Tilting modules over small Dedekind domains

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Abstract

A Dedekind domain \(R\) is called small if \(\text{card}(R) \leq 2^\omega\) and \(\text{card}(\text{Spec}(R)) \leq \omega\). Assuming Gödel’s Axiom of Constructibility \((V = L)\), we characterize tilting modules over small Dedekind domains. In particular, we prove that under \(V = L\), a class of modules, \(\mathcal{T}\), is a tilting torsion class iif there is a set \(P \subseteq \text{Spec}(R)\) such that \(\mathcal{T} = \{M \in \text{Mod}-R \mid \text{Ext}_1^R(R/p; M) = 0 \text{ for all } p \in P\}\).

Since the early 1970s, tilting theory has been developed in the setting of finite dimensional modules over finite dimensional algebras [2]. Some of its aspects have later been extended to arbitrary modules over arbitrary rings. This is the case of tilting torsion classes of modules [5] and tilting approximations [1], for example.

In parallel to the general theory, the structure of (infinitely generated) tilting modules has been described in detail for particular classes of rings and algebras. The case of abelian groups was treated in [13]. In the present paper, we extend the results of [13] to modules over small Dedekind domains. A Dedekind domain \(R\) is called small if \(\text{card}(R) \leq 2^\omega\) and the spectrum of \(R\), \(\text{Spec}(R)\), is countable.

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A (right $R$-) module $T$ is tilting if $\text{Gen}(T) = T^\perp$. Here, $\text{Gen}(T)$ denotes the class of all modules generated by $T$, that is, of all homomorphic images of the modules of the form $T^{(\kappa)}$ where $\kappa$ is a cardinal, and $T^\perp = \ker \text{Ext}(T, -) = \{ M \mid \text{Ext}(T, M) = 0 \}$. If $T$ is tilting, then $\text{Gen}(T)$ is a torsion class of modules; it is called the tilting torsion class generated by $T$. Equivalently, $T$ is a tilting module if $T$ has projective dimension $\leq 1$, $\text{Ext}(T, T^{(\kappa)}) = 0$ for any cardinal $\kappa$, and there is an exact sequence $0 \to R \to T' \to T'' \to 0$ with $T', T'' \in \text{Add}(T)$. Here, $\text{Add}(T)$ denotes the class of all direct summands of modules of the form $T^{(\kappa)}$ for a cardinal $\kappa$ [5].

For a set $P \subseteq \text{Spec}(R)$ denote by $T_P$ the class of all modules which are $p$-divisible for each $p \in P$. Here, a module $M$ is $I$-divisible for a right ideal $I$ of $R$ if $\text{Ext}(R/I, M) = 0$. As usual, $M$ is called divisible if $M$ is $p$-divisible for all principal right ideals $p$ of $R$. For Dedekind domains, divisibility is also equivalent to $p$-divisibility for all $p \in \text{Spec}(R)$.

In [18, Corollary 4.6] it was shown that $T_P$ is a tilting torsion class for any $P \subseteq \text{Spec}(R)$ and any Dedekind domain $R$. If $R = \mathbb{Z}$ then these are the only tilting torsion classes assuming Gödel’s Axiom of Constructibility ($V = L$) [13]. In the present paper, we will extend the latter result and characterize tilting modules, and tilting torsion classes of modules, over arbitrary small Dedekind domains (cf. Theorem 12 and Corollary 13).

Though we do not know whether our main results hold true in ZFC, we do show that our method of proof, that is, Theorem 11, is independent of ZFC (cf. Theorem 14).

In what follows, $R$ always denotes a Dedekind domain which is not a field and $Q$ denotes the quotient field of $R$. Then, all non-zero prime ideals of $R$ are maximal, and divisible modules coincide with the injective ones. By [15, 18.8], $Q/R \cong \bigoplus_{0 \neq p \in \text{Spec}(R)} E(R/p)$ where $E(R/p)$ denotes the injective hull of $R/p$. Moreover, the localization, $R_p$, of $R$ at a prime ideal $p$ is a noetherian valuation domain, hence a principal ideal domain. The ideals of $R_p$ are of the form $p^nR_p$. In particular, an $R_p$-module is divisible iff it is $pR_p$-divisible where $pR_p$ is the unique maximal ideal of $R_p$. We refer to [15, 10] for basic facts on modules over Dedekind domains.

We start with a simple characterization of $p$-divisible modules.

**Lemma 1.** Let $0 \neq p \in \text{Spec}(R)$ and $M$ be module. The following are equivalent:

(i) $M$ is $p$-divisible;

(ii) the localization $M_p = M \otimes R_p$ is a divisible $R_p$-module;

(iii) $pM = M$.

**Proof.** (i) $\iff$ (ii): Consider $E = \text{Ext}_R(R/p, M)$. Then $E = 0$ iff $E \otimes R_q = 0$ for all $q \in \text{Spec}(R)$ [15, Section 4.6]. Since $R/p$ is finitely generated, we have $E \otimes R_q \cong \text{Ext}_{R_q}(R/p \otimes R_q, M \otimes R_q)$ for each $q \in \text{Spec}(R)$ [15, p. 53]. As $(R/p) \otimes R_q = 0$ for $q \neq p$, we have $E = 0$ iff $\text{Ext}_{R_p}((R/p) \otimes R_p, M_p) = 0$. The latter says that $\text{Ext}_{R_p}(R_p/pR_p, M_p) = 0$ [15, Section 4.2], that is, $M_p$ is $(pR_p -)$ divisible.
Lemma 2. Let $0 \neq p \in \text{Spec}(R)$. Then any pure submodule of a $p$-divisible module is also $p$-divisible.

Proof. Let $N$ be a pure submodule of a $p$-divisible module $M$. Consider the pure-exact sequence $0 \to N \to M \to M/N \to 0$. The induced sequence $0 \to N \otimes R_p \to M \otimes R_p \to (M/N) \otimes R_p \to 0$ is pure-exact in $\text{Mod-}R_p$. Moreover, $M \otimes R_p$ is a divisible $R_p$-module by Lemma 1. It follows that $N \otimes R_p$ is divisible, so $N$ is $p$-divisible by Lemma 1.

Next, we consider localizations at “sets of primes”.

Lemma 3. Let $0 \in P \subseteq \text{Spec}(R)$. Define $R_{(P)} = \bigcap_{P \in P} R_P$. Put $S = R \setminus (\bigcup_{P \in P} P)$.

Then

1. $R_{(P)} = R_S$ where $R_S$ is the localization of $R$ at $S$;
2. $R_{(P)}/R \cong \bigoplus_{q \notin P} E(R/q)$ and $Q/R_{(P)} \cong \bigoplus_{0 \neq q \in P} E(R/p)$;
3. $J_{\text{Spec}(R) \setminus P} = \{M \in \text{Mod-}R \mid \forall sM = M \forall s \in S\} \supseteq \text{Mod-}R_{(P)}$.

Proof. First, we prove (i) and (ii). Clearly, $S$ is multiplicative and $R \subseteq R_S \subseteq R_{(P)} \subseteq Q$.

Localizing the exact sequence $0 \to R \to Q \to \bigoplus_{0 \neq p \in \text{Spec}(R)} E(R/p) \to 0$ at $S$ we get $0 \to R_S \to Q \to \bigoplus_{0 \neq p \in P} E(R/p) \to 0$ [15, 18.4].

Consider the exact sequence $0 \to R \to R_S \to R_S/R \to 0$. Localizing at $0 \neq q \in \text{Spec}(R)$ we get $0 \to R_q \to R_S \otimes_R R_q \to (R_S/R) \otimes_R R_q \to 0$. If $P \in P$ then $R_S \otimes_R R_P \cong R_P$, and $(R_S/R) \otimes_R R_P = 0$. If $q \notin P$ then $R_S \otimes_R R_q \cong Q$, so $(R_S/R) \otimes_R R_q \cong Q/R_q \cong E(R/q)$. It follows that $R_S/R \cong \bigoplus_{q \notin P} E(R/q)$.

In particular, we have $R_q = \bigoplus_{q \notin R/q E(R/p)}$ for any non-zero prime ideal $q$. Hence, $R_{(P)}/R = (\bigcap_{P \in P} R_P)/R \cong \bigoplus_{q \notin P} E(R/q) \cong R_S/R$. Since $R_S \subseteq R_{(P)}$, we conclude that $R_S = R_{(P)}$.

Finally, we prove (iii). Assume $sM = M$ for all $s \in S$. If $q \notin P$, then there exists $s \in q \setminus \bigcup_{P} P$. Since $sM = M$, also $qM = M$. By Lemma 1, $M$ is $q$-divisible for all $q \notin P$, that is $M \in J_{\text{Spec}(R) \setminus P}$.

Conversely, if $M \in J_{\text{Spec}(R) \setminus P}$ then $qM = M$ for all $q \notin P$ by Lemma 1. Let $s \in S$. Then $sR = q_1 \cdots q_n$ for some prime ideals $q_i \notin P$, so $sM = M$.

By part (i), $sM = M$ for all $M \in \text{Mod-}R_{(P)}$ and $s \in S$, so $J_{\text{Spec}(R) \setminus P} \subseteq \text{Mod-}R_{(P)}$.

Now we turn our attention to tilting torsion classes. First we show that certain tilting modules generate the tilting torsion classes $J_P (P \subseteq \text{Spec}(R))$. 


Proposition 4. Let $0 \not\in P \subseteq \text{Spec}(R)$ and $P' = \text{Spec}(R) \setminus P$. Then $\mathcal{T}_P$ is a tilting torsion class generated by the tilting module $T = \bigoplus_{p \in P} E(R/p)^{(x_p)} \oplus T'$ where $0 < x_p$ for all $p \in P$, and $T'$ is a projective generator over $R_{(p')} = \bigcap_{q \in P'} R_q$.

Proof. First we prove that $T^{\perp} = \mathcal{T}_P$. Let $0 \neq p \in \text{Spec}(R)$. By [15, 18.4], $E(R/p)$ has a filtration with successive factors isomorphic to $R/p$. So $(\bigoplus_{p \in P} E(R/p)^{(x_p)})^{\perp} = (\bigoplus_{p \in P} R/p)^{\perp} = \mathcal{T}_P$ by [8, Lemma 1]. Clearly, $T'^{\perp} = R_{(p')}$. By Lemma 3(ii), $R_{(p')}$ has a filtration with successive factors isomorphic to $R$ or $R/p$ ($p \in P$), so $T'^{\perp} \supseteq (\bigoplus_{p \in P} R/p)^{\perp} = \mathcal{T}_P$. It follows that $T^{\perp} = (\bigoplus_{p \in P} E(R/p)^{(x_p)})^{\perp} \cap T'^{\perp} = \mathcal{T}_P$.

It remains to prove that $T$ is a tilting module. By Lemma 3(iii), $R_{(p')} \in \mathcal{T}_P$, and since $T' \in \text{Mod-}R_{(p')}$, also $T' \in \mathcal{T}_P$. As $E(R/p)$ is divisible for any $p \in P$, we infer that $\mathcal{T}_P$ is a torsion class containing $T$. So $\text{Gen}(T) \subseteq \mathcal{T}_P$.

On the other hand, take $M \in \mathcal{T}_P$. Then $sM = M$ for all $s \in S = R \setminus \bigcup_{q \in P'} q$ by Lemma 3(iii). For each $p \in P$, take $x_p \in p \setminus \bigcup_{q \in P'} q$. By Lemma 3(ii) and [15, 18.4], the elements $\{x_p^n \mid n < \omega, p \in P\} \subseteq Q$ generate the $R$-module $R_{(p')}$. Since $x_p \in S$ for all $p \in P$, for each $m \in M$, we can define an $R$-homomorphism $\phi_m: R_{(p')} \to M$ such that $\phi_m(1) = m$ and $\phi_m(x_p^{-n}) = x_p^{-1} \phi_m(x_p^{-n})$ for all $n < \omega$ and $p \in P$. It follows that $\mathcal{T}_P \subseteq \text{Gen}(R_{(p')}) = \text{Gen}(T')$. Altogether, $\text{Gen}(T) = \mathcal{T}_P = T^{\perp}$, so $T$ is tilting. $\square$

Later on, we will show that the tilting torsion classes $\mathcal{T}_P$ ($P \subseteq \text{Spec}(R)$) are the only ones assuming $V = L$. In ZFC, however, we obtain the following:

Theorem 5. Let $\mathcal{T}$ be a class of modules such that $M \in \mathcal{T}$ iff $M^{**} \in \mathcal{T}$ where $M^* = \text{Hom}_R(M, Q/R)$ denotes the character module of $M$.

Then, $\mathcal{T}$ is a tilting torsion class if and only if $\mathcal{T} = \mathcal{T}_P$ for a set $P \subseteq \text{Spec}(R)$.

Proof. First, $\mathcal{T}_P$ is a tilting torsion class satisfying $M \in \mathcal{T}_P$ iff $M^{**} \in \mathcal{T}_P$ by [18, Lemma 4.3].

Conversely, assume $\mathcal{T} = T^{\perp}$ for a tilting module $T$. Let $C = T^*$. By assumption and by [18, Lemma 4.2], $C$ is a cotilting module and $\mathcal{T} = Cog(C) = T^{\perp}C$ is a cotilting torsion-free class closed under direct limits. By [7, Corollary 17] there is a set $P$ of non-zero prime ideals of $R$ such that $\mathcal{T} = \mathcal{T}_P = \{M \mid \forall p \in P : \text{Hom}_R(R/p, M) = 0\} = \{M \mid \forall p \in P : R/p \not\cong M\}$.

Observe first that $\mathcal{T}_P = \{M \mid \forall p \in P : \text{Tor}_k(M, R/p) = 0\}$ since $\text{Tor}_k(M, R/p) = 0$ iff $\text{Ext}_k(M, (R/p)^*) = 0$. Since $(R/p)^* \cong R/p$, the latter is equivalent to $\text{Ext}_k(M, R/p) = 0$, and hence to $\text{Hom}(M, E(R/p)) \to \text{Hom}(M, E(R/p)/(R/p))$ being surjective. Clearly, the latter implies $R/p \not\cong M$. On the other hand, if $R/p \not\cong M$, then $E(M) \cong Q^{(\omega)} \bigoplus_{q \neq p} E(R/q)^{(\omega)}$. Since $E(R/p)/(R/p)$ is injective, we can w.l.o.g assume that $M$ is injective, and hence that $M \cong Q^{(\omega)}$. But then $\text{Ext}_k(M, R/p) = 0$ since $R/p$ is cotorsion. So the map $\text{Hom}(M, E(R/p)) \to \text{Hom}(M, E(R/p)/(R/p))$ is surjective.

Finally, for any module $M$, standard homological identities [4, Section VI.5.] give $M \in \mathcal{T}$ iff $M^{**} \in \mathcal{T}$ iff $\text{Ext}(T, M^{**}) = 0$ iff $\text{Tor}(T, M^*) = 0$ iff $\text{Tor}(M^*, T) = 0$. 

iff \( \text{Ext}(M^*, T^*) = 0 \) iff \( \text{Ext}(M^*, C) = 0 \) iff \( M^* \in \mathcal{T}_P = \mathcal{T} \). By [18, Theorem 4.4] and by the assumption, \( M^* \in \mathcal{T}_P \) iff \( M^{**} \in \mathcal{T}_P \) iff \( M \in \mathcal{T}_P \). This proves that \( \mathcal{T} = \mathcal{T}_P \). \( \square \)

In the remainder of the paper we obtain several results which are known for \( R = \mathbb{Z} \) (see [13], and its precursors [16] and [12]). This way, we proceed step by step towards the main theorem, that is, the characterization of the tilting torsion classes under \( V = L \).

First, we need a simple fact.

**Lemma 6.** Let \( 0 \neq p \in \text{Spec}(R) \) and let \( \widehat{R}_p \) be the \( R \)-completion of the localization \( R_p \). Then \( \widehat{R}_p \) does not contain any non-zero \( p \)-divisible \( R_p \)-submodule.

**Proof.** Obviously, \( \widehat{R}_p \) does not contain any non-zero \( (pR_p) \)-divisible \( R_p \)-submodule since \( pR_p = yR_p \) for some \( y \in R_p \) and \( \bigcap_{n \in \mathbb{N}} y^n\widehat{R}_p = 0 \).

Now let \( M \) be an \( R \)-submodule of \( \widehat{R}_p \) and suppose that \( M \) is \( p \)-divisible. Then \( M_p = M \otimes R_p \) is divisible by Lemma 1. Moreover, \( M_p \) can be considered as a (divisible) \( R_p \)-submodule of \( \widehat{R}_p \). Therefore \( M_p = 0 \) and thus \( M = 0 \) since \( M \subseteq \widehat{R}_p \) is torsion-free. \( \square \)

Let \( M \) be a torsion-free \( R \)-module. Then there is a largest subring \( R' \) of \( Q \) such that \( M \) is an \( R' \)-module. \( R' \) is called the nucleus of \( M \) and denoted by \( \text{nuc}(M) \). We have the following characterization:

**Lemma 7.** Let \( M \) be a torsion-free module. Let \( P \) consist of the zero ideal and of all prime ideals \( p \) such that \( M \) is not \( p \)-divisible. Let \( S = R \setminus \bigcup_{p \in P} p \). Then \( \text{nuc}(M) = R_S = \bigcap_{p \in P} R_p \).

**Proof.** By Lemma 3(iii), \( M \in \mathcal{F}_{\text{Spec}(R) \setminus P} = \{ N \in \text{Mod-R} \mid sN = N \ \forall s \in S \} \). Since \( M \) is torsion-free, \( M \) is an \( R_S \)-module, so \( R_S \subseteq \text{nuc}(M) \).

Conversely, if \( 0 \neq p \in P \) then \( \text{nuc}(M) \) is not \( p \)-divisible by Lemma 1(iii), so \( \text{nuc}(M) \subseteq R_p \) by Lemma 1(ii). By Lemma 3(i), \( \text{nuc}(M) \subseteq \bigcap_{p \in P} R_p = R_S \). \( \square \)

Note that, in particular, if \( R' \) is a proper subring of \( Q \) containing \( R \), then \( R' = \text{nuc}(R') \) is a localization of the form above.

Next, we show that any torsion-free module has a “small” factor with the same nucleus.

**Proposition 8.** Let \( R \) be a small Dedekind domain. For any torsion-free module \( N \), there is an epimorphism \( \eta : N \to N' \) such that \( N' \) is torsion-free, \( \text{card}(N') \leq 2^{\omega} \), and \( \text{nuc}(N) = \text{nuc}(N') \).

**Proof.** The assertion is clear in case \( N \) is divisible (then \( N \cong Q^{(\kappa)} \) for a cardinal \( \kappa \), and \( \text{card}(Q) = \text{card}(R) \leq 2^{\omega} \)). Otherwise, let \( P = \{ p \in \text{Spec}(R) \mid \exists N \text{ not } p\text{-divisible}\} \cup \{0\} \), so \( \text{nuc}(N) = \bigcap_{p \in P} R_p \subseteq Q \) by Lemma 7. For each \( 0 \neq p \in P \), let \( \nu_p \) be the embedding
Proposition 9. Assume $Q$ is countably generated. Let $M$ and $N$ be torsion-free modules such that $M$ is of finite rank.

Then, $M$ is projective over $\text{nuc}(N)$; in particular $\text{nuc}(M) = \text{nuc}(N)$.

Proof. By Lemma 7 we may assume that $R = \text{nuc}(N)$ and $N$ is not $p$-divisible for any non-zero prime ideal $p$ of $R$.

Let $n$ be the rank of $M$. W.l.o.g. $F = R^n \subseteq M \subseteq Q^n$. In particular, $M/F \subseteq Q^n/R^n$ is torsion. We will show that $M/F$ is a bounded module.

Consider the exact sequence $0 \rightarrow F \rightarrow M \rightarrow M/F \rightarrow 0$. Since $M/F$ is torsion, we have $\text{Hom}(M/F, N) = 0 = \text{Hom}(M/F, N^{(\alpha)})$. By assumption, $\text{Ext}(M, N) = 0 = \text{Ext}(M, N^{(\alpha)})$. Since $N^{(\alpha)}$ is torsion-free and $F$, $M$ are of finite rank, there are canonical isomorphisms $\bigoplus_{\alpha} \text{Hom}(F, N) \cong \text{Hom}(F, N^{(\alpha)})$ and $\bigoplus_{\alpha} \text{Hom}(M, N) \cong \text{Hom}(M, N^{(\alpha)})$ such that the following diagram commutes:

$$
\begin{array}{cccc}
0 \rightarrow & \bigoplus_{\alpha} \text{Hom}(F, N) & \rightarrow & \bigoplus_{\alpha} \text{Hom}(M, N) & \rightarrow & \bigoplus_{\alpha} \text{Ext}(M/F, N) & \rightarrow & 0 \\
& \cong & & \cong & & & & \\
0 \rightarrow & \text{Hom}(F, N^{(\alpha)}) & \rightarrow & \text{Hom}(M, N^{(\alpha)}) & \rightarrow & \text{Ext}(M/F, N^{(\alpha)}) & \rightarrow & 0
\end{array}
$$

So $X := \text{Ext}(M/F, N^{(\alpha)}) \cong \bigoplus_{\alpha} \text{Ext}(M/F, N)$. Since $Q$ is countably generated, $X \cong \text{Hom}(M/F, E(N^{(\alpha)})/N^{(\alpha)})$ is $R$-complete by [10, Section V.2.8]. As in [9, Corollary 39.10], we see that $\text{Ext}(M/F, N)$ is bounded, that is, there is some $0 \neq r \in R$ such that $r \text{Ext}(M/F, N) = 0$, and thus $r \text{Hom}(M/F, E(N)/N) = 0$.

By [3, p. 249], the torsion module $M/F$ is a direct sum of its $p$-components, $M/F = \bigoplus_p (M/F)_p$. We have $\text{Hom}(R/p, E(N)/N) \cong \text{Ext}(R/p, N) \neq 0$ for all $0 \neq p \in \text{Spec}(R)$. So if $(M/F)_p \neq 0$ then $r \in p$. Since $rR = p_1 \cdots p_n$ for finitely many prime ideals $0 \neq p_i$ ($i \leq m$) there are only finitely many non-zero $p$-components of $M/F$. Moreover, $(M/F)_p$ is a bounded submodule of $E(R/p)^n \cong E_{R/p}(R/p)_p^n$, hence $(M/F)_p$ is finitely generated (cf. [15, Theorem 18.4]). Then $M$ is finitely generated and torsion-free, hence projective [3, Section 6.3.23]. \qed

The next lemma is well known, see [11, p. 537].
**Lemma 10.** Assume \( Q \) is countably generated. Let \( M \) be a torsion-free module of countable rank. Then \( M \) is projective if and only if all finite rank submodules of \( M \) are projective.

Note, if \( Q \) is countably generated then the rank and minimal number of generators coincide for any torsion-free module of infinite rank.

In order to pass from torsion-free modules of countable rank to arbitrary ones, we will employ the Gödel’s Axiom of Constructibility. Later on, we will see that the following result is actually independent of ZFC, so there is no way to prove it only by algebraic means.

**Theorem 11 (\( V = L \)).** Let \( R \) be a small Dedekind domain. Let \( M \) and \( N \) be torsion-free modules such that \( \text{nuc}(M) \supseteq \text{nuc}(N) \) and \( \text{Ext}(M, N^{(\omega)}) = 0 \).

Then \( M \) is a projective \( \text{nuc}(N) \)-module. In particular \( \text{nuc}(M) = \text{nuc}(N) \).

**Proof.** By Proposition 8 we may assume that \( \text{card}(N) \leq 2^\omega = \omega_1 \). The proof is by induction on the rank of \( M \). Since \( R \) is small, \( Q \) is countably generated (cf. [15, Theorem 18.4]). If \( M \) has countable rank then the result follows from Proposition 9 and Lemma 10.

Let \( \text{rank}(M) = \text{card}(M) = \kappa \geq \omega_1 \) and assume that the result is true for all modules of rank less than \( \kappa \). Then all \( < \kappa \)-generated submodules of \( M \) are \( \text{nuc}(N) \)-projective.

If \( \kappa \) is a singular cardinal then Shelah’s Singular Compactness Theorem implies that \( M \) is \( \text{nuc}(N) \)-projective [17, 3.11].

If \( \kappa \) is a regular cardinal, then \( M = \bigcup_{\alpha < \kappa} M_\alpha \) where \( (M_\alpha \mid \alpha < \kappa) \) is a smooth ascending chain such that each \( M_\alpha \) is \( < \kappa \)-generated. Then \( V = L \) implies that \( E = \{ \alpha < \kappa \mid \text{Ext}(M_\beta/M_\alpha, N^{(\omega)}) \neq 0 \text{ for some } \beta > \alpha \} \) is not stationary in \( \kappa \) (cf. [6, pp. 352–353]). By induction hypothesis, we also have \( E = \{ \alpha < \kappa \mid M_\beta/M_\alpha \text{ not } \text{nuc}(N) \text{-projective for some } \beta > \alpha \} \). Let \( C = \{ c_\alpha \mid \alpha < \kappa \} \) be a cub such that \( C \cap E = \emptyset \). Then \( M = \bigcup_{\alpha < \kappa} M_{c_\alpha} \) and \( M_{c_{\alpha+1}/M_{c_\alpha}} \) is \( \text{nuc}(N) \)-projective for all \( \alpha < \kappa \). Therefore \( M \) is also \( \text{nuc}(N) \)-projective. \( \square \)

Finally, we are ready to prove.

**Theorem 12 (\( V = L \)).** Let \( R \) be a small Dedekind domain and \( T \) a module. Then \( T \) is tilting if and only if \( T \) is of the form

\[
\bigoplus_{p \in P} \left( \bigoplus_{\alpha \in \kappa_p} (E/p)_{\alpha} \right) \oplus T',
\]

where \( 0 \notin P \subseteq \text{Spec}(R) \), \( \kappa_p \) are non-zero cardinals for all \( p \in P \), and \( T' \) is a projective generator over \( R_{(p')} = \bigcap_{q \in P'} R_q \) where \( P' = \text{Spec}(R) \setminus P \).

**Proof.** If \( T \) is a module of the form above then \( T \) is tilting by Proposition 4.

Conversely, assume that \( T \) is a tilting module. In particular, \( \text{Ext}(T, T) = 0 \). We prove that each \( p \)-component, \( t_p \), of the torsion part, \( t \), of \( T \) is divisible. Namely, if \( t_p \neq 0 \)
then $R/p \subseteq T$, so $\text{Ext}(R/p, T) = 0$. As $t = \bigoplus_p t_p$ is pure in $T$, $t$ is $p$-divisible by Lemma 2. By Lemma 1, $t_p$ is divisible. It follows that $t$ is divisible (= injective). Hence $t$ is a direct summand of $T$.

Thus, there are $0 \notin P \subseteq \text{Spec}(R)$ and $x_p > 0$ such that $T = \bigoplus_{p \in P} E(R/p)^{(x_p)} \oplus T'$ where $T'$ is torsion-free and $p$-divisible for all $p \in P$. Moreover $T' \neq 0$, since otherwise $Q \in T^\perp \setminus \text{Gen}(T)$.

As $\text{Ext}(T', (T')^{(\omega)}) = 0$ and $V = L$ we have that $T'$ is a projective nuc($T'$)-module by Theorem 11. By Lemma 7, there is a subset $0 \in P' \subseteq \text{Spec}(R)$ such that nuc($T'$) = $R_{(P')}$, $= \bigcap_{p \in P'} R_p$ and $P \cap P' = \emptyset$.

Next we show that $P \cup P' = \text{Spec}(R)$. Suppose that there is $q \notin P \cup P'$. We have $\text{Ext}(T', \bigoplus_{p \in \omega} R/q^n) = 0$ since $T'$ is torsion-free, $R/q^n$ is cotorsion for each $0 < n < \omega$ (cf. [14, Section 11.4]), and $T^\perp$ is closed under direct sums. Since $T'$ contains a copy of $R_{(P')}$, it also contains $R' = R_{(\text{Spec}(R) \setminus q)}$ (cf. Lemma 3). So $\text{Ext}(R', \bigoplus_{\omega} R/q^n) = 0$.

On the other hand, the exact sequence $0 \to R' \to Q \to \bigoplus_{0 \neq p \neq q} E(R/p) \to 0$ induces the long exact sequence

$$0 = \text{Ext}(\bigoplus_{0 \neq p \neq q} E(R/p), \bigoplus_{n \in \omega} R/q^n) \to \text{Ext}(Q, \bigoplus_{n \in \omega} R/q^n)
\to \text{Ext}(R', \bigoplus_{n \in \omega} R/q^n) = 0.$$

So $\text{Ext}(Q, \bigoplus_{n \in \omega} R/q^n) = 0$ and hence $\bigoplus_{n \in \omega} R/q^n$ is cotorsion. This contradicts [14, Section 11.4], since $\bigoplus_{n \in \omega} R/q^n$ is not bounded. This proves $P \cup P' = \text{Spec}(R)$.

It remains to show that $T'$ is a generator. But $T^\perp = (\bigoplus_{p \in P} E(R/p)^{(x_p)})^\perp \cap T^\perp = \mathcal{T}_P \supseteq \text{Mod}-R_{(P')}$ by Lemma 3. So Gen($T$) = Gen($R_{(P')}$), and also Hom($\bigoplus_{p \in P} E(R/p)^{(x_p)}$, $R_{(P')}$) = 0. It follows that $T'$ generates $R_{(P')}$ (as an $R$-module, and hence also as an $R_{(P')}$-module). So $T'$ is a generator for Mod-$R_{(P')}$.

As an immediate consequence we have:

**Corollary 13 ($V = L$).** Let $R$ be a small Dedekind domain and $\mathcal{T}$ a class of modules. Then $\mathcal{T}$ is a tilting torsion class if and only if there exists $P \subseteq \text{Spec}(R)$ such that $\mathcal{T} = \mathcal{T}_P$.

**Proof.** By Theorems 5 and 12. 

We do not know whether Theorem 12 and Corollary 13 hold true in ZFC or whether they are independent of ZFC. But we do know that our way of proving — via Theorem 11 — is independent. This follows from our next result, where UP denotes the Shelah’s Unifomization Principle — a combinatorial statement known to be consistent with ZFC + GCH (for more details on UP, we refer to [17, Section 2]). Indeed, our next result implies that it is consistent with ZFC that Theorem 11 fails for each Dedekind domain $R$ and some $M \in \text{Mod}-R$ in the case when $N = R$.

**Theorem 14 (UP).** Let $R$ be a Dedekind domain. Then there exists a torsion-free non-projective module $M$ such that $\text{Ext}(M, R^{(\omega)}) = 0$. 
Proof. In fact, a much stronger result holds true under UP. By [17, Lemma 2.4], for each non-right perfect ring $S$ and each cardinal $\kappa$ there is a non-projective right $S$-module $M$ such that $\text{Ext}_S(M,N) = 0$ for any right $S$-module $N$ with $\text{card}(N) < \kappa$.

Our claim follows by taking $S = R$ and $\kappa$ as the successor cardinal of $\text{card}(R^{(\omega)})$. Since $R$ is Dedekind, $\text{Ext}(R/p, R/p) \neq 0$ for each $0 \neq p \in \text{Spec}(R)$, so the module $M$ is torsion-free. \hfill \Box

References