ON CERTAIN CLASS OF MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS∗

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Abstract In the present investigation we define a new class of meromorphic functions on the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ by making use of the generalized Dziok–Srivastava operator $H_{m}^{l}[\alpha_1]$. Coefficient inequalities, growth and distortion inequalities, as well as closure results are obtained. We also establish some results concerning the partial sums of meromorphic functions and neighborhood results for functions in new class.

Key words meromorphic functions; starlike function; convolution; positive coefficients; coefficient inequalities; Dziok-Srivastava operator

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1 Introduction

Denote by $\Sigma$ the class of normalized meromorphic functions $f$ of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$$

defined on the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$. A function $f \in \Sigma$ is called meromorphic starlike of order $\rho$ ($0 \leq \rho < 1$) if

$$-\Re \left( \frac{zf'(z)}{f(z)} \right) > \rho$$

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for all $z \in \Delta := \Delta^* \cup \{0\}$. The class of all such functions is denoted by $\Sigma^*(\rho)$. Let $\Sigma_P$ be the class of functions $f \in \Sigma$ of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n; \quad a_n \geq 0. \quad (1.3)$$

Further, we denote

$$\Sigma_P^*(\rho) = \Sigma^*(\rho) \cap \Sigma_P.$$

For functions $f(z)$ given by (1.1) and $g(z) = 1/z + \sum_{n=1}^{\infty} b_n z^n$, we define the Hadamard product or convolution of $f$ and $g$ by

$$(f * g)(z) := \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n = (g * f)(z).$$

For complex parameters $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ ($\beta_j \neq 0, -1, \cdots; j = 1, 2, \cdots, m$) the generalized hypergeometric function $lF_m(z)$ is defined by

$$lF_m(z) \equiv lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!} \quad (1.4)$$

$l \leq m + 1; \ l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; z \in U$,

where $\mathbb{N}$ denotes the set of all positive integers and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1, & n = 0, \\ a(a + 1)(a + 2) \cdots (a + n - 1), & n \in \mathbb{N}; a \in \mathbb{C}. \end{cases} \quad (1.5)$$

For positive real values of $\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m$ ($l \leq m + 1; \ l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$), let

$$\mathcal{H}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) : \Sigma \to \Sigma$$

be a linear operator defined by

$$\mathcal{H}^l_m[\alpha, \beta]f(z) = \mathcal{H}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z)$$

$$= \left[z^{-1} lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)\right] \ast f(z),$$

$$\mathcal{H}^l_m[\alpha, \beta]f(z) = z^{-1} + \sum_{n=1}^{\infty} \omega_n(\alpha, \beta; l; m) a_n z^n, \quad (1.6)$$

where

$$\omega_n(\alpha, \beta; l; m) = \frac{(\alpha_1)_{n+1} \cdots (\alpha_l)_{n+1}}{(\beta_1)_{n+1} \cdots (\beta_m)_{n+1}} \frac{1}{(n + 1)!} \quad (1.7)$$

is a positive number for all $n \in \mathbb{N}_0$ so we have $\mathcal{H}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) : \Sigma_P \to \Sigma_P$. It is well known that the power series (1.6) is convergent in $\Delta^*$ as a convolution of two convergent power series. For notational simplicity, we use a shorter notation $\mathcal{H}^l_m[\alpha, \beta]$ for $\mathcal{H}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$ in the sequel.

The class $\Sigma_P^*(\alpha)$ and various other subclasses of $\Sigma$ were studied rather extensively by Clunie [5], Nehari and Netanyahu [12], Pommerenke [13, 14], Royster [15], and others (cf.,
For 0 ≤ η < 1 and 0 ≤ λ < 1/2, let $M_{\lambda, \eta}^l$ denote a subclass of $\Sigma_P$ consisting of functions of form (1.3) satisfying the condition that
\[
-R\left(\frac{z(H_m^l[\alpha, \beta]f(z))' + \lambda z^2(H_m^l[\alpha, \beta]f(z))''}{(1 - \lambda)H_m^l[\alpha, \beta]f(z) + \lambda z(H_m^l[\alpha, \beta]f(z))'}\right) > \eta \quad \text{for all} \quad z \in \Delta,
\] (1.8)
where $H_m^l[\alpha, \beta]$ is given by (1.6). Further, we can state shortly this condition by
\[
-R\left(\frac{zG(z)}{G(z)}\right) > \eta,
\] (1.9)
where
\[
G(z) = \frac{(1 - \lambda)F(z) + \lambda zF'(z)}{1 - 2\lambda}
\] (1.10)
and $F(z) = H_m^l[\alpha, \beta]f(z)$.

Note, that a class of meromorphic harmonic functions, similarly defined as $M_{\lambda, \eta}^l$ was considered in [8]. The present paper aims at providing a systematic investigation of the various interesting properties and characteristics of functions belonging to the class $M_{\lambda, \eta}^l$. Properties of a certain integral operator and its inverse defined on the new class $M_{\lambda, \eta}^l$ are also discussed.

### 2 Coefficients Inequalities

First, we give a necessary and sufficient condition for a function $f$ to be in the class $M_{\lambda, \eta}^l$.

**Lemma 1** Suppose that $\gamma \in [0, 1)$, $r \in (0, 1]$ and the function $H$ is of the form
\[
H(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n, \quad 0 < |z| < r
\] (2.1)
with $b_n \geq 0$. Then the condition
\[
-R\left(\frac{zH'(z)}{H(z)}\right) > \gamma \quad \text{for} \quad |z| < r
\] (2.2)
is equivalent to the condition
\[
\sum_{n=1}^{\infty} (n + \gamma)b_n r^{n+1} \leq 1 - \gamma.
\] (2.3)

**Proof** Let $H$ be of form (2.1) with $b_n \geq 0$ and let it satisfy (2.2). Then
\[
-R\left(\frac{zH'(z)}{H(z)}\right) = R\left\{\frac{1}{z} - \sum_{n=1}^{\infty} nb_n z^n\right\} = R\left\{\frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n\right\} > \gamma
\] (2.4)
for $|z| < r \leq 1$. By letting $z \to r^-$, we obtain

$$1 - \sum_{n=1}^{\infty} nb_n r^{n+1} \geq \gamma,$$

which gives (2.3).

Conversely, suppose that $H$ is given by (2.1) and satisfies (2.3). Then for $|z| < r \leq 1$, we have

$$|zH(z)| = \left|1 + \sum_{n=1}^{\infty} b_n z^{n+1}\right| \geq 1 - \sum_{n=1}^{\infty} b_n |z^{n+1}|$$

$$\geq 1 - \sum_{n=1}^{\infty} b_n r^{n+1} \geq 1 - (n+\gamma)b_n r^{n+1}$$

$$\geq 1 - (1 - \gamma) = \gamma \geq 0.$$}

Thus $H(z) \neq 0$ for $0 < |z| < r \leq 1$ and the function $w(z) := -\frac{zH'(z)}{H(z)}$ is analytic in $|z| < r \leq 1$. Since

$$\Re(w) > \gamma \quad \text{if and only if} \quad |w-1| < |w+1-2\gamma|,$$

to prove (2.2) it is sufficient to show that

$$\left|\frac{w-1}{w+1-2\gamma}\right| < 1 \quad \text{and} \quad |w+1-2\gamma| \neq 0 \quad \text{for} \quad |z| < r \leq 1. \quad (2.5)$$

Suppose that $|w+1-2\gamma| = 0$ for some $z_0$ with $|z_0| < r$. Then we have

$$0 = |w+1-2\gamma| = \left|\frac{z_0H'(z_0)}{H(z_0)} + 1 - 2\gamma\right|$$

$$= \left|\frac{2(1-\gamma) - \sum_{n=1}^{\infty} (n-1+2\gamma)b_n z_0^{n+1}}{1 + \sum_{n=1}^{\infty} b_n z_0^{n+1}}\right|.$$

Thus

$$\sum_{n=1}^{\infty} (n-1+2\gamma)b_n z_0^{n+1} - 2(1-\gamma) = 0.$$

The coefficients $b_n$ are nonnegative so the geometric properties of analytic functions give

$$\sum_{n=1}^{\infty} (n-1+2\gamma)b_n r^{n+1} - 2(1-\gamma) > 0.$$

Hence also

$$2\sum_{n=1}^{\infty} (n+\gamma)b_n r^{n+1} - 2(1-\gamma) > 0,$$

which contradicts with (2.3). Now we will prove the first inequality in (2.5). We see that

$$\left|\frac{w-1}{w+1-2\gamma}\right| = \left|\frac{H(z) + zH'(z)}{zH'(z) - (1-2\gamma)H(z)}\right|.$$
\[
\begin{align*}
&= \left| \frac{\sum_{n=1}^{\infty} (n+1)b_n z^{n+1}}{-2(1-\gamma) + \sum_{n=1}^{\infty} (n-1+2\gamma)b_n z^{n+1}} \right| \\
&< \frac{\sum_{n=1}^{\infty} (n+1)b_n r^{n+1}}{2(1-\gamma) - \sum_{n=1}^{\infty} (n-1+2\gamma)b_n r^{n+1}} \leq 1
\end{align*}
\]

(2.6)
because the denominator of (2.6) minus the nominator is equal to
\[
2 \left[ (1-\gamma) - \sum_{n=1}^{\infty} (n+\gamma)b_n r^{n+1} \right],
\]
which is nonnegative by (2.3). Thus the proof is completed. \(\square\)

Condition (1.2) and the above lemma with \(r=1\) give the following corollary.

**Corollary 1** Let \(f(z) \in \Sigma_{\rho}\) be given by (1.3). Then \(f\) is meromorphic starlike of order \(\rho\), \((0 \leq \rho < 1)\) if and only if
\[
\sum_{n=1}^{\infty} (n+\rho)a_n \leq 1 - \rho.
\]

(2.7)
Moreover, conditions (1.10) and (1.11), applied to Lemma 1 with \(r=1\), provide the next corollary.

**Corollary 2** Let \(f(z) \in \Sigma_{\rho}\) be given by (1.3). Then \(f \in M^l_m(\lambda, \eta)\) if and only if
\[
\sum_{n=1}^{\infty} (n+\eta)(n\lambda - \lambda + 1) \omega_n(\alpha, \beta; l; m)a_n \leq (1-\eta)(1-2\lambda).
\]

(2.8)
For brevity, throughout this paper we let
\[
d_n(\lambda, \eta) := (n+\eta)(n\lambda - \lambda + 1),
\]

(2.9)
unless otherwise stated. Our next result gives the coefficient estimates for functions in \(M^l_m(\lambda, \eta)\).

**Theorem 1** If \(f \in M^l_m(\lambda, \eta)\), then
\[
a_n \leq \frac{(1-\eta)(1-2\lambda)}{d_n(\lambda, \eta)\omega_n(\alpha, \beta; l; m)}, \quad n = 1, 2, 3, \ldots.
\]

(2.10)
The result is sharp for the functions \(f_n(z)\) given by
\[
f_n(z) = \frac{1}{z} + \frac{(1-\eta)(1-2\lambda)}{d_n(\lambda, \eta)\omega_n(\alpha, \beta; l; m)} z^n, \quad n = 1, 2, 3, \ldots.
\]

(2.11)
**Proof** If \(f \in M^l_m(\lambda, \eta)\), then by (2.8), we have, for each \(n,\)
\[
d_n(\lambda, \eta)\omega_n(\alpha, \beta; l; m)a_n \leq \sum_{n=1}^{\infty} d_n(\lambda, \eta)\omega_n(\alpha, \beta; l; m)a_n \leq (1-\eta)(1-2\lambda).
\]
Therefore, we get (2.11). Since
\[
f_n(z) = \frac{1}{z} + \frac{(1-\eta)(1-2\lambda)}{d_n(\lambda, \eta)\omega_n(\alpha, \beta; l; m)} z^n.
\]
satisfies condition (2.8), hence \( f_n(z) \in M^l_m(\lambda, \eta) \) and the equality in (2.10) is attained for this function.

**Theorem 2** Assume that there exists a positive number

\[
v = \inf_{n \in \mathbb{N}} \{ d_n(\lambda, \eta)\omega_n(\alpha, \beta; l; m) \}.
\]  

(2.12)

If \( f \in M^l_m(\lambda, \eta) \), then

\[
|f(z)| \leq \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \leq \frac{1}{r} + r \sum_{n=1}^{\infty} a_n
\]

and, by (2.8) together with (2.12), we obtain

\[
\sum_{n=1}^{\infty} a_n \leq \frac{(1 - \eta)(1 - 2\lambda)}{v}.
\]

(2.15)

Then, using (2.14) and (2.15), we have

\[
|f(z)| \leq \frac{1}{r} + \frac{(1 - \eta)(1 - 2\lambda)}{v} r.
\]

Similarly, we can obtain

\[
|f(z)| \geq \left| \frac{1}{r^2} - \frac{1}{r} \right| - \frac{(1 - \eta)(1 - 2\lambda)}{v} r.
\]

If \( v = d_1(\lambda, \eta)\omega_1(\alpha, \beta; l; m) = (1 + \eta)\omega_1(\alpha, \beta; l; m) \), then the result is sharp for the function (2.13) which satisfies (2.8), hence it is in the class \( M^l_m(\lambda, \eta) \).

**Proof** Since \( f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in M^l_m(\lambda, \eta) \), we have

\[
|f(z)| \leq \frac{1}{r^2} + \sum_{n=1}^{\infty} a_n r^n \leq \frac{1}{r^2} + r \sum_{n=1}^{\infty} a_n
\]

(2.14)

and, by (2.8) together with (2.12), we obtain

\[
\sum_{n=1}^{\infty} a_n \leq \frac{(1 - \eta)(1 - 2\lambda)}{v}.
\]

(2.15)

Similarly, we can prove the following theorem.

**Theorem 3** Suppose that there exists a positive number \( v \) described in (2.12). If \( f \in M^l_m(\lambda, \eta) \), then

\[
\left| \frac{1}{r^2} - \frac{1}{r} \right| - \frac{(1 - \eta)(1 - 2\lambda)}{v} r.
\]

Similarly, we can obtain

\[
|f(z)| \geq \left| \frac{1}{r^2} - \frac{1}{r} \right| - \frac{(1 - \eta)(1 - 2\lambda)}{v} r.
\]

If \( v = d_1(\lambda, \eta)\omega_1(\alpha, \beta; l; m) = (1 + \eta)\omega_1(\alpha, \beta; l; m) \), then the result is sharp for (2.13).

3 Closure Theorems

Let the functions \( f_k(z) \) be given by

\[
f_k(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,k} z^n, \quad k = 1, 2, \ldots, m.
\]  

(3.1)
We shall prove the following closure theorems for the class \( M^l_m(\lambda, \eta) \).

**Theorem 4** Let the function \( f_k(z) \) defined by (3.1) be in the class \( M^l_m(\lambda, \eta) \) for every \( k = 1, 2, \ldots, m \). Then the function \( f(z) \) defined by \( f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \) \((a_n \geq 0)\) belongs to the class \( M^l_m(\lambda, \eta) \), where \( a_n = \frac{1}{m} \sum_{k=1}^{m} a_{n,k} \) \((n = 1, 2, \cdots)\).

**Proof** Since \( f_k(z) \in M^l_m(\lambda, \eta) \), it follows from Corollary 2 that
\[
\sum_{n=1}^{\infty} d_n(\lambda, \eta) \omega_n(\alpha, \beta; l; m)a_n = \sum_{n=1}^{\infty} d_n(\lambda, \eta) \omega_n(\alpha, \beta; l; m) \left( \frac{1}{m} \sum_{k=1}^{m} a_{n,k} \right)
= \frac{1}{m} \sum_{k=1}^{m} \left( \sum_{n=1}^{\infty} d_n(\lambda, \eta) \omega_n(\alpha, \beta; l; m)a_{n,k} \right)
\leq (1 - \eta)(1 - 2\lambda).
\]

By Corollary 2, we have \( f(z) \in M^l_m(\lambda, \eta) \). \( \square \)

**Theorem 5** The class \( M^l_m(\lambda, \eta) \) is closed under convex linear combinations.

**Proof** Let the function \( f_k(z) \) given by (3.1) be in the class \( M^l_m(\lambda, \eta) \). Then it is sufficient to show that the function
\[
H(z) = \mu f_1(z) + (1 - \mu) f_2(z) \quad (0 \leq \mu \leq 1)
\]
is also in the class \( M^l_m(\lambda, \eta) \). Since for \( 0 \leq \mu \leq 1 \),
\[
H(z) = \frac{1}{z} + \sum_{n=1}^{\infty} [\mu a_{n,1} + (1 - \mu) a_{n,2}] z^n,
\]
we observe that
\[
\sum_{n=1}^{\infty} [\mu a_{n,1} + (1 - \mu) a_{n,2}] d_n(\lambda, \eta) \omega_n(\alpha, \beta; l; m)
= \mu \sum_{n=1}^{\infty} d_n(\lambda, \eta) \omega_n(\alpha, \beta; l; m)a_{n,1} + (1 - \mu) \sum_{n=1}^{\infty} d_n(\lambda, \eta) \omega_n(\alpha, \beta; l; m)a_{n,2}
\leq (1 - \eta)(1 - 2\lambda).
\]

By Corollary 2, we have \( H(z) \in M^l_m(\lambda, \eta) \). \( \square \)

**Theorem 6** Let \( f_0(z) = \frac{1}{z} \) and \( f_n(z) = \frac{1}{z} + \frac{(1 - \eta)(1 - 2\lambda)}{d_n(\lambda, \eta) \omega_n(\alpha, \beta; l; m)} z^n \) for \( n = 1, 2, \ldots \). Then \( f(z) \in M^l_m(\lambda, \eta) \) if and only if \( f(z) \) can be expressed in the form \( f(z) = \sum_{n=0}^{\infty} \nu_n f_n(z) \) where \( \nu_n \geq 0 \) and \( \sum_{n=0}^{\infty} \nu_n = 1 \).

**Proof** If
\[
f(z) = \sum_{n=0}^{\infty} \nu_n f_n(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \nu_n (1 - \eta)(1 - 2\lambda) d_n(\lambda, \eta) \omega_n(\alpha, \beta; l; m) z^n,
\]
then it is easy to see that it satisfies condition (2.8) and by Corollary 2 we have \( f \in M^l_m(\lambda, \eta) \). Conversely, let \( f(z) \in M^l_m(\lambda, \eta) \). From Corollary 2, we have \( a_n \leq \frac{(1 - \eta)(1 - 2\lambda)}{d_n(\lambda, \eta) \omega_n(\alpha, \beta; l; m)} \), for \( n = 1, 2, \ldots \), we may take \( \nu_n = \frac{d_n(\lambda, \eta) \omega_n(\alpha, \beta; l; m)}{(1 - \eta)(1 - 2\lambda)} a_n \), for \( n = 1, 2, \ldots \) and \( \lambda_0 = 1 - \sum_{n=1}^{\infty} \nu_n \). Then
\[
f(z) = \sum_{n=0}^{\infty} \nu_n f_n(z). \quad \square
\]
4 Radius of Starlikeness

In the following theorem we obtain the radius of starlikeness for the class $M^l_m(\lambda, \eta)$. We say that $f$ given by (1.3) is meromorphically starlike of order $\rho$ ($0 \leq \rho < 1$) in $|z| < r$ when it satisfies the condition (1.2) in $|z| < r$.

**Theorem 7** Let the function $f$ given by (1.3) be in the class $M^l_m(\lambda, \eta)$. Then, if

$$\inf_{n \geq 1} \left[ \frac{(1 - \rho)d_n(\lambda, \eta)\omega_n(\alpha, \beta; l; m)}{(n + \rho)(1 - \eta)(1 - 2\lambda)} \right]^{\frac{1}{n+1}} := r_1(\eta, \lambda, \rho)$$

(4.1)

is positive, then $f$ is meromorphically starlike of order $\rho$ in $|z| < r \leq r_1(\eta, \lambda, \rho)$.

**Proof** Let the function $f \in M^l_m(\lambda, \eta)$ be of form (1.3). If $0 < r \leq r_1(\eta, \lambda, \rho)$, then by (4.1)

$$r^{n+1} \leq \frac{(1 - \rho)d_n(\lambda, \eta)\omega_n(\alpha, \beta; l; m)}{(n + \rho)(1 - \eta)(1 - 2\lambda)}$$

(4.2)

for all $n \in \mathbb{N}$. From (4.2) we get

$$\frac{n + \rho}{1 - \rho} r^{n+1} \leq \frac{d_n(\lambda, \eta)\omega_n(\alpha, \beta; l; m)}{(1 - \eta)(1 - 2\lambda)}$$

for all $n \in \mathbb{N}$, thus

$$\sum_{n=1}^{\infty} \frac{n + \rho}{1 - \rho} a_n r^{n+1} \leq \sum_{n=1}^{\infty} \frac{d_n(\lambda, \eta)\omega_n(\alpha, \beta; l; m)}{(1 - \eta)(1 - 2\lambda)} a_n \leq 1$$

(4.3)

because of (2.8). If $f \in \Sigma_\rho$, then by Lemma 1 the function $f$ is meromorphically starlike of order $\rho$ in $|z| < r$ if and only if

$$\sum_{n=1}^{\infty} (n + \rho)a_n r^{n+1} \leq 1 - \rho.$$  

(4.4)

Therefore, (4.3) and (4.4) give that $f$ is meromorphically starlike of order $\rho$ in $|z| < r \leq r_1(\eta, \lambda, \rho)$.

Suppose that there exists a number $\tilde{r} > r_1(\eta, \lambda, \rho)$ such that each $f \in M^l_m(\lambda, \eta)$ is meromorphically starlike of order $\rho$ in $|z| < \tilde{r} \leq 1$. The function

$$f(z) = \frac{1}{z} + \frac{(1 - \eta)(1 - 2\lambda)}{d_n(\lambda, \eta)\omega_1(\alpha, \beta; l; m)} z^n$$

is in the class $M^l_m(\lambda, \eta)$, thus it should satisfy (4.4) with $\tilde{r}$:

$$\sum_{n=1}^{\infty} (n + \rho)a_n \tilde{r}^{n+1} \leq 1 - \rho,$$

(4.5)

while the left-hand side of (4.5) becomes

$$(n + \rho) \frac{(1 - \eta)(1 - 2\lambda)}{d_n(\lambda, \eta)\omega_1(\alpha, \beta; l; m)} \tilde{r}^{n+1} > (n + \rho) \frac{(1 - \eta)(1 - 2\lambda)}{d_n(\lambda, \eta)\omega_1(\alpha, \beta; l; m)} \frac{(1 - \rho)d_n(\lambda, \eta)\omega_n(\alpha, \beta; l; m)}{(n + \rho)(1 - \eta)(1 - 2\lambda)} = 1 - \rho,$$

which contradicts with (4.5). Therefore, the number $r_1(\eta, \lambda, \rho)$ in Theorem 7 cannot be replaced by a greater number. This means that $r_1(\eta, \lambda, \rho)$ is the so-called radius of meromorphic starlikeness of order $\rho$ for the class $M^l_m(\lambda, \eta)$. 


5 Integral Operators

In this section, we consider integral transforms of functions in the class \( M^l_m(\lambda, \eta) \).

**Theorem 8** Let the function \( f(z) \) given by (1.3) be in \( M^l_m(\lambda, \eta) \). Then the integral

\[
F(z) = c \int_0^1 u^c f(uz) du \quad (0 < u \leq 1, 0 < c < \infty)
\]

is in the class \( M^l_m(\lambda, \delta) \) \((0 \leq \delta < 1)\) whenever

\[
\delta \leq \frac{\eta(c+1)+1}{\eta+1}.
\]

**Proof** Let \( f(z) \in M^l_m(\lambda, \eta) \). Then

\[
F(z) = c \int_0^1 u^c f(uz) du = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c}{c+n+1} a_n z^n.
\]

It is sufficient to show that

\[
\sum_{n=1}^{\infty} \frac{c}{c+n+1} \frac{d_n(\lambda, \delta) \omega_n(\alpha, \beta; l; m)}{(1-\delta) (1-2\lambda)} a_n \leq 1. \tag{5.2}
\]

Since \( f \in M^l_m(\lambda, \eta) \), we have

\[
\sum_{n=1}^{\infty} \frac{d_n(\lambda, \eta) \omega_n(\alpha, \beta; l; m)}{(1-\eta)(1-2\lambda)} a_n \leq 1.
\]

Note that (5.2) is satisfied if

\[
\frac{c}{c+n+1} \frac{d_n(\lambda, \delta) \omega_n(\alpha, \beta; l; m)}{(1-\delta) (1-2\lambda)} \leq \frac{d_n(\lambda, \eta) \omega_n(\alpha, \beta; l; m)}{(1-\eta)}.
\]

Solving for \( \delta \), we have

\[
\delta \leq \frac{n^2 + n(\eta + 1 + cn) + \eta(c+1)}{n^2 + n(\eta + 1 + c) + c + \eta} = \Phi(n).
\]

A simple computation shows that the function \( \Phi(n) \) is increasing and

\[
\frac{\eta(c+1)+1}{\eta+c+1} = \Phi(1) \leq \Phi(n) < 1.
\]

Using this, the result follows. \( \square \)

It is easy to see that if \( 0 \leq \delta \leq \delta_1 < 1 \), then \( M^l_m(\lambda, \delta_1) \subset M^l_m(\lambda, \delta) \). Therefore the above Theorem 8 provides the following corollary.

**Corollary 3** For integral (5.1), we have

\[
F(M^l_m(\lambda, \eta)) \subset M^l_m(\lambda, \delta) \tag{5.3}
\]

with

\[
\delta \leq \frac{\eta(c+1)+1}{\eta+c+1} < 1.
\]
If we replace the class \( M'_m(\lambda, \delta) \) in (5.3) with a smaller class \( M'_m(\lambda, \delta_1) \) such that
\[
\delta_1 > \frac{\eta(c + 1) + 1}{\eta + c + 1},
\] (5.4)
then (5.3) becomes false.

**Proof** The inclusion relation (5.3) follows directly from Theorem 8. For the proof of sharpness (5.3) notice that for the function
\[
f(z) = \frac{1}{z} + \frac{(1 - \eta)(1 - 2\lambda)}{(1 + \eta)\omega_1(\alpha, \beta; l; m)} z,
\]
satisfies (2.8) so it is in the class \( M'_m(\lambda, \eta) \), moreover, we have
\[
F(z) = \frac{1}{z} + \frac{c(1 - \eta)(1 - 2\lambda)}{(c + 2)(1 + \eta)\omega_1(\alpha, \beta; l; m)} z.
\]
By condition (2.8), the above function \( F \) is in the class \( M'_m(\lambda, \delta_1) \) if and only if
\[
\frac{d_1(\lambda, \delta_1)\omega_1(\alpha, \beta; l; m)}{(1 - \delta_1)(1 - 2\lambda)} \leq \frac{c(1 - \eta)(1 - 2\lambda)}{(c + 2)(1 + \eta)\omega_1(\alpha, \beta; l; m)} \leq 1
\]
or equivalently
\[
\frac{(1 + \delta_1)}{(1 - \delta_1)} \frac{c(1 - \eta)}{(c + 2)(1 + \eta)} \leq 1.
\]
Solving the above inequality with respect to \( \delta_1 \) we obtain
\[
\delta_1 \leq \frac{\eta(c + 1) + 1}{\eta + c + 1},
\]
which contradicts with (5.4).

**Theorem 9** Let \( f(z) \) given by (1.3) be in \( M'_m(\lambda, \eta) \) and
\[
\bar{F}(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c + n + 1}{c} a_n z^n, \quad c > 0.
\] (5.5)
Then \( \bar{F}(z) \) is in \( M'_m(\lambda, \delta) \) (0 \( \leq \delta \leq 1 \)) for \(|z| \leq r(\eta, \lambda; \delta; n)\), where
\[
r(\eta, \lambda; \delta; n) = \inf_{n \in \mathbb{N}} \left( \frac{c(1 - \delta)(n + \eta)}{(1 - \eta)(c + n + 1)(n + \delta)} \right)^{1/(n+1)}.
\] (5.6)

**Proof** By Lemma 1 the function \( \bar{F} \) satisfies (1.9) for \(|z| < r\) if and only if
\[
\sum_{n=1}^{\infty} \frac{(c + n + 1)d_n(\lambda, \delta)\omega_n(\alpha, \beta; l; m)}{c(1 - \delta)(1 - 2\lambda)} a_n r^{n+1} \leq 1.
\] (5.7)
Since \( f \in M'_m(\lambda, \eta) \), by Corollary 2, we have
\[
\sum_{n=1}^{\infty} d_n(\lambda, \eta)\omega_n(\alpha, \beta; l; m) a_n \leq (1 - \eta)(1 - 2\lambda).
\]
Thus inequality (5.7) is satisfied if
\[
\frac{(c + n + 1)d_n(\lambda, \delta)\omega_n(\alpha, \beta; l; m)}{c(1 - \delta)(1 - 2\lambda)} a_n r^{n+1} \leq \frac{d_n(\lambda, \eta)\omega_n(\alpha, \beta; l; m) a_n}{(1 - \eta)(1 - 2\lambda)}.
\]
for all \( n \in \mathbb{N} \). Solving with respect to \( r \) we get
\[
 r(\eta, \lambda, \delta) \leq \left( \frac{c(1 - \delta)(n + \eta)}{(1 - \eta)(c + n + 1)(n + \delta)} \right)^{1/(n+1)} = r(\eta, \lambda, \delta; n)
\]
for all \( n \in \mathbb{N} \). The number sequence \( \{r(\eta, \lambda, \delta; n)\} \) is such that \( \lim_{n \to \infty} r(\eta, \lambda, \delta; n) = 1 \) and \( r(\eta, \lambda, \delta; n) > 0 \) so the infimum in (5.6) exists and is in \((0, 1]\). Therefore, we obtain the desired result. \( \square \)

6 Neighborhoods for the Class \( M^I_m(\lambda, \eta) \)

Following the earlier works on neighborhoods of analytic functions by Goodman [7] and Ruscheweyh [16], we begin with introducing here the \( \delta \)-neighborhood of a function \( f \in \Sigma_P \) of the form (1.3) by means of the definition below:
\[
 N_\delta(f) := \left\{ g \in \Sigma_P : g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \quad \text{and} \quad \sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta \right\}.
\]
(6.1)

Particularly for the identity function \( e(z) = \frac{1}{z} \), we have
\[
 N_\delta(e) := \left\{ g \in \Sigma_P : g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \quad \text{and} \quad \sum_{n=2}^{\infty} n|b_n| \leq \delta \right\}.
\]
(6.2)

**Theorem 10** Suppose that there exists a positive number
\[
 s = \inf_{n \in \mathbb{N}} \left\{ \frac{d_n(\lambda, \eta)\omega_n(\alpha, \beta; l; m)}{n} \right\}.
\]
(6.3)

Then \( M^I_m(\lambda, \eta) \subset N_\delta(e) \), where
\[
 \delta := \frac{(1 - \eta)(1 - 2\lambda)}{s}.
\]
(6.4)

**Proof** For function \( f \in M^I_m(\lambda, \eta) \), by Corollary 2, we have
\[
 (1 - \eta)(1 - 2\lambda) \geq \sum_{n=1}^{\infty} d_n(\lambda, \eta)\omega_n(\alpha, \beta; l; m)a_n \geq \sum_{n=1}^{\infty} nsa_n,
\]
which, in view of definition (6.2), proves Theorem 10. \( \square \)

**Definition 1** A function \( f \in \Sigma_P \) is said to be in the class \( M^I_m(\lambda, \eta, \gamma) \), \( 0 \leq \gamma < 1 \), if there exists a function \( g \in M^I_m(\lambda, \eta) \) such that
\[
 \left| \frac{f'(z)}{g'(z)} - 1 \right| < \gamma \quad (z \in \Delta^*).
\]
(6.5)

**Theorem 11** Suppose that there exists a positive number \( v \) given by (2.12). If \( g \in M^I_m(\lambda, \eta) \) and
\[
 \gamma = \frac{\delta v}{|v - (1 - \eta)(1 - 2\lambda)|},
\]
(6.6)

then
\[
 N_\delta(g) \subset M^I_m(\lambda, \eta, \gamma).
\]
Proof Let $f \in N_\delta(g)$. Then we find from (6.1) that

$$
\sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta,
$$

(6.7)

where

$$
g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n, \quad f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n.
$$

Since $g \in M^l_m(\lambda, \eta)$, then using Theorem 3, we obtain

$$
\frac{|f'(z)| - 1}{|g'(z)|} < \frac{\sum_{n=1}^{\infty} n|a_n - b_n|}{\delta} \leq \frac{\delta}{1 - \frac{(1-\eta)(1-2\lambda)}{\epsilon}} = \gamma,
$$

provided $\gamma$ is given by (6.6). Hence, by definition, $f \in M^l_m(\lambda, \eta, \gamma)$ for $\gamma$ given by (6.6), which completes the proof.

7 Partial Sums

Silverman [17] determined sharp lower bounds for the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. As a natural extension, one is interested to search results analogous to those of Silverman for meromorphic univalent functions. In this section, motivated essentially by the work of Silverman [17] and Cho and Owa [4] we will investigate the ratio of a function of the form

$$
f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n,
$$

(7.1)

to its sequence of partial sums $f_k(z) = \frac{1}{z} + \sum_{n=1}^{k} a_n z^n$ when the coefficients are sufficiently small so that satisfy the condition analogous to

$$
\sum_{n=1}^{\infty} d_n(\lambda, \eta) \omega_n(\alpha, \beta; l; m) a_n \leq (1 - \eta)(1 - 2\lambda).
$$

More precisely, we will determine sharp lower bounds for $\Re\{f(z)/f_k(z)\}$ and $\Re\{f_k(z)/f(z)\}$. In this connection we make use of the well known result that $\Re\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0 (z \in \Delta)$ if and only if $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$ satisfies the inequality $|\omega(z)| \leq 1$. Unless otherwise stated, we assume that $f$ is of the form (1.3) and its sequence of partial sums is denoted by

$$
f_k(z) = \frac{1}{z} + \sum_{n=1}^{k} a_n z^n \quad k = 1, 2, 3, \ldots.
$$

(7.2)

Further, for simplicity, we denote in the sequel

$$
\omega_n = \omega_n(\alpha, \beta; l; m),
$$

$$
d_n = d_n(\lambda, \eta).
$$
Theorem 12 Suppose that a function $f$ is in the class $M^l_m(\lambda, \eta)$ and suppose that all of its partial sums (7.2) don’t vanish in $\Delta^*$. Moreover, suppose that

$$2 - 2 \sum_{n=1}^{k} |a_n| - \frac{d_{k+1}\omega_{k+1}}{(1 - \eta)(1 - 2\lambda)} \sum_{n=k+1}^{\infty} |a_n| > 0 \quad \text{for all } k \in \mathbb{N}. \quad (7.3)$$

Then

$$\Re \left\{ \frac{f(z)}{f_k(z)} \right\} \geq \frac{d_{k+1}\omega_{k+1} - (1 - \eta)(1 - 2\lambda)}{d_{k+1}\omega_{k+1}} \quad (7.4)$$

for all $k \in \mathbb{N}$ and all $z \in \Delta^*$, where

$$d_n\omega_n = \begin{cases} (1 - \eta)(1 - 2\lambda), & \text{if } n = 1, 2, 3, \ldots, k, \\ d_{k+1}\omega_{k+1}, & \text{if } n = k + 1, k + 2, \ldots. \end{cases} \quad (7.5)$$

Result (7.4) is sharp for the function given by

$$f(z) = \frac{1}{z} + \frac{(1 - \eta)(1 - 2\lambda)}{d_{k+1}\omega_{k+1}} z^{k+1}. \quad (7.6)$$

Proof Define the function $w(z)$ by

$$w(z) = \frac{1 + w(z)}{1 - w(z)} = \frac{d_{k+1}\omega_{k+1}}{(1 - \eta)(1 - 2\lambda)} \left[ \frac{f(z)}{f_k(z)} - \frac{d_{k+1}\omega_{k+1} - (1 - \eta)(1 - 2\lambda)}{d_{k+1}\omega_{k+1}} \right]$$

$$= 1 + \sum_{n=1}^{k} a_n z^{n+1} + \left( \frac{d_{k+1}\omega_{k+1}}{(1 - \eta)(1 - 2\lambda)} \right) \sum_{n=k+1}^{\infty} a_n z^{n+1}$$

$$= 1 + \sum_{n=1}^{k} a_n z^{n+1} + \left( \frac{d_{k+1}\omega_{k+1}}{(1 - \eta)(1 - 2\lambda)} \right) \sum_{n=k+1}^{\infty} a_n z^{n+1}. \quad (7.7)$$

For (7.4), it suffices to show that $|w(z)| \leq 1$. Now, from (7.7) we can write

$$w(z) = \frac{\left( \frac{d_{k+1}\omega_{k+1}}{(1 - \eta)(1 - 2\lambda)} \right) \sum_{n=k+1}^{\infty} a_n z^{n+1}}{2 + 2 \sum_{n=1}^{k} a_n z^{n+1} + \left( \frac{d_{k+1}\omega_{k+1}}{(1 - \eta)(1 - 2\lambda)} \right) \sum_{n=k+1}^{\infty} a_n z^{n+1}}. \quad (7.8)$$

Hence, by (7.3), we obtain

$$|w(z)| \leq \frac{\left( \frac{d_{k+1}\omega_{k+1}}{(1 - \eta)(1 - 2\lambda)} \right) \sum_{k=1}^{\infty} |a_n|}{2 - 2 \sum_{n=1}^{k} |a_n| - \left( \frac{d_{k+1}\omega_{k+1}}{(1 - \eta)(1 - 2\lambda)} \right) \sum_{n=k+1}^{\infty} |a_n|}. \quad (7.9)$$

Now $|w(z)| \leq 1$ if and only if

$$2 - 2 \sum_{n=1}^{k} |a_n| - \frac{d_{k+1}\omega_{k+1}}{(1 - \eta)(1 - 2\lambda)} \sum_{n=k+1}^{\infty} |a_n| \leq 0$$

which is equivalent to

$$\sum_{n=1}^{k} |a_n| + \frac{d_{k+1}\omega_{k+1}}{(1 - \eta)(1 - 2\lambda)} \sum_{n=k+1}^{\infty} |a_n| \leq 1.$$
From condition (2.8), it is sufficient to show that
\[
\sum_{n=1}^{k} |a_n| + \frac{d_{k+1}\omega_{k+1}}{(1-\eta)(1-2\lambda)} \sum_{n=k+1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} \frac{d_n\omega_n}{(1-\eta)(1-2\lambda)} |a_n|,
\]
which is equivalent to
\[
\sum_{n=1}^{k} \frac{d_n\omega_n - (1-\eta)(1-2\lambda)}{(1-\eta)(1-2\lambda)} |a_n| + \sum_{n=k+1}^{\infty} \frac{d_n\omega_n - d_{k+1}\omega_{k+1}}{(1-\eta)(1-2\lambda)} |a_n| \geq 0, \quad (7.8)
\]
which is true through (7.5). To see that the function \( f \) given by (7.6) gives the sharp result, we observe that \( f \) satisfies (2.8) so \( f \in M_{\eta}^{l}(\lambda, \eta) \) and for \( z = re^{i\pi/(k+2)} \) we have
\[
\frac{f(z)}{f_k(z)} = 1 + \frac{(1-\eta)(1-2\lambda)}{d_{k+1}\omega_{k+1}} z^{k+2} = 1 - \frac{(1-\eta)(1-2\lambda)}{d_{k+1}\omega_{k+1}} r^{k+2}
\]
\[
\rightarrow \frac{d_{k+1}\omega_{k+1} - (1-\eta)(1-2\lambda)}{d_{k+1}\omega_{k+1}} \text{ as } r \rightarrow 1^{-}, \quad (7.9)
\]
which shows that bound (7.4) is the best possible for each \( k \in \mathbb{N} \). \( \square \)

**Theorem 13** Under the assumptions of Theorem 12, we have
\[
\Re \left\{ \frac{f_k(z)}{f(z)} \right\} \geq \frac{d_{k+1}\omega_{k+1}}{d_{k+1}\omega_{k+1} + (1-\eta)(1-2\lambda)} \quad (z \in \Delta^*). \quad (7.10)
\]
Result (7.10) is sharp for the function given by (7.6).

**Proof** As in the previous proof, let
\[
\frac{1+w(z)}{1-w(z)} = \frac{d_{k+1}\omega_{k+1} + (1-\eta)(1-2\lambda)}{(1-\eta)(1-2\lambda)} \left[ \frac{f_k(z)}{f(z)} - \frac{d_{k+1}\omega_{k+1}}{d_{k+1}\omega_{k+1} + (1-\eta)(1-2\lambda)} \right]
\]
\[
= 1 + \sum_{n=1}^{k} a_n z^{n+1} - \left( \frac{d_{k+1}\omega_{k+1}}{(1-\eta)(1-2\lambda)} \right) \sum_{n=k+1}^{\infty} a_n z^{n+1}
\]
\[
= 1 + \sum_{n=1}^{\infty} a_n z^{n+1},
\]
it suffices to show that \( |w(z)| \leq 1 \), which leads us to the inequality
\[
|w(z)| \leq \frac{\left( \frac{d_{k+1}\omega_{k+1} + (1-\eta)(1-2\lambda)}{(1-\eta)(1-2\lambda)} \right) \sum_{n=k+1}^{\infty} |a_n|}{2 - 2 \sum_{n=1}^{k} |a_n| - \left( \frac{d_{k+1}\omega_{k+1} - 1+\eta}{(1-\eta)(1-2\lambda)} \right) \sum_{n=k+1}^{\infty} |a_n|} \leq 1.
\]
This last inequality is equivalent to
\[
\sum_{n=1}^{k} |a_n| + \frac{d_{k+1}\omega_{k+1}}{(1-\eta)(1-2\lambda)} \sum_{n=k+1}^{\infty} |a_n| \leq 1.
\]
Make use of (2.8) we get (7.10). Finally, equality holds in (7.10) for the extremal function \( f \) given by (7.6) if we put \( z = r \) in (7.9).

**Concluding Remarks** By specializing the parameters \( l, m, \lambda \), the various results presented in this paper would provide interesting extensions and generalizations of those considered
earlier in [1–3, 5, 11, 18–20]. In fact, by appropriately selecting these arbitrary sequences, with
the Fox-Wright generalization $\psi_m$ of the hypergeometric function $F_m$, the results presented in
this paper would find further applications for the class of meromorphic functions. Theorems 1
to 13 would thus eventually lead us further to new results for the class of functions (defined analog-
ously to the class $M_m^\lambda(\lambda, \eta)$) by associating with the Fox-Wright generalized hypergeometric
function $\psi_m$.

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