ON TENSE MV-ALGEBRAS

MICHAL BOTUR, JAN PASEKA

Abstract. The main aim of this article is to study tense MV-algebras which are just MV-algebras with new unary operations $G$ and $H$ which express a universal time quantifiers. Tense MV-algebras were introduced by D. Diagonescu and G. Georgescu. Using a new notion of an fm-function between MV-algebras we settle a half of their Open problem about representation for some classes of tense MV-algebras, i.e., we show that any tense semisimple MV-algebra is induced by a time frame analogously to classical works in this field of logic. As a by-product we obtain a new characterization of extremal states on MV-algebras.

1. Introduction

Propositional logic usually do not incorporate the dimension of time. To obtain the so-called tense logic, the propositional calculus is enriched by adding new unary operators $G$ and $H$ (and new derived operators $F := \neg G \neg$ and $P := \neg H \neg$, where $\neg$ denotes the classical negation connective) which are called tense operators. The operator $G$ usually expresses the quantifier ‘it will still be the case that’ and $H$ expresses ‘it has always been the case that’. Hence, $F$ and $P$ are in fact tense existential quantifiers.

If $T$ is non-empty set and $\rho$ a binary relation on $T$, the couple $(T, \rho)$ is called a time frame. For a given logical formula $\phi$ of our propositional logic and for $t \in T$ we say that $G(\phi)(t)$ is valid if $\phi(s)$ is valid for any $s \in T$ with $s \rho t$. Analogously, $H(\phi)(t)$ is valid if $\phi(s)$ is valid for any $s \in T$ with $t \rho s$. Thus $F(\phi)(t)$ is valid if there exists $s \in T$ with $s \rho t$ and $\phi(s)$ is valid and analogously $P(\phi)(t)$ is valid if there exists $s \in T$ with $s \rho t$ and $\phi(s)$ is valid in the propositional logic.

Study of tense operators was originated in 1980’s, see e.g. a compendium [3]. Recall that for a classical propositional calculus represented by means of a Boolean algebra $B = (B, \lor, \land, \neg, 0, 1)$, tense operators were axiomatized in [3] by the following axioms:

(B1) $G(1) = 1, H(1) = 1$,
(B2) $G(x \land y) = G(x) \land G(y), H(x \land y) = H(x) \land H(y)$,
(B3) $\neg G \neg H(x) \leq x, \neg H \neg G(x) \leq x$.

For Boolean algebras, the axiom (B3) is equivalent to

(B3') $G(x) \lor y = x \lor H(y)$.

To introduce tense operators in non-classical logics, some more axioms must be added on $G$ and $H$ to express connections with additional operations or logical connectives. For example, for intuitionistic logic (corresponding to Heyting algebras) it was done in [5], for algebras of logic of quantum mechanics see [6] and [7], for so called basic algebras it was done in [2], for other interesting algebras the reader is referred to [13], [14] and [16].
Among algebras connected with many-valued logic, let us mention MV-algebras and Lukasiewicz-Moisil algebras. Tense operators for the previous cases were introduced and studied in [9] and [11]. Contrary to Boolean algebras where the representation problem through a time frame is solved completely, authors in [11] only mention that this problem for MV-algebras was not treated. Hence, our main goal is to find a suitable time frame for given tense operators on a semisimple MV-algebra, i.e., to solve the representation problem for semisimple MV-algebras.

This problem was solved by the first author for such tense MV-algebras that the tense operators $G$ and $H$ preserve all powers of the operations $\oplus$ and $\odot$. The second author generalized his results replacing original term $t_q$ (see [19]) constructed for any rational $q$ by the Teheux’s term (see Section 2.2 of this paper or [19]). Here, we present more general concept of used ideas for obtaining stronger results. The main representation theorem for semisimple tense MV-algebras and second author’s results are corollaries of this.

The paper is divided as follows. In Section 2 we recall the basic fundamental results on MV-algebras and tense MV-algebras, and in this way fix the notation and terminology. Afterwards we summarize some folklore results on MV-terms $t_r(x)$ produced only from operations of the form $x \oplus x$ and $x \odot x$. Then in Section 3 we introduce a notion of a semi-state on an MV-algebra and we show that any semi-state is a meet of extremal states. Also, we give a new characterization of extremal states on MV-algebras. In Section 4 we introduce the notions of an fm-function between MV-algebras (strong fm-function between MV-algebras). We establish a canonical construction of strong fm-function between MV-algebras.

In Section 5 we solve the representation problem for fm-function between semisimple MV-algebras. Moreover, we show that in this case they coincide with strong fm-functions. Finally we prove the representation theorem for semisimple tense MV-algebras.

2. Preliminaries

2.1. MV-algebras and tense operators. The concept of MV-algebras was introduced by C.C. Chang in [8] as algebraic counterpart of the Łukasiewicz multi valued logic (see [17]). Recall, that by an MV-algebra is meant an algebra $A = (A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the axioms:

(MV1) $x \oplus y = y \oplus x$,  
(MV2) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,  
(MV3) $x \oplus 0 = x$,  
(MV4) $\neg\neg x = x$,  
(MV5) $x \oplus 1 = 1$, where $1 := \neg 0$,  
(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

The order relation $\leq$ can be introduced on any MV-algebra $A$ by the stipulation $x \leq y$ if and only if $\neg x \oplus y = 1$.

Moreover, the ordered set $(A, \leq)$ can be organized into a bounded lattice $(A, \lor, \land, 0, 1)$ where $x \lor y = \neg(\neg x \oplus y) \oplus y$ and $x \land y = \neg(\neg x \lor \neg y)$.

Besides of these, we can introduce two more interesting operation $\odot$ and $\rightarrow$ by setting $x \odot y := \neg(\neg x \oplus \neg y)$ and $x \rightarrow y := \neg x \oplus y$.

1It is not published.
Those operations are connected by the adjointness property
\[ x \circ y \leq z \text{ if and only if } x \leq y \to z. \]

An MV-algebra is said to be linearly ordered (or an MV-chain) if its order is linear.
Given a positive integer \( n \in \mathbb{N} \), we let \( n \times x = x \oplus x \oplus \cdots \oplus x, \) \( n \) times, \( x^n = x \oplus x \oplus \cdots \oplus x, \) \( n \) times, \( 0x = 0 \) and \( x^0 = 1. \)

In every MV-algebra the following equalities hold:
\begin{align*}
\text{(D1)} \quad & a \oplus \bigvee_{i \in I} x_i = \bigvee_{i \in I} (a \oplus x_i), \\
\text{(D2)} \quad & a \oplus \bigwedge_{i \in I} x_i = \bigwedge_{i \in I} (a \oplus x_i),
\end{align*}
whenever the respective sides are defined.

An element \( a \) of an MV-algebra \( A \) is said to be Boolean if \( a \oplus a = a. \) We say that an MV-algebra \( A \) is Boolean if every element of \( A \) is Boolean. For an MV-algebra \( A, \) the set \( B(A) \) of all Boolean elements is a Boolean algebra.

Morphisms of MV-algebras (shortly \( MV \)-morphisms) are defined as usual, i.e., they are functions which preserve the binary operations \( \oplus \) and \( \circ, \) the unary operation \( \neg \) and the constants 0 and 1.

Tense MV-algebras were introduced by D. Diagonescu and G. Georgescu in [11].

**Definition 1.** Let us have an MV-algebra \( A = (A, \oplus, \neg, 0). \) We say that \((A, G, H)\) is a tense MV-algebra and \( G \) and \( H \) are tense operators if \( G \) and \( H \) are a unary operations on \( A \) satisfying:
\begin{itemize}
  \item[(i)] \( G(1) = H(1) = 1, \)
  \item[(ii)] \( G(x) \circ G(y) \leq G(x \circ y), \quad H(x) \circ H(y) \leq H(x \circ y), \)
  \item[(iii)] \( G(x) \oplus G(y) \leq G(x \oplus y), \quad H(x) \oplus H(y) \leq H(x \oplus y), \)
  \item[(iv)] \( G(x) \circ G(x) = G(x \circ x), \quad H(x) \circ H(x) = H(x \circ x), \)
  \item[(v)] \( G(x) \circ G(x) = G(x \circ x), \quad H(x) \circ H(x) = H(x \circ x), \)
  \item[(vi)] \( \neg G \neg H(x) \leq x, \quad \neg H \neg G(x) \leq x. \)
\end{itemize}

Applying the axioms (i) and (iii), we get immediately monotonicity of the operators \( G \) and \( H. \) Thus, if \( x \leq y \) for any \( x, y \in A \) then \( G(x) \leq G(y) \) and \( H(x) \leq H(y). \)

We note that the original definition of tense MV-algebras [11, Proposition 5.1, Remark 5.1] use alternative inequalities
\begin{itemize}
  \item[(ii') \( G(x \to y) \leq G(x) \to G(y), \quad H(x \to y) \leq H(x) \to H(y), \)
  \item[(vi') \( x \leq G \neg H \neg x, \quad x \leq H \neg G \neg x. \)
\end{itemize}

Monotonicity of the operators \( G \) and \( H \) and adjointness property give equivalence of (ii) and (ii'). Using double negation law and antitonicity of the negation we obtain equivalence of (vi) and (vi')

The following theorem describes the most important construction of tense MV-algebras.

**Theorem 1.** [11] Let \( A \) be a linearly ordered complete MV-algebra and let \( T \) be any set with a binary relation \( \rho \subseteq T^2. \) Then \((A^T, G^*, H^*)\) where operations \( G^* \) and \( H^* \) are calculated point-wise
\[ G^*(x)(i) := \bigwedge_{i \in T} x(j) \quad \text{and} \quad H^*(x)(i) := \bigwedge_{j \in T} x(j) \]
is a tense MV-algebra. In this case we say that the tense MV-algebra \((A^T, G^*, H^*)\) is induced by the frame \((T, \rho).\)

We will prove that any couple of tense operators on any semisimple MV-algebra can be embedded into \([0,1]^T, G^*, H^*)\) where \( G^* \) and \( H^* \) are tense operators induced by some
time frame \((T, \rho)\). For related results on more general operators on any semisimple MV-algebra see [19].

2.2. Dyadic numbers and MV-terms. The contents of this part summarizes the basic results about some folklore results of some MV-terms from [22] and [18]. The techniques described here have been used already in [10], [12], [15] and [21].

We remark some concepts introduced by B. Teheux in [22]. The set \(\mathbb{D}\) of dyadic numbers is the set of the rational numbers that can be written as a finite sum of power of 2. If \(a\) is a number of \([0, 1]\), a dyadic decomposition of \(a\) is a sequence \(a^* = (a_i)_{i \in \mathbb{N}}\) of elements of \(\{0, 1\}\) such that \(a = \sum_{i=1}^{\infty} a_i 2^{-i}\). We denote by \(a_i^*\) the \(i\)th element of any sequence (of length greater than \(i\)) \(a^*\). If \(a\) is a dyadic number of \([0, 1]\), then \(a\) admits a unique finite dyadic decomposition, called the dyadic decomposition of \(a\). If \(a^*\) is a dyadic decomposition of a real \(a\) and if \(k\) is a positive integer then we denote by \(\gamma a^*\gamma_k\) the finite sequence \((a_1, \ldots, a_k)\) defined by the first \(k\) elements of \(a^*\) and by \(\cup a^*_\cup k\) the dyadic number \(\sum_{i=1}^k a_i 2^{-i}\). We denote by \(f_0(x)\) and \(f_1(x)\) the terms \(x \oplus x\) and \(x \odot x\) respectively, and by \(T_\mathbb{D}\) the clone generated by \(f_0(x)\) and \(f_1(x)\).

We also denote by \(g\), the mapping between the set of finite sequences of elements of \([0, 1]\) (and thus of dyadic numbers in \([0, 1]\)) and \(T_\mathbb{D}\) defined by:

\[
g(a_1, \ldots, a_k) = f_{a_k} \circ \cdots \circ f_{a_1}
\]

for any finite sequence \((a_1, \ldots, a_k)\) of elements of \([0, 1]\). If \(a = \sum_{i=1}^k a_i 2^{-i}\), we sometimes write \(g_a\) instead of \(g(a_1, \ldots, a_k)\).

**Lemma 1.** [22, Lemma 1.14] If \(a^* = (a_i)_{i \in \mathbb{N}}\) and \(x^* = (x_i)_{i \in \mathbb{N}}\) are dyadic decompositions of two elements of \(a, x \in [0, 1]\), then, for any positive integer \(k \in \mathbb{N}\),

\[
g_{\gamma a^*_\gamma k}(x) = \begin{cases} 1 & \text{if } x > \sum_{i=1}^k a_i 2^{-i} + 2^{-k} \\ 0 & \text{if } x < \sum_{i=1}^k a_i 2^{-i} \end{cases}
\]

Note that for any finite sequence \((a_1, \ldots, a_k)\) of elements of \([0, 1]\) such that \(a_k = 0\) we have that \(g(a_1, \ldots, a_k) = g(a_1, \ldots, a_{k-1}) \oplus g(a_1, \ldots, a_{k-1})\) and clearly any dyadic number \(a\) corresponds to such a sequence \((a_1, \ldots, a_k)\).

As an immediate consequence, we get

**Corollary 1.** [22, Corollary 1.15 (1)] Let us have the standard MV-algebra \([0, 1]\), \(x \in [0, 1]\) and \(r \in (0, 1) \cap \mathbb{D}\). Then there is a term \(t_r\) in \(T_\mathbb{D}\) such that

\[
t_r(x) = 1 \text{ if and only if } r \leq x.
\]

2.3. Filters, ultrafilters and the term \(t_r\). The aim of this part is to show that any filter \(F\) in an MV-algebra \(A\) which does not contain the element \(t_r(x)\) for some dyadic number \(r \in (0, 1) \cap \mathbb{D}\) and an element \(x \in A\) can be extended to an ultrafilter \(U\) containing \(F\) such that \(t_r(x) \notin U\).

A filter of an MV-algebra \(A\) is a subset \(F \subseteq A\) satisfying:

\[
\begin{align*}
(F1) \quad & 1 \in F \\
(F2) \quad & x \in F, \ y \in A, \ x \leq y \Rightarrow y \in F \\
(F3) \quad & x, y \in F \Rightarrow x \odot y \in F.
\end{align*}
\]

A filter is said to be proper if \(0 \notin F\). Note that there is a one-to-one correspondence between filters and congruences on MV-algebras. A filter \(Q\) is prime if it satisfies the following conditions:

\[
(P1) \quad 0 \notin Q.
\]
(P2) For each \(x, y\) in \(A\) such that \(x \vee y \in Q\), either \(x \in Q\) or \(y \in Q\).

In this case the corresponding factor MV-algebra \(A/Q\) is linear.

A filter \(U\) is maximal (and in this case it will be also called an ultrafilter) if \(0 \notin U\) and for any other filter \(F\) of \(A\) such that \(U \subseteq F\), then either \(F = A\) or \(F = U\). There is a one-to-one correspondence between ultrafilters and MV-morphisms from \(A\) into \([0, 1]\) (extremal states). For any ultrafilter \(A \in T\) we identify the class \(x/A\) with its image in the standard algebra and thus with its image in interval \([0, 1]\) of real numbers.

In what follows we work mostly with MV-morphisms into \([0, 1]\) instead of ultrafilters.

**Lemma 2.** Let \(A\) be a linearly ordered MV-algebra, \(s : A \rightarrow [0, 1]\) an MV-morphism, \(x \in A\) such that \(s(x) = 1\). Then \(x \oplus x = 1\).

**Proof.** Assume that \(x \leq \neg x\). Then \(1 = s(x) \odot s(x) = s(x \odot x) \leq s(x \odot \neg x) = s(0) = 0\) which is absurd. Therefore \(\neg x < x\) and we have that \(x \oplus x \geq x \oplus \neg x = 1\). \(

**Proposition 1.** Let \(A\) be a linearly ordered MV-algebra, \(s : A \rightarrow [0, 1]\) an MV-morphism, \(x \in A\). Then \(s(x) = 1\) iff \(t_r(x) = 1\) for all \(r \in (0, 1) \cap \mathbb{D}\).

Equivalently, \(s(x) < 1\) iff there is a dyadic number \(r \in (0, 1) \cap \mathbb{D}\) such that \(t_r(x) \neq 1\). In this case, \(s(x) < r\).

**Proof.** In what follows we may assume that \(x \neq 0\) since \(s(0) = 0\) and \(t_r(0) = 0\) for all \(r \in (0, 1) \cap \mathbb{D}\). Note first that \(s(t_r(x)) = t_r(s(x))\) since \(s\) is an MV-morphism. Then \(s(x) = 1\) iff \(r \leq s(x)\) for all \(r \in (0, 1) \cap \mathbb{D}\) iff \(t_r(s(x)) = 1\) for all \(r \in (0, 1) \cap \mathbb{D}\) iff \(s(t_r(x)) = 1\) for all \(r \in (0, 1) \cap \mathbb{D}\).

Assume now that \(t_r(x) = 1\) for all \(r \in (0, 1) \cap \mathbb{D}\). Then evidently \(s(t_r(x)) = 1\) for all \(r \in (0, 1) \cap \mathbb{D}\) and by the above considerations we have that \(s(x) = 1\).

Conversely, let \(s(x) = 1\) and \(r \in (0, 1) \cap \mathbb{D}\). Then \(t_r(x) = t(x) \oplus t(x)\) such that \(t(x)\) is some term from the clone \(T_D\) constructed entirely from the operations \((-) \odot (-)\) and \((-) \odot (-)\). Therefore \(t(t(x)) = t(s(x)) = t(1) = 1\). By Lemma 2 we get that \(t_r(x) = t(x) \oplus t(x) = 1\).

**Proposition 2.** Let \(A\) be an MV-algebra, \(x \in A\) and \(F\) be any filter of \(A\). Then there is an MV-morphism \(s : A \rightarrow [0, 1]\) such that \(s(F) \subseteq \{1\}\) and \(s(x) < 1\) if and only if there is a dyadic number \(r \in (0, 1) \cap \mathbb{D}\) such that \(t_r(x) \notin F\).

**Proof.** Assume first that there is an MV-morphism \(s : A \rightarrow [0, 1]\) such that \(s(F) \subseteq \{1\}\) and \(s(x) < 1\). Then there is a dyadic number \(r \in (0, 1) \cap \mathbb{D}\) such that \(s(x) < r < 1\). By Corollary 2 we get that \(s(t_r(x)) = t_r(s(x)) \neq 1\). Hence \(t_r(x) \notin F\).

Now, let there be a dyadic number \(r \in (0, 1) \cap \mathbb{D}\) such that \(t_r(x) \notin F\). Then there is a filter \(K\) of \(A, F \subseteq K\), \(t_r(x) \notin K\) such that \(K\) is maximal with this property. Evidently, \(K\) is a prime filter of \(A\). Hence the factor algebra \(A/K\) is linearly ordered and we have a surjective MV-morphism \(g : A \rightarrow A/K, g(K) \subseteq \{1\}\). Let us denote by \(U_K\) the maximal filter of \(A/K\) and by \(s_K : A/K \rightarrow [0, 1]\) the corresponding MV-morphism. Because \(t_r(x) \notin K\) we get that \(t_r(g(x)) = g(t_r(x)) \neq 1\).

It follows from Proposition 1 that \(s_K(t_r(g(x))) < r < 1\). This yields that \(s_K(g(t_r(x))) < r < 1\). Let us put \(s = s_K \circ g\). Then \(s : A \rightarrow [0, 1]\) is an MV-morphism, \(s(t_r(x)) < r < 1\). Evidently \(s(x) < r < 1\) otherwise we would have also \(1 = s(t_r(x)) < 1\), a contradiction. Clearly, \(s(F) \subseteq s(K) = s_K(g(K)) \subseteq s_K(\{1\}) = \{1\}\).
Corollary 2. Let $A$ be an MV-algebra, $x \in A$ and $F$ be any filter of $A$ such that $t_r(x) \notin F$ for some dyadic number $r \in (0,1) \cap \mathbb{D}$. Then there is an ultrafilter $U$ of $A$ such that $F \subseteq U$ and $x/U < r < 1$.

3. Semistates on MV-algebras

In this section we characterize arbitrary meets of MV-morphism into a unit interval as so-called semi-states.

**Definition 2.** Let $A$ be an MV-algebra. A map $s : A \to [0,1]$ is called

(i) a semi-state on $A$ if

(ii) $x \leq y$ implies $s(x) \leq s(y)$,

(iii) $s(x) = 1$ and $s(y) = 1$ implies $s(x \odot y) = 1$,

(iv) $s(x) \odot s(x) = s(x \odot x)$,

(v) $s(x) + s(x) = s(x \oplus x)$.

(ii) a strong semi-state on $A$ if it is a semistate such that

(vi) $s(x) \odot s(y) \leq s(x \odot y)$,

(viii) $s(x) + s(y) \leq s(x \odot y)$,

(ix) $s(x^n) = s(x)^n$ for all $n \in \mathbb{N}$,

(x) $n \times s(x) = s(n \times x)$ for all $n \in \mathbb{N}$,

Note that any MV-morphism into a unit interval is a strong semi-state.

**Lemma 3.** Let $A$ be an MV-algebra, $S$ a non-empty set of semi-states (strong semi-states) on $A$. Then the point-wise meet $t = \bigwedge S : A \to [0,1]$ is a semi-state (strong semi-state) on $A$.

**Proof.** Let us check the conditions (i)-(vi) from Definition 2.

(i): Clearly, $t(1) = \bigwedge \{s(1) \mid s \in S\} = \bigwedge \{1 \mid s \in S\} = 1$.

(ii): Assume $x \leq y$. Then $t(x) = \bigwedge \{s(x) \mid s \in S\} \leq \bigwedge \{s(y) \mid s \in S\} = t(y)$.

(iii): Let $t(x) = \bigwedge \{s(x) \mid s \in S\} = 1$ and $t(y) = \bigwedge \{s(y) \mid s \in S\} = 1$. It follows, that for all $s \in S$, $s(x) = 1 = s(y)$. Hence also $s(x \odot y) = 1$. This yields that $t(x \odot y) = \bigwedge \{s(x \odot y) \mid s \in S\} = 1$.

(iv, v): Since $[0,1]$ is linearly ordered we have (by taking in the respective part of the proof either the minimum of $s_1(x)$ and $s_2(x)$ or the maximum of $s_1(x)$ and $s_2(x)$)

$$t(x) \odot t(x) = \bigwedge \{s_1(x) \mid s_1 \in S\} \odot \bigwedge \{s_2(x) \mid s_2 \in S\} = \bigwedge \{s_1(x) \odot s_2(x) \mid s_1, s_2 \in S\}$$

$$\geq \bigwedge \{s_1(x) \odot s(x) \mid s \in S\} = \bigwedge \{s(x \odot x) \mid s \in S\} = t(x \odot x),$$

$$t(x) \odot t(x) = \bigwedge \{s_1(x) \mid s_1 \in S\} \odot \bigwedge \{s_2(x) \mid s_2 \in S\} = \bigwedge \{s_1(x) \odot s_2(x) \mid s_1, s_2 \in S\}$$

$$\leq \bigwedge \{s(x) \odot s(x) \mid s \in S\} = \bigwedge \{s(x \odot x) \mid s \in S\} = t(x \odot x),$$

$$t(x) \oplus t(x) = \bigwedge \{s_1(x) \mid s_1 \in S\} \oplus \bigwedge \{s_2(x) \mid s_2 \in S\} = \bigwedge \{s_1(x) \oplus s_2(x) \mid s_1, s_2 \in S\}$$

$$\geq \bigwedge \{s(x) \oplus s(x) \mid s \in S\} = \bigwedge \{s(x \oplus x) \mid s \in S\} = t(x \oplus x),$$

and

$$t(x) \oplus t(x) = \bigwedge \{s(x) \mid s \in S\} \oplus \bigwedge \{s(x) \mid s \in S\} = \bigwedge \{s(x) \oplus s(x) \mid s_1, s_2 \in S\}$$

$$\leq \bigwedge \{s(x) \odot s(x) \mid s \in S\} = \bigwedge \{s(x \odot x) \mid s \in S\} = t(x \odot x).$$

(vi): Let us compute the following

$$t(x) \odot t(y) = \bigwedge \{s_1(x) \mid s_1 \in S\} \odot \bigwedge \{s_2(y) \mid s_2 \in S\} = \bigwedge \{s_1(x) \odot s_2(y) \mid s_1, s_2 \in S\}$$

$$\leq \bigwedge \{s(x) \odot s(y) \mid s \in S\} \leq \bigwedge \{s(x \odot y) \mid s \in S\} = t(x \odot y).$$
Proposition 3. Let the Introduction yields that

\[ \bigwedge \{ s(x) \mid s \in S \} \leq \bigwedge \{ s(x) \leq s(y) \mid s, y \in S \} = \bigwedge \{ s(x) \} = t(x \oplus y). \]

Proof. Clearly, \( t(x \leq t(y) \leq \bigwedge \{ s(x) \mid s \in S \} \leq \bigwedge \{ s(x) \leq t(x \oplus y) \mid s \in S \} = t(x \oplus y). \)

Lemma 4. Let \( A \) be an MV-algebra, \( s, t \) semi-states on \( A \). Then \( t \leq s \) iff \( t(x) = 1 \) implies \( s(x) = 1 \) for all \( x \in A \).

Proof. Clearly, \( t \leq s \) yields the condition \( t(x) = 1 \) implies \( s(x) = 1 \) for all \( x \in A \).

Assume now that \( t(x) = 1 \) implies \( s(x) = 1 \) for all \( x \in A \) is valid and that there is \( y \in A \) such that \( s(y) < t(y) \). Thus, there is a dyadic number \( r \in (0,1) \cap \mathbb{D} \) such that \( s(y) < r < t(y) \). By Corollary 1 there is a term \( t_r \) in \( T_{\mathbb{D}} \) such that \( t_r(s(y)) < 1 \) and \( t_r(t(y)) = 1 \). It follows that \( s(t_r(y)) = t_r(s(y)) < 1 \) and \( t(t_r(y)) = t_r(t(y)) = 1 \). The last condition yields that \( s(t_r(y)) = 1 \), a contradiction. □

Proposition 3. Let \( A \) be an MV-algebra, \( t \) a semi-state on \( A \) and \( S_t = \{ s : A \to [0,1] \mid s \text{ is an MV-morphism}, s \geq t \} \). Then \( t = \bigwedge S_t \).

Proof. Clearly, \( t \leq \bigwedge S_t \). Assume that there is \( x \in A \) such that \( t(x) < \bigwedge S_t(x) \). Thus, there is a dyadic number \( r \in (0,1) \cap \mathbb{D} \) such that \( t(x) < r < \bigwedge S_t(x) \). Again by Corollary 1 there is a term \( t_r \) in \( T_{\mathbb{D}} \) such that \( t(t_r(x)) = t_r(t(x)) < 1 \). Let us put \( F = \{ z \in A \mid t(z) = 1 \} \). The set \( F \) is by the condition (iii) a filter of \( A \), \( t_r(x) \notin F \). Hence there is by Proposition 2 an MV-morphism \( s : A \to [0,1] \) such that \( s(F) \subseteq \{ 1 \} \) and \( s(x) < r < \bigwedge S_t(x) \), a contradiction. □

Corollary 1. Any semi-state on an MV-algebra \( A \) is a strong semi-state.

Corollary 2. The only semi-state \( s \) on an MV-algebra \( A \) with \( s(0) \neq 0 \) is the constant function \( s(x) = 1 \) for all \( x \in A \).

Corollary 3. The only semi-state \( s \) on the standard MV-algebra \([0,1] \) with \( s(0) = 0 \) is the identity function.
Remark 1. It is transparent that all the preceding notions and results including Proposition 2 can be dualized. In particular, any dual semi-state, i.e., a map \( s : A \rightarrow [0,1] \) satisfying conditions (i),(ii),(iv), (v) and the dual condition (iii) \( s(x) = 0 \) and \( s(y) = 0 \) implies \( s(x \oplus y) = 0 \) is a join of extremal states on \( A \).

**Proposition 4.** Let \( A \) be an MV-algebra, \( s \) a state on \( A \). Then the following conditions are equivalent:

(a) \( s \) is a morphism of MV-algebras,
(b) \( s \) satisfies the condition \( s(x \wedge x') = s(x) \wedge s(x)' \) for all \( x \in A \),
(c) \( s \) satisfies the condition (iv) from Definition 2,
(d) \( s \) satisfies the condition (viii) from Definition 2.

**Proof.** (a) \( \Rightarrow \) (b): It is evident.

(b) \( \Rightarrow \) (c): Clearly, any state satisfies conditions (i) and (ii) from Definition 2. Let us check the condition (iii). Assume that \( s(x) = 1 = s(y) \). Then \( s(x \oplus y) = s(x') \oplus s(y')' = s(x' + x \wedge y')' = s(x' + x \wedge y') \geq (s(x') + s(x \wedge y'))' = (0 + 0)' = 1 \).

By the assumption (b) we have that the condition (iv) is satisfied and for any state the condition (v) is equivalent with (iv). It follows that \( s \) is a semi-state. By Corollary 1 \( s \) is a strong semi-state, i.e., (viii) is satisfied.

(d) \( \Rightarrow \) (a): It follows from [20, Lemma 3.1].

\( \square \)

4. Functions between MV-algebras and their construction

This section studies the notion of an fm-function between MV-algebras (strong fm-function between MV-algebras). The main purpose of this section is to establish in some sense a canonical construction of strong fm-function between MV-algebras. This construction is an ultimate source of numerous examples.

**Definition 3.** By an **fm-function between MV-algebras** \( G \) is meant a function \( G : A_1 \rightarrow A_2 \) such that \( A_1 = (A_1; \oplus_1, \odot_1, \neg_1, 0_1, 1_1) \) and \( A_2 = (A_2; \oplus_2, \odot_2, \neg_2, 0_2, 1_2) \) are MV-algebras and

\[(FM1) \ G(1_1) = 1_2, \]
\[(FM2) \ x \leq_1 y \text{ implies } G(x) \leq_2 G(y), \]
\[(FM3) \ G(x) = 1_2 = G(y) \text{ implies } G(x \odot_1 y) = 1_2, \]
\[(FM4) \ G(x) \odot_2 G(x) = G(x \odot_1 x), \]
\[(FM5) \ G(x) \oplus_2 G(x) = G(x \odot_1 x). \]

If moreover \( G \) satisfies conditions

\[(FM6) \ G(x) \odot_2 G(y) \leq G(x \odot_1 y), \]
\[(FM7) \ G(x) \oplus_2 G(y) \leq G(x \odot_1 y), \]
\[(FM8) \ G(x) \wedge_2 G(y) = G(x \wedge_1 y), \]
\[(FM9) \ G(x^n) = G(x)^n \text{ for all } n \in \mathbb{N}, \]
\[(FM10) \ n \times_2 G(x) = G(n \times_1 x) \text{ for all } n \in \mathbb{N}, \]

we say that \( G \) is a **strong fm-function between MV-algebras**.

If \( G : A_1 \rightarrow A_2 \) and \( H : B_1 \rightarrow B_2 \) are fm-functions between MV-algebras, then a morphism between \( G \) and \( H \) is a pair \( (\varphi, \psi) \) of morphism of MV-algebras \( \varphi : A_1 \rightarrow B_1 \) and \( \psi : A_2 \rightarrow B_2 \) such that \( \psi(G(x)) = H(\varphi(x)) \), for any \( x \in A_1 \).

Note that (FM8) yields (FM2), (FM9) yields (FM4) and (FM10) yields (FM5). Also, a composition of fm-functions (strong fm-functions) is an fm-function (a strong fm-function) again and any morphism of MV-algebras is an fm-function (a strong fm-function).
The notion of an fm-function generalizes both the notions of a semi-state and of a \( \odot \)-operator from [19] which is an fm-function \( G \) from \( A_1 \) to itself such that (FM6) is satisfied.

According to both (FM4) and (FM5), \( G|_{B(A_1)} : B(A_1) \to B(A_2) \) is an fm-function (a strong fm-function) whenever \( G \) has the respective property.

**Lemma 5.** Let \( G : A_1 \to A_2 \) be an fm-function between MV-algebras, \( r \in (0,1) \cap \mathbb{D} \). Then \( t_r(G(x)) = G(t_r(x)) \) for all \( x \in A_1 \).

*Proof.* Note that \( G(x) \oplus_2 G(x) = G(x \oplus_1 x) \) by (FM5) and \( G(x) \odot_2 G(x) = G(x \odot_1 x) \) by (FM4). Then, since \( t_r \in T_\mathbb{D} \) is defined inductively using only the operations \((-) \oplus (-)\) and \((-) \odot (-)\), we get \( t_r(G(x)) = G(t_r(x)) \). \( \square \)

By a frame is meant a triple \((S, T, R)\) where \( S, T \) are non-void sets and \( R \subseteq S \times T \). Having an MV-algebra \( M = (M; \oplus, \odot, \neg, 0, 1) \) and a non-void set \( T \), we can produce the direct power \( M^T = (M^T; \oplus, \odot, \neg, o, j) \) where the operations \( \oplus, \odot \) and \( \neg \) are defined and evaluated on \( p, q \in M^T \) componentwise. Moreover, \( o, j \) are such elements of \( M^T \) that \( o(t) = 0 \) and \( j(t) = 1 \) for all \( t \in T \). The direct power \( M^T \) is again an MV-algebra.

The notion of frame allows us to construct new examples of MV-algebras with a strong operator.

**Theorem 2.** Let \( M \) be a linearly ordered complete MV-algebra, \((S, T, R)\) be a frame and \( G^* \) be a map from \( M^T \) into \( M^S \) defined by

\[
G^*(p)(s) = \bigwedge \{ p(t) \mid t \in T, sRt \},
\]

for all \( p \in M^T \) and \( s \in S \). Then \( G^* \) is a strong fm-function between MV-algebras which has a left adjoint \( P^* \). In this case, for all \( q \in M^S \) and \( t \in T \),

\[
P^*(q)(t) = \bigvee \{ q(s) \mid s \in T, sRt \}
\]

and \( P^* : (M^S)^{op} \to (M^T)^{op} \) is a strong fm-function between MV-algebras.

*Proof.* The conditions (FM1)-(FM10) can be easily shown by the same considerations as in [19, Theorem 3.4] and/or Lemma [3];

Moreover, for any \( p \in M^T \) and \( q \in M^S \), we can compute:

\[
q(s) \leq G^*(p)(s) \text{ for all } s \in T \iff q(s) \leq \bigwedge \{ p(t) \mid t \in T, sRt \} \text{ for all } s \in T
\]

\[
\iff q(s) \leq p(t) \text{ for all } s, t \in T, sRt
\]

\[
\iff \bigvee_{i \in I} \{ q(s) \mid s \in T, sRt \} \leq p(t) \text{ for all } t \in T
\]

\[
\iff P^*(q)(t) \leq p(t) \text{ for all } t \in T.
\]

This yields that \( q \leq G^*(p) \) iff \( P^*(q) \leq p \). Then \( P^* \) is a left adjoint of \( G^* \). Hence \( G^* \) preserves arbitrary meets. \( \square \)

We say that \( G^* : M^T \to M^S \) is the canonical strong fm-function between MV-algebras induced by the frame \((S, T, R)\) and the MV-algebra \( M \).

5. The main theorem and its applications

Before proving our main theorem, we remark that semisimple MV-algebras [10] are just subdirect products of the simple MV-algebras. Any simple MV-algebra is uniquely embeddable into the standard MV-algebra on the interval \([0, 1]\) of reals. It is known that...
an MV-algebra is semisimple if and only if the intersection of the set of its maximal (prime) filters is equal to the set \{1\}. Note also that any complete MV-algebra is semisimple.

Hence a semisimple MV-algebra \(A\) is embedded into \([0, 1]^T\) (see \[1\]) where \(T\) is the set of all ultrafilters (morphisms into the standard MV-algebra) and \(\pi_F(x) = x(F) = x/F \in [0, 1]\) for any \(x \in S \subseteq [0, 1]^T\) and any \(F \in T\); here \(\pi_F : [0, 1]^T \to [0, 1]\) is the respective projection onto \([0, 1]\).

**Theorem 3.** Let \(G : A_1 \to A_2\) be an fm-function between semisimple MV-algebras, \(T\) a set of all MV-morphism from \(A_1\) to the standard MV-algebra \([0, 1]\) and \(S\) a set of all MV-morphism from \(A_2\) to \([0, 1]\).

Further, let \((S, T, \rho_G)\) be a frame such that the relation \(\rho_G \subseteq S \times T\) is defined by

\[
sp_G t \text{ if and only if } s(G(x)) \leq t(x) \text{ for any } x \in A_1.
\]

Then \(G\) is representable via the canonical strong fm-function \(G^* : [0, 1]^T \to [0, 1]^S\) between MV-algebras induced by the frame \((S, T, \rho_G)\) and the standard MV-algebra \([0, 1]\), i.e., the following diagram of fm-functions commutes:

\[
\begin{array}{ccc}
A_1 & \xrightarrow{G} & A_2 \\
\downarrow i^T_{A_1} & & \downarrow i^S_{A_2} \\
[0, 1]^T & \xrightarrow{G^*} & [0, 1]^S.
\end{array}
\]

**Proof.** Assume that \(x \in A_1\) and \(s \in S\). Then \(i^S_{A_2}(G(x))(s) = s(G(x)) \leq t(x)\) for all \(t \in T\), \((s, t) \in \rho_G\). It follows that \(i^S_{A_2}(G(x)) \leq G^*(i^T_{A_1}(x))\).

Note that \(s \circ G\) is a semi-state on \(A_1\) and by Proposition 3 we get that

\[
s \circ G = \bigwedge\{t : A_1 \to [0, 1] \mid t \text{ is an MV-morphism, } t \geq s \circ G\} = \bigwedge\{t \in T \mid (s, t) \in \rho_G\}.
\]

This yields that actually \(i^S_{A_2}(G(x)) = G^*(i^T_{A_1}(x))\). \(\square\)

**Proposition 5.** For any MV-algebra \(A_1\), any semisimple MV-algebra \(A_2\) with a set \(S\) of all MV-morphism from \(A_2\) to \([0, 1]\) and any map \(G : A_1 \to A_2\) the following conditions are equivalent:

(i) \(G\) is an fm-function between MV-algebras.

(ii) \(G\) is a strong fm-function between MV-algebras.

**Proof.** (i) \(\implies\) (ii): Note that the composition \(\pi_s \circ i^S_{A_2} \circ G\) is a strong semi-state for any \(s \in T\). It follows that \(i^S_{A_2} \circ G\) is a strong fm-function between MV-algebras. Since the embedding \(i^S_{A_2} : A_2 \to [0, 1]^S\) reflects order we obtain that conditions (FM6)-(FM10) are satisfied.

(ii) \(\implies\) (i): It is evident. \(\square\)

**Open problem 1.** Find MV-algebras \(A_1\) and \(A_2\) with an fm-function \(G\) between them such that \(G\) is not a strong fm-function.

Note that our approach of using semi-states in the above proof of Theorem 3 also covers the main result of the paper \[19\] which is Theorem 4.5 from \[19\].
Theorem 5. Let \( A \) be a semisimple MV-algebra with tense operators \( G \) and \( H \). Then \((S, G, H)\) can be embedded into the tense MV-algebra \([0, 1]^T, G^*, H^*\) induced by the frame \((T, \rho_G)\), where \( T \) is the set of all maximal proper filters and the relation \( \rho_G \) is defined by

\[
A\rho_G B \text{ if and only if } G(x)/A \leq x/B \text{ for any } x \in S.
\]

**Proof.** First, let us define a second relation \( \rho_H \subseteq T^2 \) by the stipulation:

\[
B\rho_H A \text{ if and only if } H(x)/B \leq x/A \text{ for any } x \in S.
\]

**Claim 1.** The equality \( \rho_G = \rho_H^{-1} \) holds.

**Proof.** Let us suppose that \( A\rho_G B \) for some \( A, B \in T \). Due to Definition \( \text{vi} \) we have \( G^\neg H(x) \leq x \) and \( \neg x/A \leq G^\neg H(x)/A \). \( A\rho_G B \) yields \( G^\neg H(x)/A \leq \neg H(x)/B \) and together \( \neg x/A \leq \neg H(x)/B \) yields \( H(x)/B \leq x/A \) for any \( x \in S \).

Due to the definition of \( \rho_H \) we have \( B\rho_H A \) and \( \rho_G \subseteq \rho_H^{-1} \). Analogously we can prove the second inclusion.

The remaining part follows from Theorem 3. Basically, the obtained equations \( G^*(x) = G(x) \) and \( H^*(x) = H(x) \) finish the proof.

**Theorem 6.** a) If \([0, 1]^T, G^*, H^*\) is a tense MV-algebra induced by a time frame \((T, \rho)\), then

(i) if \( \rho \) is reflexive then \( G^*(x) \leq x \) and \( H^*(x) \leq x \) hold for any \( x \in [0, 1]^T \),

(ii) if \( \rho \) is symmetric then \( G^*(x) = H^*(x) \) holds for any \( x \in [0, 1]^T \),

(iii) if \( \rho \) is transitive then \( G^*G^*(x) \geq G^*(x) \) and \( H^*H^*(x) \geq H^*(x) \) hold for any \( x \in [0, 1]^T \).

b) Let \((S, G, H)\) be a semisimple tense MV-algebra and \((T, \rho_G)\) the time frame which induces the tense MV-algebra \([0, 1]^T, G^*, H^*\) by Theorem 5. Then

(i) if \( G(x) \leq x \) and \( H(x) \leq x \) hold for any \( x \in S \) then \( \rho_G \) is reflexive,

(ii) if \( G(x) = H(x) \) holds for any \( x \in S \) then \( \rho_G \) is symmetric,

(iii) if \( GG(x) \geq G(x) \) and \( HH(x) \geq H(x) \) hold for any \( x \in S \) then \( \rho_G \) is transitive.

**Proof.** a) If the relation \( \rho \) is reflexive, then \( iqi \) yields \( G^*(x)(i) = \bigwedge_{i=pq} x(j) \leq x(i) \) for any \( i \in T \). The part for \( H^* \) we can prove analogously.

aii) If \( \rho \) is symmetric then \( G^*(x)(i) = \bigwedge_{i=pq} x(j) = \bigwedge_{j=pi} x(j) = H^*(x)(i) \) for any \( i \in T \) which clearly yields \( G^* = H^* \).

aiii) If \( \rho \) is transitive then \( \{x(k) \mid i,p,j,k \} \subseteq \{x(k) \mid i,p,k \} \) and then

\[
G^*G^*(x)(i) = \bigwedge_{i=pq} G^*(x)(j) = \bigwedge_{i=pq,j=pk} x(k) = \bigwedge_{i=pk} \{x(k) \mid i,p,j,k \} = \bigwedge_{i,p,k} x(k) = G^*(x)(i)
\]
holds for any $i \in T$.

b) We remark that relation $\rho_G$ in Theorem 5 is defined by

$$A \rho_G B \text{ if and only if } G(x)/A \leq x/B \text{ for any } x \in A.$$  

bi) If $G(x) \leq x$ for any $x \in A$ then $G(x)/A \leq x/A$ holds for any $x \in A$ and thus $A \rho_G A$. Together $\rho_G$ is reflexive.

bii) The Claim 1 in the proof of Theorem 5 shows that $G(x)/A \leq x/B$ for any $x \in A$ if and only if $H(x)/B \leq x/A$ for any $x \in A$. If $G = H$ holds then $G(x)/A \leq x/B$ for any $x \in A$ if and only if $G(x)/B \leq x/A$ for any $x \in A$ and consequently $A \rho_G B$ holds if and only if $B \rho_G A$ holds. Thus the relation $\rho_G$ is symmetric.

biii) Let us suppose that $G(x) \leq GG(x)$ for any $x \in A$. If $A \rho_G B$ and $B \rho_G C$ hold then any $x \in A$ satisfies $G(x)/A \leq GG(x)/A \leq G(x)/B \leq x/C$ which yields $A \rho_G C$. Thus the relation $\rho_G$ is transitive. □

Remark 2. Note that one can extend the number of fm-functions between MV-algebras arbitrarily and our results remain valid. Similarly as for semi-states in Remark 1 we could introduce the notion of a dual (strong) fm-function and all the preceding results would also remain valid in this dual setting.

6. Concluding remarks

We have settled a half of the Open problem 5.1 in [11] using a more general approach of fm-functions. The remaining part asks about the existence of a representation theorem for any tense MV-algebra via Di Nola representation theorem for MV-algebras. We hope that our results will be a next step in obtaining a general representation theorem for tense MV-algebras.

We expect that our method can be easily applied to modal or similar operators that may be treated as universal quantifiers on various types of MV-algebras

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Palacký University Olomouc, Faculty of Sciences, tř. 17.listopadu 1192/12, Olomouc 771 46, Czech Republic
E-mail address: michal.botur@upol.cz

Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic
E-mail address: paseka@math.muni.cz