by the definition of (1.5)

\[
\langle \varphi, \varphi \rangle = \langle T(\tau), \varphi \rangle - (\tau_{ij}, e_{ij}(\varphi))_0 + (\tau_{ij,j}, \varphi_i)_0 \quad \forall \varphi \in \mathbf{V}, \gamma \varphi = \varphi.
\]

Hence

\[
\langle \varphi^*, \varphi \rangle \leq \| \tau \|_{H^{1/2}(\Omega)} \| \varphi \| \quad \forall \varphi \in \mathbf{V}, \gamma \varphi = \varphi,
\]

so that

\[
\| \varphi^* \|_{-1/2, \Gamma_K} \leq \| \tau \|_{H^{1/2}(\Omega)} \quad \forall \tau \in H(\text{div}, \Omega), T(\tau) = \varphi^*,
\]

by virtue of (1.2). Let \( u = u(\varphi^*) \) be the solution of (1.5) and \( \sigma = \varepsilon(u(\varphi^*)) \in H(\text{div}, \Omega) \). Evidently we have

\[
\| \sigma \|_{H^{1/2}(\Omega)} = \| u(\varphi^*) \|.
\]

Inserting \( v = u(\varphi^*) \) into (1.5) and using (1.6'), (1.2), we obtain

\[
\langle \varphi^*, \gamma u(\varphi^*) \rangle = \| u(\varphi^*) \|^2 = \| \sigma \|_{H^{1/2}(\Omega)} \| u(\varphi^*) \|
\]

\[
\geq \| \sigma \|_{H^{1/2}(\Omega)} \| \gamma u(\varphi^*) \|_{1/2, \Gamma_K}.
\]

Hence

\[
\| \varphi^* \|_{-1/2, \Gamma_K} \geq \| \sigma \|_{H^{1/2}(\Omega)},
\]

which, together with (1.6) and (1.6'), proves the lemma.

**Lemma 1.3.** It holds that

\[
\| \varphi^* \|_{-1/2, \Gamma_K} = \sup_{\varphi} \frac{\langle \varphi^*, \gamma \varphi \rangle}{\| \varphi \|} \quad \forall \varphi^* \in H^{-1/2}(\Gamma_K).
\]
Proof. We have
\[
\langle \psi^*, yv \rangle \leq \| \psi^* \|_{-1/2, \Gamma_K} \| yv \|_{1/2, \Gamma_K}
\]
\[
\leq \| \psi^* \|_{-1/2, \Gamma_K} \| v \| \quad \forall \ v \in V,
\]
so that
\[
\sup_{v} \frac{\langle \psi^*, yv \rangle}{\| v \|} \leq \| \psi^* \|_{-1/2, \Gamma_K}.
\]  \tag{1.7}

Let \( u = u(\phi^*) \) be the solution of (1.5). Then for \( \sigma = e(u(\phi^*)) \)
\[
\langle \phi^*, yu(\phi^*) \rangle = \| u(\phi^*) \|^2 = \| u(\phi^*) \| \| \sigma \|_{H(\text{div}, Q)}
\]
\[
= \| u(\phi^*) \| \| \phi^* \|_{-1/2, \Gamma_K}
\]
by virtue of Lemma 1.2. From this and (1.7), the assertion follows.

In a way similar to the proof of Lemma 1.1, we can prove

**Lemma 1.4.** \( \bar{T} \) maps \( H(\text{div}, Q) \) onto \( W' \).

The dual spaces \( H^{-1/2}(\Gamma_K) \) and \( W' \) are mutually isomorphic. Indeed, let \( \phi^* \in H^{-1/2}(\Gamma_K) \) be an arbitrary element. By virtue of Lemma 1.1, there exists a \( \tau \in H(\text{div}, Q) \) such that \( T(\tau) = \phi^* \). Let us set \( \mu^* = \bar{T}(\tau) \in W' \) and write \( \mu^* = \beta^* \phi^*, \beta^*: H^{-1/2}(\Gamma_K) \rightarrow W' \). It is easy to see that \( \mu^* \) does not depend on the choice of \( \tau \in H(\text{div}, Q) \), satisfying \( T(\tau) = \phi^* \). Comparing the Green's formulas (i), (ii) and using the fact that \( H^{-1/2}(\Gamma_K) \) and \( W \) are isomorphic, we obtain the assertion. The relation between the dual norms in \( H^{-1/2}(\Gamma_K) \) and \( W' \) is given by

**Lemma 1.5.** It holds that
\[
\| \phi^* \|_{-1/2, \Gamma_K} = \| \beta^* \phi^* \|_{W'} \quad \forall \ \phi^* \in H^{-1/2}(\Gamma_K).
\]

Proof. Follows immediately from the definition of dual norms and the relation
\[
\langle \phi^*, \phi \rangle = \langle \beta^* \phi^*, \beta \phi \rangle \quad \forall \ \phi \in H^{1/2}(\Gamma_K), \ \forall \ \phi^* \in H^{-1/2}(\Gamma_K).
\]

As \( H^{1/2}(\Gamma_K) \) is dense in \( L^2(\Gamma_K) \), the same holds for \( W \):

**Lemma 1.6.** \( W \) is dense in \( L^2(\Gamma_K) \).

Next, let us suppose that \( Q \) is a bounded polygonal domain, the vertices of which will be denoted by \( A_1, \ldots, A_m \). Another result, useful in what follows, is
**Lemma 1.7.** Let

\[ \mathcal{W}_i = \{ \varphi \in L^2(\Gamma_K) \mid \exists v \in V, \varphi = v_i, v_n = 0 \text{ on } \Gamma_K \}. \]

Then \( \mathcal{W}_i \) is dense in \( L^1(\Gamma_K) = \{ v \in L^1(\Gamma) \mid v = 0 \text{ on } \Gamma_u \} \) in \( L^1(\Gamma_K) \)-norm.

The proof is evident.

**Remark 1.1.** If \( Q \) is a domain with sufficiently smooth boundary, then \( W = H^{1/2}(\Gamma_K) \). Moreover, if \( \varphi = (\varphi_1, \varphi_2) \in W \) and \( \lambda \) is an arbitrary real number, the couple \( (\varphi_1, \lambda \varphi_2) \in W \). This is not true if \( Q \) is a polygonal domain (or a domain with Lipschitz boundary, in general). All difficulties arise from the jumps of the outward normal at the vertices of \( Q \).

Finally, let

\[ W'_+ = \{ \varphi_1^* \in W' \mid \langle \varphi_1^*, v_n \rangle \geq 0 \ \forall \ v \in V, v_n \geq 0 \text{ on } \Gamma_K \} \]

and

\[ W'_- = -W'_+ \]

be the closed convex cones of non-negative and non-positive functionals, respectively, in \( W' \).

Let us introduce

\[ K = \{ v \in V \mid v_n \leq 0 \text{ on } \Gamma_K \}. \]

\( K \) is a closed, convex subset of \( V \).

**By a variational solution of the Signorini problem with friction we call an element \( u \in K \) such that**

\[ \mathcal{F}(u) \leq \mathcal{F}(v) \quad v \in K, \quad (\mathcal{F}) \]

where

\[ \mathcal{F}(v) = \frac{1}{2} (\tau_{ij}(v), \varepsilon_{ij}(v))_0 + \int_{\Gamma_K} g |v_i| \, ds - (F_i, v_i)_0, \]

where \( g \in L^\infty(\Gamma_K), g \geq 0 \text{ a.e. on } \Gamma_K, F = (F_1, F_2) \in (L^2(Q))^2 \) is a vector of body forces, \( \varepsilon_{ij}(v) = \frac{1}{2} (\partial v_i / \partial x_j + \partial v_j / \partial x_i) \) is the strain tensor, corresponding to \( v \in V \) and

\[ \tau_{ij}(v) = c_{ijkl} e_{kl}(v), \quad i, j = 1, 2 \quad (1.8) \]

is the stress tensor, associated to \( g(v) \) by means of the generalized Hooke's law (1.8). As far as the coefficients \( c_{ijkl} \) are concerned, we suppose that
\[ c_{ijkl} \in L^\infty(Q), \quad i,j,k,l = 1,2. \quad (1.9) \]
\[ c_{ijkl} = c_{jilk} = c_{klij} \quad \text{hold a.e. in } Q. \quad (1.10) \]
there exists a positive constant \( a_0 \) such that
\[ c_{ijkl} \xi_{ij} \xi_{kl} \geq a_0 \xi_{ij} \xi_{ij} \quad \forall \xi_{ij} = \xi_{ij} \text{ and a.e. in } Q. \quad (1.11) \]

**Theorem 1.1.** Let (1.9) - (1.11) be satisfied. Then there exists a unique solution \( u \) of (\( \mathcal{P} \)). The solution can be equivalently characterized through the relation

\[
(\tau_{ij}(u), \varepsilon_{ij}(v) - \varepsilon_{ij}(u))_0 + \int_{\Gamma_h} g(|v_i| - |u_i|) \, ds \geq (F_i, v_i - u_i)_0 \quad \forall v \in K. \quad (\mathcal{P}')
\]

For the proof see [2].

Let us analyze the problem (\( \mathcal{P} \)), using the Green's formula (ii). Choosing \( v = u + \psi \), \( \psi = (\psi_1, \psi_2) \in (\mathcal{W}(Q))^2 \) in (\( \mathcal{P}' \)), we obtain

\[ \tau \in H(\text{div}, Q) \quad \text{and} \quad \tau_{ij,j} + F_i = 0 \quad \text{a.e. in } Q, \; i = 1,2. \quad (1.12) \]

Consequently, (\( \mathcal{P}' \)) implies

\[
\left\langle T_i(u), v_n - u_n \right\rangle + \left\langle T_i(u), v_i - u_i \right\rangle + \int_{\Gamma_h} g(|v_i| - |u_i|) \, ds \geq 0 \quad (1.13)
\]

for every \( v \in K \). Let \( v \) be of the form \( v = u \mp \psi \) with \( \psi \in V, \; \psi_n = 0 \; \Gamma_h \). Then \( v \in K \) and (1.13) yields

\[
\left\langle T_i(u), \mp \psi_i \right\rangle + \int_{\Gamma_h} g(|u_i| \mp |\psi_i| - |u_i|) \, ds \geq 0
\]

so that

\[
\mp \left\langle T_i(u), \psi_i \right\rangle + \int_{\Gamma_h} g |\psi_i| \, ds \geq 0.
\]

As a consequence,

\[
\left| \left\langle T_i(u), \psi_i \right\rangle \right| \leq \int_{\Gamma_h} g |\psi_i| \, ds. \quad (1.14)
\]

The mapping \( \Phi: \mathcal{W} \rightarrow R \), defined by means of

\[
\Phi(\psi_i) = \left\langle T_i(u), \psi_i \right\rangle \quad \forall \psi_i \in \mathcal{W}.
\]
is a linear functional on \( W'_t \), bounded in \( L^1(\Gamma_K) \)-norm. By virtue of Lemma 1.7, \( \Phi \) can be extended from \( W'_t \) onto \( L^1(\Gamma_K) \) with the preservation of the norm. Taking into account the form of a general linear functional on \( L^1(\Gamma_K) \) and (1.14), we conclude that

\[
T_i(\mathbf{u}) \in L^\infty(\Gamma_K), \\
|T_i(\mathbf{u})| \leq g \quad \text{a.e. on } \Gamma_K.
\]

As \( u_t \leq |u_t| \), (1.15) yields

\[
T_i(\mathbf{u}) u_t + g |u_t| \geq 0 \quad \text{a.e. on } \Gamma_K.
\]

Let \( \psi^n \in V \) be a sequence of functions, satisfying \( \psi^n \cdot \mathbf{n} = 0 \) on \( \Gamma_K \), \( \psi^n \cdot \mathbf{t} \rightarrow -u_t \) in \( L^1(\Gamma_K) \) (the existence of such a sequence follows from Lemma 1.7). We have

\[
\langle \langle T_i(\mathbf{u}), \psi^n \cdot \mathbf{t} \rangle \rangle + \int_{\Gamma_K} g |u_t + \psi^n \cdot \mathbf{t} - |u_t|| \, ds \\
\rightarrow -\langle \langle T_i(\mathbf{u}), u_t \rangle \rangle - \int_{\Gamma_K} g |u_t| \, ds \geq 0
\]

in view of the fact that the left-hand side of (1.17) is non-negative. From this and (1.16) we find that

\[
T_i(\mathbf{u}) u_t + g |u_t| = 0 \quad \text{a.e. on } \Gamma_K.
\]

Therefore (1.13) together with (1.18) gives

\[
\langle \langle T_n(\mathbf{u}), v_n - u_n \rangle \rangle + \langle \langle T_i(\mathbf{u}), v_t \rangle \rangle + \int_{\Gamma_K} g |v_t| \, ds \geq 0 \quad \forall \mathbf{v} \in K.
\]

Inserting \( \mathbf{v} = 0 \) and \( \mathbf{v} = 2\mathbf{u} \) into the latter inequality, we obtain

\[
\langle \langle T_n(\mathbf{u}), u_n \rangle \rangle = 0.
\]

Finally, (1.13), (1.18) and (1.19) lead to

\[
\langle \langle T_n(\mathbf{u}), v_n \rangle \rangle + \langle \langle T_i(\mathbf{u}), v_t \rangle \rangle + \int_{\Gamma_K} g |v_t| \, ds \geq 0 \quad \forall \mathbf{v} \in K.
\]

Next, let \( \mathbf{v} \in V \) be such that \( v_n \leq 0, \, v_t = 0 \) on \( \Gamma_K \). Then from (1.20) we conclude that

\[
\langle \langle T_n(\mathbf{u}), v_n \rangle \rangle \leq 0.
\]
This inequality, however, does not imply that \( T_n(u) \in W'_+ \). From the latter, (1.21) has to be satisfied for all \( v \in K \), satisfying \( v_\tau = 0 \) on \( \Gamma_K \); see also Remark 1.1.

In order to obtain \( T_n(u) \in W'_+ \), it suffices that \( \Gamma_K \) consists of one straight line segment. In such a case, it is readily seen that if \( (\varphi_1, \varphi_2) \in W \), then also \( (\varphi_1, 0) \in W \), so that (1.21) implies
\[
T_n(u) \in W'_+.
\] (1.22)

**Remark 1.2.** If \( T_n(u) \in L^{2}(\Gamma_K) \), then (1.19) and (1.22) mean the usual pointwise relations \( T_n(u)u_\tau = 0 \) and \( T_n(u) \leq 0 \) a.e. on \( \Gamma_K \), respectively.

**Remark 1.3.** Let \( Q \) be a general polygonal domain and \( \Gamma_K \) arbitrary. Then (1.21) implies (1.22) if, for example, \( T_n(u) \in H^{1/2,1}(\Gamma_K) \) for some \( \varepsilon > 0 \) or if \( g \in L^\infty(\Gamma_K) \) is equal to zero in some neighbourhood of all vertices of \( Q \), which lie on \( \Gamma_K \).

We are able to give a mixed formulation of the problem \((\mathcal{P})\). To this end we introduce the set \( \Lambda = \Lambda_1 \times \Lambda_2 \subseteq W' \), where
\[
\Lambda_1 = W'_+,
\]
\[
\Lambda_2 = \{ \mu_2 \in L^2(\Gamma_K) \mid ||\mu_2|| \leq 1 \text{ a.e. on supp } g, \mu_2 = 0 \text{ on } \Gamma_K - \text{supp } g \}.
\]

One can verify, that \( \Lambda \) is closed and convex in \( W' \).

Let \( \mathcal{L} : H^1(Q) \times W' \to R \) be a Lagrangian, defined through the relation
\[
\mathcal{L}(v, \mu) = \frac{1}{2}(\tau_{ij}(v), e_{ij}(v))_0 + \langle \mu, \delta v \rangle_g - (F_i, v_i)_0,
\]
\[
v = (v_1, v_2) \in H^1(Q), \quad \mu = (\mu_1, \mu_2) \subset W',
\]
where \( \langle \mu, \delta v \rangle_g = \langle \mu_1, v_n \rangle + \langle \mu_2 g, v_\tau \rangle \).

We say that \( \{w, \lambda\} \in V \times \Lambda \) is a saddle-point of \( \mathcal{L} \) on \( V \times \Lambda \) if
\[
\mathcal{L}(w, \mu) \leq \mathcal{L}(w, \lambda) \leq \mathcal{L}(v, \lambda) \quad \forall v \in V, \forall \mu \in \Lambda. \quad (1.23)
\]

The following theorem shows the relation between the solution of \((\mathcal{P})\) and (1.23).

**Theorem 1.2.** Let \( T_n(u) \in W'_+ \). Then there exists a unique saddle-point \( \{w, \lambda_1, \lambda_2\} \in V \times \Lambda_1 \times \Lambda_2 \) of \( \mathcal{L} \) on \( V \times \Lambda \) and it holds that
\[
w = u, \quad \lambda_1 = -T_n(u), \quad g\lambda_2 = -T_\tau(u),
\]
where \( u \) denotes the solution of \((\mathcal{P})\).
Proof. Let \( \{w, \lambda\} \in V \times \Lambda, \lambda = (\lambda_1, \lambda_2) \) satisfy (1.23). An equivalent formulation of (1.23) is
\[
\begin{align*}
(\tau_{ij}(w), \varepsilon_{ij}(v))_0 + \langle \lambda, \delta v \rangle_g &= (F_i, v_i)_0 \quad \forall v \in V, \\
\langle \mu - \lambda, \delta w \rangle_g &\leq 0 \quad \forall \mu \in \Lambda.
\end{align*}
\] (1.24)_a

From (1.24)_a and the Green's formula (ii), \( \lambda_1 = -T_n(w), \) \( g\lambda_2 = -T_i(w) \) follows. Let us analyze (1.24)_b. Introducing \( \mu_2 = \lambda_2 \) and \( \mu_1 = 0, 2 \lambda_1 \) into (1.24)_b, we have
\[
\langle \lambda_1, w_n \rangle = 0, \quad \langle \mu_1, w_n \rangle \leq 0 \quad \forall \mu_1 \in \Lambda_1.
\] (1.25)

Hence we conclude that \( w_n \leq 0 \) a.e. on \( \Gamma_K, \) i.e., \( w \in K. \) Analogously, introducing \( \mu_1 = \lambda_1 \) into (1.24)_b, we have
\[
\langle \mu_2, w_i \rangle \leq \langle g\lambda_2, w_i \rangle \quad \forall \mu_2 \in \Lambda_2.
\]

The choice \( \mu_2 = \text{sign } \lambda \) leads to
\[
\langle g, |w_i| \rangle \leq \langle g\lambda_2, w_i \rangle.
\]

Using this, (1.25), the definitions of \( \Lambda_1, \Lambda_2 \) and (1.24)_a, we obtain
\[
(\tau_{ij}(w), \varepsilon_{ij}(v-w))_0 + \int_{\Gamma_K} g(|v_i| - |w_i|) \, ds \geq (F_i, v_i - w_i)_0 \quad \forall v \in V,
\]
i.e., \( w \) is a solution of (\( \mathcal{P} \)), \( w = u. \) The uniqueness is a consequence of Theorem 1.1.

Conversely, let \( u \in K \) be the solution of (\( \mathcal{P} \)). Then \( \{u, -T_n(u), -g^{-1}T_i(u)\} \in V \times \Lambda \times \Lambda \) follows from the interpretation of (\( \mathcal{P} \)) and the assumptions of the theorem. A direct verification of (1.23), with the use of (1.18), (1.19) and (1.22) is straightforward.

The inequalities (1.23) (or equivalently (1.24)_a, (1.24)_b) will be called a mixed formulation of (\( \mathcal{P} \)) and will be denoted by (\( \mathcal{P} \)), in what follows.

2. APPROXIMATION OF (\( \mathcal{P} \))

Let \( \{\mathcal{G}_h\}, h \to 0+ \) be a regular family of triangulations of \( \bar{Q}, \) with usual requirements on the mutual position of triangles, compatible with the decomposition of \( \Gamma \) into \( \Gamma_u, \Gamma_K \) and moreover compatible with the boundary of

\[\text{If } g = 0, \text{ then } T_i(u) = 0 \text{ as follows from (1.15)_b. In such a case we define } 0/0 = 0.\]
supp $g$ in $\Gamma_K$. By $h$ we denote the maximal diameter of all triangles $T \in \mathscr{T}_h$, and by $a_1, \ldots, a_{m(h)}$ all vertices of $\mathscr{T}_h$, lying on $\Gamma_K$. We define the sets

$$V_h = \{v_h \in (C(\overline{Q}))^2 \cap V | v_h|_T \in (P_1(T))^2 \forall T \in \mathscr{T}_h\}, \quad (2.1)$$

$$W_h' = \{\mu_h \in L^2(\Gamma_K) | \mu_h|_{a_i a_{i+1}} \in (P_0(a_i a_{i+1}))^2, i = 1, \ldots, m-1\}. \quad (2.2)$$

$$A_{1h} = \{\mu_{1h} \in L^2(\Gamma_K) | \mu_{1h}|_{a_i a_{i+1}} \in P_0(a_i a_{i+1}), \mu_{1h} \geq 0 \text{ on } \Gamma_K\}, \quad (2.3)$$

$$A_{2h} = \{\mu_{2h} \in L^2(\Gamma_K) | \mu_{2h}|_{a_i a_{i+1}} \in P_0(a_i a_{i+1}), \quad \|\mu_{2h}\| \leq 1 \text{ on supp } g, \mu_{2h} = 0 \text{ on } \Gamma_K - \text{supp } g\}, \quad (2.4)$$

$$A_h = A_{1h} \times A_{2h}, \quad (2.5)$$

where $P_1(T)$ and $P_0(a_i a_{i+1})$ denotes the space of linear polynomials on $T$ and of constant functions on $a_i a_{i+1}$, respectively. $A_h$ is a finite-dimensional approximation of $A$ (an internal one, i.e., $A_h \subset A$, $\forall h \in (0, 1)$).

In the sequel, we shall consider a more general construction of $V_h$ and $W_h'$. Let $\{\mathscr{F}_h\}, H \to 0^+$ be a family of partitions of $\Gamma$, independent of $\{\mathscr{T}_h\}$, nodes of which will be denoted by $a_1, \ldots, a_{m(H)}$, i.e.,

$$\Gamma = \bigcup_{i=1}^{m(H)} a_i a_{i+1}, \quad a_{m+1} = a_1. \quad (2.1)$$

We suppose that $\mathscr{F}_h$ is compatible with the decomposition of $\Gamma$ into $\Gamma_u, \Gamma_k$ and with the boundary of supp $g$ in $\Gamma_K$. Let us consider regular families of $\{\mathscr{F}_h\}$ in the following sense,

there exists a positive constant $\beta_0$ such that $\min H_i/H \geq \beta_0$,

where $H_i$ is the length of $a_i a_{i+1}, H = \max H_i$. To every $\mathscr{F}_h$ sets $W_h', A_{1h}, A_{2h}$ and $A_h$ will be associated, which are defined as in (2.2)-(2.5), with only a minor change: the subscript $h$ has to be replaced by $H$. In the sequel, the equality $h = H$ means that the partition of $\Gamma$ is generated by the triangulation $\mathscr{F}_h$ of $\overline{Q}$.

The problem of finding a saddle-point \{\(u_h, \lambda_H\)\} of (3) on $V_h \times A_h$ will be called an approximation of (3) and will be denoted by (3h\(h\)). Similarly to the continuous case, \{\(u_h, \lambda_H\)\} $\in V_h \times A_h$ can be characterized equivalently through the relations

$$\langle \tau_{ij}(u_h), e_{ij}(v_h) \rangle_0 + \langle \lambda_H, \delta v_h \rangle_0 = (F_i, v_{ih})_0 \quad \forall v_h \in V_h, \quad (3_{hh})$$

$$\langle \mu_H - \lambda_H, \delta u_h \rangle_0 \leq 0 \quad \forall \mu_H \in A_h, \quad (3_{hh})$$

where $\langle \cdot, \cdot \rangle_0$ is represented by the standard scalar product in $(L^2(\Gamma_K))^2$. 


Let us introduce the closed convex subset $K_{hH}$ of $V_h$,

$$K_{hH} = \{ v_h \in V_h \mid \langle \mu_{1H}, v_h \rangle \leq 0 \ \forall \mu_{1H} \in A_{1H} \}.$$  \hfill (2.6)

$K_{hH}$ contains all functions from $V_h$ such that the mean values of their normal components on any $a_{iH}$ are non-positive. $K_{hH}$ can be regarded as an external approximation of $K$.

Let $(u_h, \lambda_{hH}) \in V_h \times \Lambda_{hH}$ be a solution of $(\mathcal{P}_{hH})$. In view of the definition of $(\mathcal{P}_{hH})$, we see that $u_h \in K_{hH}$ and it is a solution of the variational inequality

$$\langle (\tau_{ij}(u_h), e_{ij}(v_h)) - (\gamma_{ij}(u_h))_0 + \langle g_{ij}, v_h - u_h \rangle \rangle \geq (F_i, v_{ih} - u_{ih})_0 \ \forall \ v_h \in K_{hH}. \hfill (2.7)$$

Now we prove the following existence theorem:

**Theorem 2.1.** To every $h, H \in (0, 1)$ there exists a solution $(u_h, \lambda_{hH})$ of $(\mathcal{P}_{hH})$, the first component of which is uniquely determined.

**Proof.** The existence of the solution follows from the $V$-ellipticity of the form $(\tau_{ij}(v_h), e_{ij}(v_h))_0$, the boundedness of $A_{1H}$ and the fact that there exists a function $\tilde{v}_h \in V_h$ such that $\tilde{v}_h \cdot n < 0$ on $\Gamma_K$ (see [3, Proposition 2.2, p. 161]). The uniqueness of $u_h$ follows from $V$-ellipticity mentioned above and from (2.7).

Let us emphasize that the second component is not unique, in general. If the relation

$$\langle \mu_{1H}, v_{hn} \rangle + \langle g_{2H}, v_{hn} \rangle = 0 \ \forall \ v_h \in V_h

\Rightarrow \mu_{h} = (\mu_{1H}, \mu_{2H}) = 0 \hfill (2.8)$$

holds, $\lambda_{hH}$ is uniquely determined. It is not difficult to find an example of $V_h$ and $\Lambda_{hH}$, in which (2.8) is not satisfied (see [4] with a simple unilateral boundary value problem).

### 3. Convergence Results

Let us study the convergence of $(\mu_{h}, \lambda_{hH})$ to $(\mu, \lambda)$ without any assumptions on the smoothness of $u$. We shall suppose that $h \to 0+$ if and only if $H \to 0+$.

**Theorem 3.1.** For every $v \in K$ let there exist a sequence $(v_h)_h \in K_{hH}$ such that $v_h \to v$ in $V$ for $h \to 0+$. Then

$$u_h \to u \ \text{in} \ H^1(Q), \hfill \lambda_{2H} \to \lambda_2 \ \text{in} \ L^2(\Gamma_K) \ (\text{weakly}).$$
Proof. Recall that the first component $u_h$ is the solution of the minimization problem (2.7). Since $\lambda_{2H}$ is bounded in $L^2(\Gamma_k)$ by the definition of $A_{2H}$, $\lambda_{2H}$ is bounded in $H^1(Q)$, as follows from (2.7). There exist subsequences $\{\lambda_{2H}\} \subset \lambda_{2H}$ and an element $\lambda_2^* \in H^1(Q) \times L^2(\Gamma_k)$ such that

$$
\lambda_{2H} \rightharpoonup \lambda_2^* \quad \text{in} \quad L^2(\Gamma_k).
$$

By the definition of $A_{2H}$, we have $\lambda_2^* \in \Lambda_2$. Let us prove that $u^* \in K$. To this end, it is sufficient to prove that $u^*_n \leq 0$ a.e. on $\Gamma_k$, or equivalently

$$
\langle \mu_1, u^*_n \rangle \leq 0
$$

for every $\mu_1 \in L^2(\Gamma_k) = \{ \mu_1 \in L^2(\Gamma_k), \mu_1 \geq 0 \text{ a.e. on } \Gamma_k \}$.

Let $\mu_1 \in L^2(\Gamma_k)$ be an arbitrary, fixed element. It is easy to see that there exists a sequence $\{\mu_{1H}\}, \mu_{1H} \subset \Lambda_{1H}$ such that

$$
\mu_{1H} \rightharpoonup \mu_1, \quad H \to 0^+, \quad \text{in} \quad L^2(\Gamma_k).
$$

On the other hand, we have

$$
\langle \mu_{1H}, u_{hH} \rangle \leq 0,
$$

since $u_h \in K_{hH}$. Passing to the limit with $h, H \to 0^+$ and using (3.1) and (3.3), we arrive at (3.2).

Next, we show that $u^*$ is a solution of the Signorini problem with friction. Let $v \in K$ be an arbitrary element. By the assumption there exists $\{\nu_h\}$, $\nu_h \in K_{hH}$ and such that

$$
\nu_h \rightharpoonup v \quad \text{in} \quad H^1(Q).
$$

Since the mapping $\gamma: V \to L^2(\Gamma_k)$ is completely continuous, passing to the limit with $h, H \to 0^+$, we obtain from (2.7), (3.1) and (3.4)

$$
(\tau_{ij}(u^*), e_{ij}(u^*))_0 - (\tau_{ij}(u^*), e_{ij}(v))_0 + \langle g\lambda_2^*, u^*_t \rangle - \langle g\lambda_2^*, v_t \rangle \leq (F_i, u^*_t - v_t)_0.
$$

Using the definition of $(\mathcal{P}_{hH})$, in particular $(\mathcal{P}_{hH})$, with $\mu_{1H} = \lambda_{1H}$, we obtain

$$
\langle g\mu_{2H}, u_{hH} \rangle \leq \langle g\lambda_{2H}, u_{hH} \rangle \quad \forall \mu_{2H} \in \Lambda_{2H}.
$$

Since $A_{2H}$ is dense in $A_2$ with $L^2(\Gamma_k)$-norm, one can construct a sequence $\mu_{2H}, \mu_{2H} \in A_{2H}$ such that $\mu_{2H} \rightharpoonup \text{sign } u^*_t$. From this and (3.6) we deduce

$$
\int_{\Gamma_h} g |u^*_t| \, ds = \langle g \text{ sign } u^*_t, u^*_t \rangle \leq \langle g\lambda_2^*, u^*_t \rangle.
$$
On the other hand,

$$\langle g\lambda_z^*, v_i \rangle \leq \int_{\Gamma_K} g|v_i| \, ds.$$ 

Combining (3.5), (3.7) and the latter estimate, we obtain

$$(\tau_{ij}(u^*), \varepsilon_{ij}(u^* - v))_0 + \int_{\Gamma_K} g(|u_i^*| - |v_i|) \, ds \leq (F_i, u_i^* - v_i)_0.$$ 

Since $v \in K$ is arbitrary and $u^* \in K, u^* = u$ follows. From (3.5) we conclude that $T_r(u) = g\lambda_z^*$ on $\Gamma_K$. As the solution $u$ of $(\mathcal{S})$ is uniquely determined, the whole sequences $\{u_h\}$ and $\{\lambda_2^H\}$ tend weakly to $u$ and $\lambda_2$, respectively.

It remains to verify that $u_h$ tends strongly to $u$. Let us introduce the quadratic functional

$$\mathcal{J}_H(v) = \frac{1}{2}(\tau_{ij}(v), \varepsilon_{ij}(v))_0 - (F_i, v_i)_0 + \langle g\lambda_{2H}^+, v_i \rangle$$

$$= \mathcal{J}_0(v) + \langle g\lambda_{2H}^+, v_i \rangle.$$ 

With respect to (2.7), we see that $u_h \in K_{hh}$ solves the problem

$$\mathcal{J}_H(u_h) \leq \mathcal{J}_H(v_h) \quad \forall v_h \in K_{hh}.$$ 

Therefore, applying the Taylor formula to $\mathcal{J}_H$ at $u$, we obtain

$$\mathcal{J}_H(v_h) \geq \mathcal{J}_H(u_h) = \mathcal{J}_0(u) + \langle g\lambda_{2H}^+, u_i \rangle + \mathcal{J}_0'(u, u_h - u)$$

$$+ \langle g\lambda_{2H}^+, u_{hi} - u_i \rangle + \frac{1}{2}(\tau_{ij}(u_h - u), \varepsilon_{ij}(u_h - u))_0$$

$$\geq \mathcal{J}_0(u) + \mathcal{J}_0'(u, u_h - u) + \langle g\lambda_{2H}^+, u_{hi} \rangle + (\alpha/2) \|u_h - u\|^2$$

$$\forall v_h \in K_{hh}.$$ 

Choosing $v_h \in K_{hh}$ such that $v_h \to u$, we obtain

$$(\alpha/2) \|u_h - u\|^2 \leq \mathcal{J}_0(v_h) - \mathcal{J}_0(u) + \langle g\lambda_{2H}^+, v_{hi} \rangle - \mathcal{J}_0'(u, u_h - u)$$

$$\langle g\lambda_{2H}^+, u_{hi} \rangle > 0, \quad h, H \to 0,$$

where the continuity of $\mathcal{J}_0$, the weak convergence of $u_h$ to $u$ and $\lambda_{2H}$ to $\lambda_2$ have been used.

**Remark 3.1.** Since the imbeding of $L^2(\Gamma_K)$ into $W'$ is compact, we obtain $\lambda_{2H} \to \lambda_2$ in $W'$ strongly.

**Remark 3.2.** The question arises, when the system $\{K_{hh}\}$ is dense in $K$. A sufficient condition for this is

$$K \cap (C^\infty(\bar{Q}))^2 = K.$$ 

(3.8)
Indeed, if (3.8) holds, the system of closed convex subsets $\mathcal{H}_h$ of $V_h$, defined by

$$
\mathcal{H}_h = \{v_h \in V_h \mid (v_h \cdot n)(a_i) \leq 0 \text{ for } i = 1, \ldots, m\}
$$

(3.9)
is dense in $K$. As $\mathcal{H}_h \subset \mathcal{K}_h \forall h, H \in (0, 1)$, the required density holds. If $Q$ is polygonal and there exists only a finite number of points $\Gamma_h \cap \Gamma_u$, (3.8) is true (see [6] for the proof).

4. Error Estimates

The aim of this section is to estimate the distance between the approximate displacement field $u_h$ and the exact solution $u$. Since the mixed formulation (3.8) also allows the independent approximation of the normal and tangential forces on $\Gamma_u$, it is natural to estimate the distance between $\lambda_{iH}$ and $\lambda_i$ $(i = 1, 2)$ as well.

We start with an equivalent formulation of (3.8). Let $\mathcal{H} = V \times W'$ be a Hilbert space, equipped with the norm of graph

$$
\|\tau\| = (\|v\|_1^2 + \|\mu\|_{W'}^2)^{1/2} \quad \tau = (v, \mu) \in \mathcal{H}.
$$

Let $A$ be the bilinear form on $\mathcal{H} \times \mathcal{H}$, defined by

$$
A(\tau, \tau') = (\tau_{ij}(u), \varepsilon_{ij}(v))_0 + \langle \lambda, \delta v \rangle_g - \langle \mu, \delta u \rangle_g,
$$

$$
\mathcal{H} = (u, \lambda), \quad \tau = (v, \mu) \in \mathcal{H} \tag{4.1}
$$

and let $F$ be a linear functional over $\mathcal{H}$, defined by

$$
F(\tau') = (F_i, v_i)_0 \quad \tau' = (v, \mu) \in \mathcal{H}. \tag{4.2}
$$

According to the definition (4.1) we see that

$$
\alpha \|v\|_1^2 \leq (\tau_{ij}(v), \varepsilon_{ij}(v))_0 = A(\tau, \tau') \quad \forall \tau = (v, \mu) \in \mathcal{H}, \tag{4.3}
$$

there exists a positive constant $M > 0$ such that

$$
|A(\mathcal{U}, \tau')| \leq M \|\mathcal{U}\|_{\mathcal{H}} \|\tau\|_{\mathcal{H}} \quad \forall \mathcal{U}, \tau' \in \mathcal{H}. \tag{4.4}
$$

The set $\mathcal{K} = V \times \Lambda$ is the closed and convex in $\mathcal{H}$. We shall consider the problem

$$
to \text{ find } \mathcal{U} = (u, \lambda) \in \mathcal{K} \text{ such that } \quad A(\mathcal{U}, \tau' - \mathcal{U}) \geq F(\tau' - \mathcal{U}) \forall \tau' \in \mathcal{K}. \tag{3.8^*}
$$

Using (1.24), we can see that (3.8) and (3.8*) are equivalent. Approximation of (3.8*) will be defined in the usual way: introducing the closed, convex
subsets $\mathcal{H}(h, H) = V_h \times \Lambda_H$, we look for an element $\mathcal{H}_h = (u_h, \lambda_h) \in \mathcal{H}(h, H)$, satisfying

$$\mathcal{A}(\mathcal{H}_h, \mathcal{T}_h - \mathcal{H}_h) \geq \mathcal{F}(\mathcal{T}_h - \mathcal{H}_h) \quad \forall \mathcal{T}_h \in \mathcal{H}(h, H). \quad (\mathcal{F}_{hH}^*)$$

It is easy to verify that $(\mathcal{F}_{hH}^*)$ and $(\mathcal{F}_{hH}^*)$ are equivalent.

**Lemma 4.1.** Let $\mathcal{H}$ and $\mathcal{H}_h$ be the solution of $(\mathcal{F}^*)$ and $(\mathcal{F}_{hH}^*)$, respectively. Then

$$\|u - u_h\|^2_1 \leq c(\mathcal{A}(\mathcal{H} - \mathcal{H}_h, \mathcal{H} - \mathcal{T}_h)$$

$$+ \mathcal{A}(\mathcal{H}, \mathcal{T}_h - \mathcal{H}) + \mathcal{F}(\mathcal{T}_h - \mathcal{H}_h)) \quad \forall \mathcal{T}_h \in \mathcal{H}(h, H). \quad (4.5)$$

**Proof.** From (4.3), the definitions of $(\mathcal{F}^*)$ and $(\mathcal{F}_{hH}^*)$ it follows that

$$a \|u - u_h\|^2_1 \leq \mathcal{A}(\mathcal{H} - \mathcal{H}_h, \mathcal{H} - \mathcal{T}_h)$$

$$+ \mathcal{A}(\mathcal{H}, \mathcal{T}_h - \mathcal{H}) - \mathcal{A}(\mathcal{H}_h, \mathcal{T}_h - \mathcal{H}_h) = \mathcal{A}(\mathcal{H} - \mathcal{H}_h, \mathcal{H} - \mathcal{T}_h)$$

$$+ \mathcal{A}(\mathcal{H}, \mathcal{T}_h - \mathcal{H}) + \mathcal{A}(\mathcal{H}_h, \mathcal{H}_h - \mathcal{T}_h)$$

$$\leq \mathcal{A}(\mathcal{H} - \mathcal{H}_h, \mathcal{H} - \mathcal{T}_h) + \mathcal{F}(\mathcal{H}_h - \mathcal{T}_h)$$

In order to derive an estimate for the distance between $\lambda_h$ and $\lambda$, we make the following assumption, concerning the function $g$. We shall suppose that $g$ is a piecewise constant, non-negative function, defined on $\Gamma_K$.

First, we introduce another formulation of $(\mathcal{F})$. To this end let us define $A^2 = gA^2$, i.e.,

$$A^2 = \{u_2 \in L^2(\Gamma_K) \mid |u_2| \leq g \text{ a.e. on } \Gamma_K\}.$$

To simplify the notations, we omit the superscript $g$ and write simply $A^2$. Let $\Lambda = A_1 \times A_2$. We look for an element $(w, \lambda) \in V \times \Lambda$, $\lambda = (\lambda_1, \lambda_2)$ such that

$$(\tau_{ij}(w), e_{ij}(v))_0 + \langle \lambda, \delta v \rangle = (F_i, v)_0 \quad \forall v \in V,$$

$$\langle \mu - \lambda, \delta w \rangle \leq 0 \quad \forall \mu \in \Lambda, \quad (4.6)$$

where $\langle \mu, \delta v \rangle = \langle \mu, \delta v \rangle$ with $g \equiv 1$ on $\Gamma_K$. Like in Theorem 1.2 we can prove that $w = u$, $\lambda_1 = -T_n(u)$, $\lambda_2 = -T_t(u)$, where $u$ is the solution of $(\mathcal{F})$.

In order to study the approximation of (4.6), we change the definition of $A_{2H}$:

$$A_{2H} = \mu_2 \in L^2(\Gamma_K) \mid |\mu_2| \leq g \text{ on } \Gamma_K,$$

$$\Lambda_H = A_{1H} \times A_{2H}.$$
We suppose that all discontinuities of $g$ coincide with the nodes of $\mathcal{F}_H$. Then $\Lambda_H \subseteq \Lambda \ \forall H \in (0, 1)$. The approximations of (4.6) are defined in a natural way (cf. (5.3)).

Assume that the condition

there exists a positive number $\beta$ independent of $h, H$ and such that

$$\sup_{v_h} \frac{\langle \mu_H, \delta v_h \rangle}{\|v_h\|} \geq \beta \|\mu_H\|_W, \quad \forall \mu_H \in W_H$$

(4.7)

**Lemma 4.2.** Let (4.7) be satisfied. Then

$$\|\lambda - \lambda_H\|_W \leq c\{|u - u_h|\} + \inf_{\lambda_H} \|\lambda - \mu_H\|_W$$

(4.8)

$$\|\lambda - \lambda_H\|_{L^2(\Gamma_K)} \leq c\{(\inf_{\lambda_H} (\|\lambda - \mu_H\|_{L^2(\Gamma_K)} + H^{-1/2} \|\lambda - \mu_H\|_W))$$

$$+ H^{-1/2} \|\lambda - \lambda_H\|_W \} \quad \text{if } \lambda \in L^2(\Gamma_K).$$

(4.9)

**Proof:** Let $\mu_H \in \Lambda_H$ be arbitrary. Using (4.6) and the definition of its approximation, we obtain

$$\langle \mu_H - \lambda_H, \delta v_h \rangle$$

$$= \langle \mu_H, \delta v_h \rangle + (r_{ij}(u_h), \varepsilon_{ij}(v_h))_0 - (F, v_{ih})_0$$

$$= \langle \mu_H, \delta v_h \rangle + (r_{ij}(u_h - u), \varepsilon_{ij}(v_h))_0 - \langle \lambda, \delta v_h \rangle$$

$$\leq c\{|u - u_h|\} + \|\lambda - \mu_H\|_W \|v_h\|,$$

where (1.4) and (1.2) have also been employed.

Therefore

$$\beta \|\lambda_H - \mu_H\|_W \leq \sup_{v_h} \frac{\langle \lambda_H - \mu_H, \delta v_h \rangle}{\|v_h\|}$$

$$\leq c\{|u - u_h|\} + \|\lambda - \mu_H\|_W.$$

Making use of the triangle inequality

$$\|\lambda - \lambda_H\|_W \leq \|\lambda - \mu_H\|_W + \|\mu_H - \lambda_H\|_W,$$

(4.8) follows. Equation (4.9) is a direct consequence of the inverse inequality between $L^2(\Gamma_K)$ and $H^{-1/2}(\Gamma_K)$,

$$\|\mu_H\|_{L^2(\Gamma_K)} = \|\beta^{-1} \mu_H\|_{L^2(\Gamma_K)} \leq cH^{-1/2} \|\beta^{-1} \mu_H\|_{-1/2, \Gamma_K}$$

$$= cH^{-1/2} \|\mu_H\|_W,$$

(see Lemma 1.5).
The main difficulty consists in the verification of (4.7). To this end we introduce the following definition.

**Definition 4.1.** Let us consider the elliptic boundary value problem

\[(\mathcal{P})\]

\[\begin{align*}
(c_{ij}(w), c_{ij}(v))_0 + (w_i, v_i)_0 &= \langle \varphi^*, \gamma v \rangle \quad \forall v \in V.
\end{align*}\]

We say that problem \((\mathcal{P})\) is **regular** if there exists a positive \(\varepsilon\) such that for every \(\varphi^* \in H^{-1/2 + \varepsilon}(\Gamma_K)\) the solution \(w \in H^{1+\varepsilon}(Q) \cap V\) and

\[\|w\|_{1+\varepsilon} \leq c(\varepsilon) \|\varphi^*\|_{-1/2, \Gamma_K}\]

(4.10)

holds with a positive constant \(c\), depending on \(\varepsilon\) only.

This property of \((\mathcal{P})\) plays an important role, as follows from

**Lemma 4.3.** Let \((\mathcal{P})\) be regular and the ratio \(h/H\) be sufficiently small. Then (4.7) holds.

**Proof.** Let \(\mu_H \in \Lambda_H\) be given. Then

\[\sup_{\delta v_h} \left\langle \mu_H, \delta v_h \right\rangle = \sup_{\delta v_h} \frac{\left\langle \beta^{-1} \mu_H, \gamma v_h \right\rangle}{\|v_h\|} \geq \frac{\left\langle \beta^{-1} \mu_H, \gamma w_h \right\rangle}{\|w_h\|} = \|w_h\|,
\]

where \(w_h \in V_h\) is the Galerkin approximation of the solution \(w \in V\) of \((\mathcal{P})\) with \(\varphi^* = \beta^{-1} \mu_H\). But we may write

\[\|w_h\| \geq \|w\| - \|w_h - w\| = \|\beta^{-1} \mu_H\|_{-1/2, \Gamma_K} - \|w_h - w\| = \|\mu_H\|_{W'} - \|w_h - w\|,
\]

as follows from Lemma 1.2. As \(\beta^{-1} \mu_H \in H^{-1/2 + \varepsilon}(\Gamma_K)\) \(\forall \varepsilon \in (0, 1)\), regularity of \((\mathcal{P})\) yields that \(w \in H^{1+\varepsilon}(Q) \cap V\). Using the well-known interpolating properties of \(V_h\) and (4.10), we obtain

\[\|w - w_h\| \leq c h^\varepsilon \|w\|_{1+\varepsilon} \leq c(\varepsilon) h^\varepsilon \|\beta^{-1} \mu_H\|_{-1/2, \Gamma_K}.
\]

Since \(Q\) is a polygonal domain and \(\mu_H\) is constant on \(a_i a_{i+1}\), we have \(\beta^{-1} \mu_H|_{a_i a_{i+1}} \in (P_0(a_i a_{i+1}))^2\), as well. Making use of

\[\|\beta^{-1} \mu_H\|_{-1/2 + \varepsilon, \Gamma_K} \leq c H^{-\varepsilon} \|\beta^{-1} \mu_H\|_{-1/2, \Gamma_K} = c H^{-\varepsilon} \|\mu_H\|_{W'}.
\]
(i.e. the inverse inequality between \( H^{-1/2+\epsilon}(\Gamma_K) \) and \( H^{-1/2}(\Gamma_K) \)), together with (4.11) and (4.12), we obtain

\[
\| \mu_h \|_{w'} = \sup_{v_h} \frac{\langle \mu_h, \delta v_h \rangle}{\| v_h \|} + c(\epsilon)(h/H)^{\epsilon} \| \mu_h \|_{w'}.
\]

If \( h/H \) is sufficiently small, we are led to (4.7).

**Remark 4.1.** Under the assumptions of Lemma 4.3, we have

\[
\sup_{v_h} \frac{\langle \mu_h, \delta v_h \rangle}{\| v_h \|} > cH^{1/2} \| \mu_h \|_{L^2(\Gamma_h)}.
\]  

**Proof.** From the inverse inequality between \( L^2(\Gamma_K) \) and \( H^{-1/2}(\Gamma_K) \) we have

\[
\| \mu_h \|_{L^2(\Gamma_h)} = \| \beta^{-1} \mu_h \|_{L^2(\Gamma_h)} \leq cH^{-1/2} \| \beta^{-1} \mu_h \|_{-1/2, \Gamma_h},
\]

\[
= cH^{-1/2} \| \mu_h \|_{w'}.
\]

From this and (4.7), the relation (4.13) follows.

In order to derive the rate of convergence of \( \{ u_h, \lambda_h \} \), we shall modify (4.5), using the definitions of \( \mathcal{A}, \mathcal{F} \), the inequality \( 2ab = \epsilon a^2 + 1/cb^2 \) and the Green's formula (ii). Thus we obtain

\[
\| u - u_h \|_1^2 \leq cM_\varepsilon \| \mathcal{A} - \mathcal{A}_h \|_\mathcal{F}^2 + 1/\varepsilon \| \mathcal{A} - \mathcal{A}_h \|_\mathcal{F}^2 + (\tau_{ij}(u), \varepsilon \delta(v_h - u))_0 \\
+ \langle \lambda, \delta(v_h - u) \rangle - \langle \mu_h, \lambda, \delta u \rangle + (F_i, u_i - v_{ih})_0 \\
\leq cM_\varepsilon \| u - u_h \|_1^2 + \| \lambda - \mu_h \|_{w'}^2 + 1/\varepsilon \| u - v_h \|_1^2 \\
+ 1/\varepsilon \| \lambda - \mu_h \|_{w'}^2 - (\tau_{ij}(u), v_{ih} - u_i)_0 + \langle \tilde{T}(u), \delta(v_h - u) \rangle \\
+ \langle \lambda, \delta(v_h - u) \rangle - \langle \mu_h, \lambda, \delta u \rangle + (F_i, u_i - v_{ih})_0.
\]

Since \( \tau_{ij}(u) + F_i = 0 \) holds a.e. in \( Q \) and \( \lambda = -\tilde{T}(u) \), from this inequality and (4.8) with \( \varepsilon > 0 \) sufficiently small, it follows that

\[
\| u - u_h \|_1^2 \leq c \inf_{\mathcal{V}_h} \| u - v_h \|_1^2 + \| \lambda - \mu_h \|_{w'}^2 - \langle \mu_h, \lambda, \delta u \rangle. \quad (4.14)
\]

The first result on the rate of convergence is

**Theorem 4.1.** Let the assumptions of Lemma 4.3 be satisfied. If \( u \in H^{1+q}(Q) \) for some \( q > 0 \) and \( T_h(u) \in L^2(\Gamma_h) \), then

\[
\| u - u_h \|_1 = O(H^q), \quad H \to 0^+ \quad (4.15)
\]
and
\[ \| \lambda - \lambda_H \|_{W^*} = O(H^{\frac{3}{2}}), \quad H \to 0+, \]  
(4.16)

where \( \hat{q} = \min(q, 1/4) \).

Proof. In (4.14) let us insert \( v_h = r_h u \), where \( r_h u \in V_h \) is the piecewise linear Lagrange interpolate of \( u \). We have
\[ \| u - r_h u \|_h^2 \leq c h^2 \| u \|_{1+q}^2. \]  
(4.17)

Since \( \lambda_1 = -T_h(u) \in L^2(\Gamma_K) \) by the assumptions and \( \lambda_2 = -T_i(u) \in L^\infty(\Gamma_K) \), we have \( \beta^* \Lambda = \hat{\mu} = (\hat{\mu}_1, \hat{\mu}_2) \in L^2(\Gamma_K) \). Let \( \tilde{\mu}_H \in W_H^* \) be such that \( \tilde{\mu}_H|_{\alpha, \alpha_i, \pm} \) is \( L^2(\alpha, \alpha_i, \pm) \)-projection of \( \mu|_{\alpha, \alpha_i, \pm} \) onto \( (\mu|_{\alpha, \alpha_i, \pm})^2 \), \( i = 1, \ldots, m \). It is easy to verify that \( \beta^* \tilde{\mu}_H|_{\alpha, \alpha_i, \pm} \) is \( L^2(\alpha, \alpha_i, \pm) \)-projection of \( \lambda|_{\alpha, \alpha_i, \pm} \) onto \( (\mu|_{\alpha, \alpha_i, \pm})^2 \). Moreover, as \( \lambda_1 \geq 0 \) and \( \lambda_2 \leq g|_{\alpha, \alpha_i, \pm} = g \in R \), the same holds for \( \beta^* \tilde{\mu}_H \). Hence \( \beta^* \tilde{\mu}_H \in \Lambda_H \) and
\[ \| \lambda - \beta^* \tilde{\mu}_H \|_{W^*} = \| \beta^*-1 \lambda - \tilde{\mu}_H \|_{1/2, \Gamma_K} = O(H). \]  
(4.18)

Using (4.18), we may write
\[ <\beta^* \tilde{\mu}_H - \lambda, \delta u> \]
\[ \leq \| \beta^* \tilde{\mu}_H - \lambda \|_{W^*} \| \delta u \|_{W} \]
\[ = \| \tilde{\mu}_H - \beta^*-1 \lambda \|_{1/2, \Gamma_K} \| \delta u \|_{W} = O(H^{1/2}). \]  
(4.19)

Combining (4.19), (4.17), (4.18) and the fact that \( h \leq cH, c > 0 \), we derive (4.15). The estimate (4.16) now follows from (4.15), (4.18) and (4.8).

Sharper estimates can be obtained under stronger regularity assumptions on \( u \) as follows.

**Theorem 4.2.** Let \( (\mathcal{P}) \) be regular and the ratio \( h/H \) sufficiently small. Let the solution \( u \) of \( (\mathcal{P}) \) belong to \( H^2(Q) \cap K \). Moreover, let the set of points of \( \Gamma_K \), where \( u_n \) and \( u_i \) change from \( u_n < 0 \) to \( u_n = 0 \) and from \( u_i = 0 \) to \( u_i \neq 0 \), respectively, be finite. Then
\[ \| u - u_h \|_1 = O(H^{1/2}), \quad H \to 0+, \]  
(4.20)

\[ \| \lambda - \lambda_H \|_{W^*} = O(H^{1/2}), \quad H \to 0+. \]  
(4.21)

**Proof.** Considering the proof of Theorem 4.1, it suffices to improve the estimate (4.19). We may write
\[ <\beta^* \tilde{\mu}_H - \lambda, \delta u> \]
\[ = \int_{\Gamma_K} (\tilde{\mu}_H + T_i(u)) u_n \, ds + \int_{\Gamma_K} (\tilde{\mu}_H + T_i(u)) u_i \, ds. \]  
(4.22)
Let

\[ \Gamma_0^n = \{ x \in \Gamma_K \mid u_n = 0 \} \]
\[ \Gamma_-^n = \{ x \in \Gamma_K \mid u_n < 0 \} \]
\[ \Gamma_0^i = \{ x \in \Gamma_K \mid u_i = 0 \} \]
\[ \Gamma_+^i = \{ x \in \Gamma_K \mid u_i > 0 \} \]
\[ \Gamma_-^i = \{ x \in \Gamma_K \mid u_i < 0 \} \]

Let the segment \( a_i a_{i+1} \subset \Gamma_0^n \). Then

\[ \int_{a_i a_{i+1}} (\tilde{\mu}_{Hn} + T_n(u)) u_n \, ds = 0, \]  

(4.23)

follows from the definition of \( \Gamma_0^n \). If \( a_i a_{i+1} \subset \Gamma_-^n \), we have \( T_n(u) = 0 \) a.e. on \( a_i a_{i+1} \) and \( \tilde{\mu}_{Hn} \), being the orthogonal \( L^2 \)-projection of \((-T_n(u))\), is equal to zero on \( a_i a_{i+1} \). Thus (4.23) holds again. Let \( I_1 \) denote the set of all \( a_i a_{i+1} \), the interior of which contains both points from \( \Gamma_-^n \) and from \( \Gamma_0^n \). Then

\[ \int_{a_i a_{i+1}} (\tilde{\mu}_{Hn} + T_n(u)) u_n \, ds \leq \| \tilde{\mu}_{Hn} \|_{L^2(a_i a_{i+1})} \| u_n \|_{L^2(a_i a_{i+1})} = O(H). \]

Here we used the fact that \( u_n|_{a_i a_{i+1}} \in H^1(a_i a_{i+1}) \subset C(a_i a_{i+1}) \) and is equal to zero at some interior point of \( a_i a_{i+1} \). The number of elements of \( I_1 \) is bounded above independently of \( H \), by assumption. Therefore

\[ \int_{\Gamma_K} (\tilde{\mu}_{Hn} + T_n(u)) u_n \, ds = \sum_{a_i a_{i+1} \in I_1} \int_{\Gamma_K} (\tilde{\mu}_{Hn} + T_n(u)) u_n \, ds = O(H). \]

(4.24)

We estimate the second term on the right-hand side of (4.22) in a similar way. If \( a_i a_{i+1} \subset \Gamma_0^i \), then

\[ \int_{a_i a_{i+1}} (\tilde{\mu}_{Hl} + T_l(u)) u_i \, ds = 0. \]

(4.25)

If \( a_i a_{i+1} \subset \Gamma_+^i \), then \( T_l(u) u_i + g | u_i | = 0 \) a.e. on \( a_i a_{i+1} \) (see (1.18)) implies \( T_l(u) = -g \), so that \( \tilde{\mu}_{Hl} = g \) on \( a_i a_{i+1} \). If \( a_i a_{i+1} \subset \Gamma_-^i \), \( T_l(u) = g \), \( \tilde{\mu}_{Hl} = -g \) on \( a_i a_{i+1} \) and (4.25) holds again. Let \( I_2 \) be the set of \( a_i a_{i+1} \), the interior of which contains points, belonging to two different sets \( \Gamma_0^i, \Gamma^i, \Gamma^i \), at least. Since \( u_i \in H^1(a_i a_{i+1}) \subset C(a_i a_{i+1}) \), each segment \( a_i a_{i+1} \in I_2 \) contains points
in which $u_t$ is equal to zero. Arguing in the same manner as for $u_n$, we obtain

$$\|u_t\|_{L^2(a,a_{i+1})} = O(H),$$

so that

$$\int_{T_K} (\tilde{\mu}_{H,t} + T_t(u)) u_t \, ds = O(H).$$

(4.26)

Equation (4.20) is a consequence of (4.17), (4.18), (4.24) and (4.26). Equation (4.21) follows from (4.20), (4.8) and (4.18).

**Remark 4.2.** It is readily seen that supplementary regularity conditions on $T_n(u), T_t(u)$ allow to improve (4.20) and (4.21). Moreover, it would be possible to derive an error estimate for $\lambda_H$ in $L^2$-norm, making use of (4.9).

**Remark 4.3.** Finally, we give a hint for the derivation of error estimates in the case of a general bounded and measurable function $g$ on $T_K$. To this end we define

$$\Lambda_{2H} = \{\mu_{2H} \in L^2(T_K) \mid \mu_{2H}\rvert_{a_i a_{i+1}} \in P_0(a_i a_{i+1}),$$

$$\mu_{2H}\rvert_{a_i a_{i+1}} \leq g_i, i = 1,\ldots, m\},$$

where $g_i$ is the mean value of $g$ on $a_i a_{i+1}$. The set $\Lambda_{2H}$ is an exterior approximation of $\Lambda$, in general, i.e., $\Lambda_{2H} \not\subset \Lambda$. In such a case, more general estimates than (4.5) and (4.8) are needed to evaluate the distance between $(u_H, \lambda_H)$ and $(u, \lambda)$ (see [4]). Nevertheless, some results similar to those of (4.20) and (4.21) can still be obtained.

**REFERENCES**


