$$T(0,t) = T_i + (T_\infty - T_i)\left[1 - \mathrm{erfc}\left(\frac{h}{k}\sqrt{\kappa t}\right)\exp\left(\frac{h^2\kappa t}{k^2}\right)\right] \tag{34}$$

or

$$T(0,t) = T_i + (T_\infty - T_i)\left\{1 - \exp\left(\frac{h^2\kappa t}{k^2}\right)\left[1 - \mathrm{erf}\left(\frac{h}{k}\sqrt{\kappa t}\right)\right]\right\} \tag{35}$$

The solution (35) is also used for the experimental determination of the heat transfer coefficient h on the basis of measured surface temperature at a given time point t_p.

The temperature distribution in the semi-infinite solid is shown in Fig. 3 as a function of time t and coordinate x. The presented formulas can be used for calculating temperature distribution in components with finite dimensions at the initial moment of heating or cooling.

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Transient Heat Conduction in Sphere

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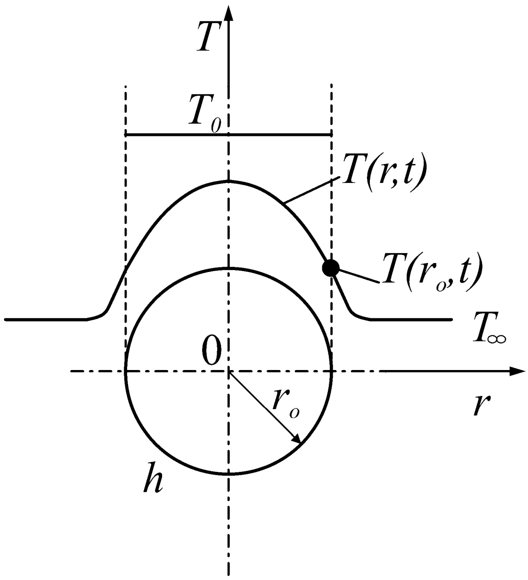
Overview

This entry covers the mathematical description of physical backgrounds governing the transient heat conduction process in sphere. The most common case, the heat transfer between the sphere's body and its surroundings due to convection, is presented. The well-known analytical approach [1, 2] is introduced. The analytical methods are commonly used because they are more accurate and easier to program than numerical methods. In contrary to the numerical solution, it is possible to present the analytical solution as a continuous function. Consequently, the function which satisfies the differential equation can be parameterized. Therefore, it is possible to study the influence of particular parameters on the solution. The additional contribution to this entry is the presentation of charts [1, 3] which allow obtaining the temperature of sphere's center, outer surface, and the average temperature. The mathematical feedback explained in this entry may be used for modeling the engineering phenomena's like hardening of steel or to determine the thermal stresses during the hardening. The mathematical model of transient heat conduction in sphere may also be used to simulate the food boiling process – e.g., potato's boiling.

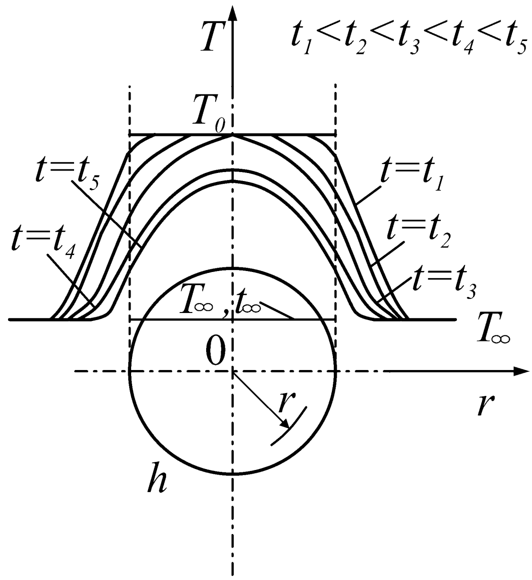
Physical Backgrounds of Transient Heat Conduction in Sphere

The transient heat conduction in sphere occurs, during the contact between the sphere's material and the surrounding, in which temperature differs from the temperature of the material of sphere. Because of the finite thermal diffusivity and relatively large radius, the transient heat conduction in sphere cannot be treated as the conduction in a lumped body. The temperature distribution inside sphere depends on time and the distance from the center of sphere to the point at which the temperature is measured. Consider the model of hot steel sphere immersed in quenching bath (see Fig. 1).

The fluid temperature is significantly lower than the temperature of metal. At time $t = 0$, the sphere, with the initial temperature T_0, is placed in a fluid which temperature is equal to T_∞, and for a time $t > 0$, the sphere is kept in the fluid.



Transient Heat Conduction in Sphere, Fig. 1 Cooling of the sphere which temperature changes from the initial temperature $\boldsymbol{T_0}$ to the ambient temperature T_∞



Transient Heat Conduction in Sphere, Fig. 2 Transient temperature profiles for a sphere – convection boundary conditions

The heat transfer occurs between the sphere and the environment with constant heat transfer coefficient h. The temperature in the sphere is assumed to be symmetric about its center point ($r = 0$). The radiation heat transfer is neglected or incorporated into the heat transfer coefficient h. At the time $t = 0$, the temperature of entire surface of the sphere is equal to T_0. However, with increasing time $t_1 > 0$ and $t_2 > t_1$, the temperature of wall starts to drop (see Fig. 2) due to the heat transfer between the sphere and the surrounding medium. This creates the temperature gradient across the wall and begins the heat transfer from the inner parts of the sphere's walls to the outers. For the short period of time, the changes of temperature at $r = 0$ are negligible, and it is possible to assume that the temperature $T(0,t)$ remains T_0. After a period of time when the t reaches t_3, the temperature at $r = 0$ starts to drop. In successive time intervals, the temperature profile gets flatter and flatter and finally becomes uniform reaching T_∞ value.

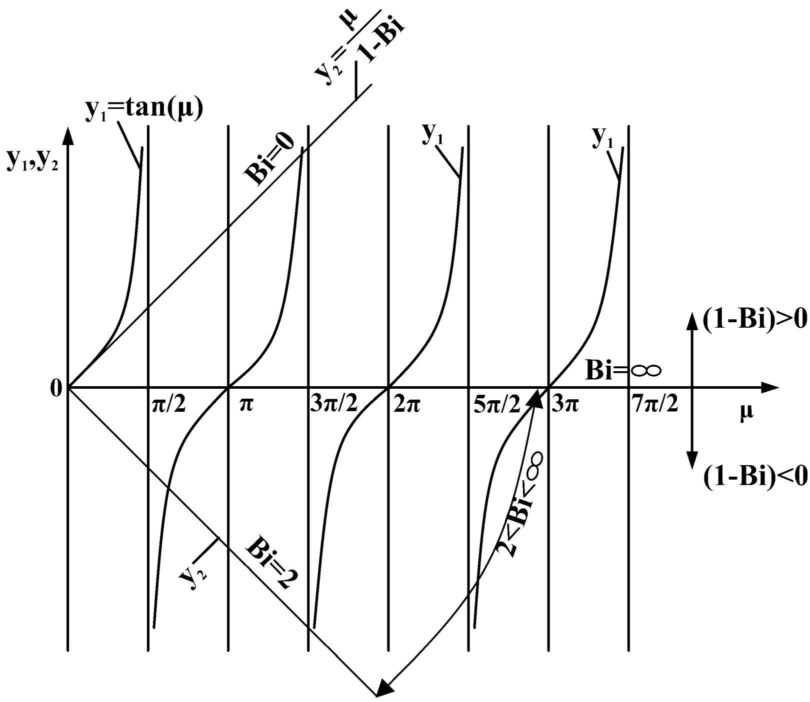
The heat released from the sphere is absorbed by the surrounding medium. Assuming that there is no heat source in the sphere domain, the temperature of sphere's wall decreases, and after an infinitely long time, $(t \rightarrow t_\infty)$ reaches the temperature of medium. This means that there is no heat transfer, because the temperature difference does not exist. The wall of the sphere reaches the thermal equilibrium with the surrounding medium. The formulas presented in the following are also valid for the situation when the fluid temperature is higher than the initial sphere temperature.

Mathematical Model of Transient Heat Conduction in Sphere

The physical principles which describe the problem of transient heat conduction in sphere are modeled using the partial differential equation, which can be solved analytically. The method proposed in this entry is the separation of variables.

The following assumptions are considered:

- One-dimensional heat transfer conduction
- Constant diffusivity ($\kappa = \text{const.}$)
- Constant thermal conductivity ($k = \text{const.}$)
- Uniform heat transfer coefficient ($h = \text{const.}$) and ambient temperature ($T_\infty = \text{const.}$)



Transient Heat Conduction in Sphere, Fig. 3 Graphical method for determining roots of the characteristic equation $\tan(\mu) = \frac{\mu}{1-\mathrm{Bi}}$

According to the heat conduction law, the temperature distribution inside the sphere is given by [4]

$$\frac{\partial \theta}{\partial t} = \kappa\left(\frac{\partial^2 \theta}{\partial r^2} + \frac{2}{r}\frac{\partial \theta}{\partial r}\right) \tag{1}$$

with boundary conditions

$$-k\frac{\partial \theta}{\partial r}\bigg|_{r=r_o} = h\theta|_{r=r_o} \tag{2}$$

$$\frac{\partial \theta}{\partial r}\bigg|_{r=0} = 0 \tag{3}$$

and initial condition

$$\theta|_{t=0} = \theta_0 \tag{4}$$

where

$$\theta = T - T_\infty \tag{5}$$

and

$$\theta_0 = T_0 - T_\infty \tag{6}$$

Introducing the new variable

$$\vartheta = \theta \cdot r \tag{7}$$

the initial boundary value problem (1–4) may be written as follows:

$$\frac{\partial \vartheta}{\partial t} = \kappa\frac{\partial^2 \vartheta}{\partial r^2} \tag{8}$$

$$-k\frac{\partial \vartheta}{\partial r}\bigg|_{r=r_0} = \left(h - \frac{k}{r_o}\right)\vartheta|_{r=r_o} \tag{9}$$

$$\vartheta|_{r=0} = 0 \tag{10}$$

$$\vartheta|_{t=0} = r\theta_0 \tag{11}$$

Equation (8) is solved using the separation of variables. The variable ϑ is a function of r and t; hence, it may be written as

$$\vartheta(r,t) = \varphi(t)\psi(r) \quad (12)$$

Substituting (12) to (8), the following form is obtained:

$$\frac{1}{\kappa}\psi\frac{d\varphi}{dt} = \varphi\frac{d^2\psi}{dr^2} \quad (13)$$

After dividing both sides of (13) by $\varphi(t)\psi(x)$, (13) is transformed into

$$\frac{1}{\kappa}\frac{1}{\varphi}\frac{d\varphi}{dt} = \frac{1}{\psi}\frac{d^2\psi}{dr^2} \quad (14)$$

The equality (14) should be satisfied for any values of x and t; therefore, both sides of (14) should be equal to the constant. Its sign, due to the finite value of ϑ for increasing time, should be negative. Denoting the aforementioned constant as $-w^2$, (14) becomes

$$\frac{1}{\kappa}\frac{1}{\varphi}\frac{d\varphi}{dt} = \frac{1}{\psi}\frac{d^2\psi}{dr^2} = -w^2 \quad (15)$$

Hence, the system of two ordinary differential equations

$$\frac{d\varphi}{dt} + \kappa w^2\varphi = 0 \quad (16)$$

and

$$\frac{d^2\psi}{d^2t} + w^2\psi = 0 \quad (17)$$

is obtained.

The general solutions of (16) and (17) are given below:

$$\varphi = C_1 \cdot e^{-\kappa w^2 t} \quad (18)$$

and

$$\psi = C_2 \cdot \cos(wr) + C_3 \cdot \sin(wr) \quad (19)$$

Substituting (18) and (19) into (12), the following formula for ϑ is obtained:

$$\vartheta(r,t) = \varphi(t)\psi(r) = e^{-\kappa w^2 t}[A \cdot \sin(wr) + B \cdot \cos(wr)] \quad (20)$$

where $A = C_1 \cdot C_3$ and $B = C_1 \cdot C_2$.

Because the solution $\vartheta(r,t)$ must satisfy boundary conditions (10), then

$$\vartheta(0,t) = e^{-\kappa w^2 t}B = 0 \quad (21)$$

and $B = 0$. Consequently, the solution of (8) is given by

$$\vartheta(r,t) = Ae^{-\kappa w^2 t}\sin(wr) \quad (22)$$

After substituting (22) into (9), the characteristic equation is obtained:

$$-kAw\cos(wr_o) = \left(h - \frac{k}{r_o}\right) \cdot A \cdot \sin(wr_0)$$
$$\tan(wr_o) = \frac{wr_o}{1 - \mathrm{Bi}} \quad (23)$$

where $\mathrm{Bi} = \frac{\mathrm{hr_o}}{\mathrm{k}}$.

Taking into account that $\mu = wr_o$, the (23) can be written in the form that is easier to represent in a graphical way:

$$\tan(\mu) = \frac{\mu}{1 - \mathrm{Bi}} \quad (24)$$

The graphical method for determination of roots of the characteristic equation is presented in Fig. 3.

On the basis of Fig. 3, it is possible to localize the regions where the roots are situated. If the denominator of the (24) is greater than zero, $(1-\mathrm{Bi}) > 0$, then the n^{th} root is located in the interval

$$(n-1)\pi \le \mu_n \le \frac{\pi}{2} + (n-1)\pi, \quad n = 1, 2\ldots, \text{ if } (1 - \mathrm{Bi}) > 0 \quad (25)$$

If the denominator of the (24) is less than zero, $(1-\mathrm{Bi}) < 0$, then the n^{th} root is located in the interval

$$\frac{\pi}{2} + (n-1)\pi \leq \mu_n \leq \pi + (n-1)\pi, \quad n = 1, 2\ldots, \text{ if } (1 - \mathrm{Bi}) < 0 \tag{26}$$

If the Biot number Bi equals 1, then

$$\mu_n = (2n-1)\frac{\pi}{2}, \qquad n = 1, 2\ldots, \text{ for Bi} = 0 \tag{27}$$

If the Biot number Bi tends to infinity, then

$$\mu_n = n\pi, \qquad n = 1, 2\ \ldots, \text{ if } \mathrm{Bi} \to \infty \tag{28}$$

The first six roots of the characteristic (24) obtained on the basis of computer programs published in [1] are listed in a Table 1.

In order to satisfy the initial boundary condition (11), (22) is written as follows:

$$\vartheta(r,t) = \sum_{n=1}^{\infty} A_n e^{-\kappa w^2 t} \sin(wr) = \sum_{n=1}^{\infty} A_n e^{-\kappa w^2 t} \sin\left(\frac{\mu_n}{r_o} r\right) \tag{29}$$

where $w = \frac{\mu_n}{r_o}$, after substituting (29) into (11), the following equality is obtained:

$$r(T_o - T_\infty) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{\mu_n}{r_o} r\right) \tag{30}$$

Multiplying both sides of (30) by $\sin(\frac{\mu_m}{r_o} r)$ and integrating from 0 to r_o gives

$$(T_o - T_\infty) \int_0^{r_0} r \sin\left(\mu_m \frac{r}{r_o}\right) dr = \sum_{n=1}^{\infty} A_n \int_0^{r_o} \sin\left(\frac{\mu_n}{r_o} r\right) \sin\left(\frac{\mu_m}{r_o} r\right) dr \tag{31}$$

The right side integral is

$$\int_0^{r_0} \sin\left(\frac{\mu_n}{r_o} r\right) \sin\left(\frac{\mu_m}{r_o} r\right) dr = \frac{r_o(\mu_n \sin\mu_m \cos\mu_n - \mu_m \sin\mu_n \cos\mu_m)}{\mu_m^2 - \mu_n^2} \tag{32}$$

Multiplying both sides of the characteristic equation (24) by $\sin\mu_n$ and substituting $\mu_n = \mu_m$ gives

$$\sin\mu_m \sin\mu_n (1 - \mathrm{Bi}) = \mu_\mathrm{m} \cos\mu_\mathrm{m} \sin\mu_\mathrm{n} \tag{33}$$

Similarly, the multiplication of both sides of the characteristic equation (24) by $\sin\mu_m$ and substitution μ_n instead of μ gives

$$\sin\mu_m \sin\mu_n (1 - \mathrm{Bi}) = \mu_\mathrm{n} \cos\mu_\mathrm{n} \sin\mu_\mathrm{m} \tag{34}$$

The right sides of (33) and (34) are equal. Therefore, when $m \neq n$, the integral given by (32) equals 0. In the summation term – the right side of (31) – the only nonzero components are these for which $m = n$. This shows the orthogonally of functions $\sin\mu_n$ and $\sin\mu_m$. Hence, if $m = n$, then

$$\int_0^{r_0} \sin\left(\frac{\mu_n}{r_o} r\right) \sin\left(\frac{\mu_m}{r_o} r\right) dr = \int_0^{r_0} \sin^2\left(\frac{\mu_n}{r_o} r\right) dr = \frac{r_0}{2\mu_n}(\mu_n - \sin\mu_n \cos\mu_n) \tag{35}$$

In case that $m \neq n$,

$$\int_0^{r_0} \sin\left(\frac{\mu_n}{r_o} r\right) \sin\left(\frac{\mu_m}{r_o} r\right) dr = 0 \tag{36}$$

Taking into account (35) and (36) in (31) and noting that the integral on the right side of (31) vanishes when $m \neq n$, (31) becomes

Transient Heat Conduction in Sphere, Table 1 First six roots of the characteristic equation $\tan(\boldsymbol{\mu}) = \frac{\boldsymbol{\mu}}{1-\mathrm{Bi}}$ obtained for different values of Bi

Bi	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6
0	0,00E+00	4.4934	7.7253	10.904	14.066	17.221
0.001	0.054767	4.4936	7.7254	10.904	14.066	17.221
0.002	0.077444	4.4939	7.7255	10.904	14.066	17.221
0.003	0.094839	4.4941	7.7256	10.904	14.066	17.221
0.004	0.1095	4.4943	7.7258	10.904	14.066	17.221
0.005	0.12241	4.4945	7.7259	10.905	14.067	17.221
0.006	0.13408	4.4947	7.726	10.905	14.067	17.221
0.007	0.14481	4.495	7.7262	10.905	14.067	17.221
0.008	0.15479	4.4952	7.7263	10.905	14.067	17.221
0.009	0.16417	4.4954	7.7264	10.905	14.067	17.221
0.01	0.17303	4.4956	7.7265	10.905	14.067	17.221
0.011	0.18146	4.4959	7.7267	10.905	14.067	17.221
0.012	0.18951	4.4961	7.7268	10.905	14.067	17.221
0.013	0.19723	4.4963	7.7269	10.905	14.067	17.222
0.014	0.20465	4.4965	7.7271	10.905	14.067	17.222
0.015	0.21181	4.4967	7.7272	10.905	14.067	17.222
0.03	0.2991	4.5001	7.7291	10.907	14.068	17.222
0.04	0.34503	4.5023	7.7304	10.908	14.069	17.223
0.05	0.38537	4.5045	7.7317	10.909	14.07	17.224
0.06	0.42173	4.5068	7.733	10.91	14.07	17.224
0.07	0.45506	4.509	7.7343	10.911	14.071	17.225
0.08	0.486	4.5112	7.7356	10.911	14.072	17.225
0.09	0.51497	4.5134	7.7369	10.912	14.073	17.226
0.1	0.54228	4.5157	7.7382	10.913	14.073	17.227
0.11	0.56818	4.5179	7.7395	10.914	14.074	17.227
0.12	0.59286	4.5201	7.7408	10.915	14.075	17.228
0.13	0.61645	4.5223	7.7421	10.916	14.075	17.228
0.14	0.63908	4.5246	7.7434	10.917	14.076	17.229
0.15	0.66086	4.5268	7.7447	10.918	14.077	17.229
0.3	0.92079	4.5601	7.7641	10.932	14.088	17.238
0.4	1.0528	4.5822	7.777	10.941	14.095	17.244
0.5	1.1656	4.6042	7.7899	10.95	14.102	17.25
0.6	1.2644	4.6261	7.8028	10.959	14.109	17.256
0.7	1.3525	4.6479	7.8156	10.968	14.116	17.261
0.8	1.432	4.6696	7.8284	10.977	14.123	17.267
0.9	1.5044	4.6911	7.8412	10.986	14.13	17.273
1	1.5708	4.7124	7.854	10.996	14.137	17.279
1.1	1.632	4.7335	7.8667	11.005	14.144	17.285
1.2	1.6887	4.7544	7.8794	11.014	14.151	17.29
1.3	1.7414	4.7751	7.892	11.023	14.158	17.296
1.4	1.7906	4.7956	7.9045	11.032	14.165	17.302
1.5	1.8366	4.8158	7.9171	11.041	14.172	17.308
3	2.2889	5.087	8.0962	11.173	14.276	17.393
4	2.4556	5.2329	8.2045	11.256	14.343	17.449
5	2.5704	5.354	8.3029	11.335	14.408	17.503
6	2.6537	5.4544	8.3913	11.409	14.47	17.556

(*continued*)

Transient Heat Conduction in Sphere, Table 1 (continued)

Bi	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6
7	2.7165	5.5378	8.4703	11.477	14.529	17.607
8	2.7654	5.6078	8.5406	11.541	14.585	17.656
9	2.8044	5.6669	8.6031	11.599	14.637	17.703
10	2.8363	5.7172	8.6587	11.653	14.687	17.748
11	2.8628	5.7606	8.7083	11.703	14.733	17.791
12	2.8851	5.7981	8.7527	11.748	14.777	17.832
13	2.9041	5.8309	8.7924	11.79	14.818	17.87
14	2.9206	5.8597	8.8282	11.828	14.856	17.907
15	2.9349	5.8852	8.8605	11.863	14.892	17.941
30	3.0372	6.0766	9.1201	12.169	15.225	18.287
40	3.0632	6.1273	9.1933	12.262	15.333	18.409
50	3.0788	6.1582	9.2384	12.32	15.403	18.489
60	3.0893	6.1788	9.269	12.36	15.452	18.545
70	3.0967	6.1937	9.2909	12.389	15.487	18.586
80	3.1023	6.2048	9.3075	12.411	15.514	18.618
90	3.1067	6.2135	9.3204	12.428	15.535	18.643
100	3.1102	6.2204	9.3308	12.441	15.552	18.663
110	3.113	6.2261	9.3393	12.453	15.566	18.68
120	3.1154	6.2309	9.3464	12.462	15.578	18.694
130	3.1174	6.2349	9.3524	12.47	15.588	18.706
140	3.1192	6.2383	9.3576	12.477	15.596	18.716
150	3.1207	6.2413	9.362	12.483	15.604	18.725

$$(T_o - T_\infty)\int_0^{r_0} r\sin\left(\mu_n \frac{r}{r_o}\right)dr = A_n \int_0^{r_o} \sin\left(\frac{\mu_n}{r_o} r\right)\sin\left(\frac{\mu_n}{r_o} r\right)dr \quad (37)$$

The coefficient A_n is obtained after some manipulations:

$$A_n = \frac{(T_o - T_\infty)\int_0^{r_0} r\sin(\mu_n \frac{r}{r_o})dr}{\int_0^{r_o} \sin(\frac{\mu_n}{r_o} r)\sin(\frac{\mu_n}{r_o} r)dr} = \frac{2r_0(T_o - T_\infty)(\sin\mu_n - \mu_n\cos\mu_n)}{\mu_n(\mu_n - \sin\mu_n\cos\mu_n)} \quad (38)$$

Substituting (38) into (29), the solution $\vartheta(x,t)$ is given according to the following formula:

$$\vartheta(r,t) = \sum_{n=1}^{\infty} A_n e^{-\kappa w^2 t}\sin\left(\frac{\mu_n}{r_o} r\right) = \frac{2r_o(T_o - T_\infty)(\sin\mu_n - \mu_n\cos\mu_n)}{\mu_n(\mu_n - \sin\mu_n\cos\mu_n)} \times \sin\left(\frac{\mu_n}{r_o} r\right)e^{-kw^2 t} \quad (39)$$

According to (7) $\vartheta = \theta r$; thus,

$$\theta = \sum_{n=1}^{\infty} \frac{2(T_o - T_\infty)(\sin\mu_n - \mu_n\cos\mu_n)}{(\mu_n - \sin\mu_n\cos\mu_n)} \frac{\sin(\frac{\mu_n}{r_o} r)}{\mu_n \frac{r}{r_o}} e^{-\kappa w^2 t} \quad (40)$$

Assuming the following dimensionless variables $R = \frac{r}{r_o}$ and $Fo = \frac{\kappa t}{r_o^2}$,

$$\theta = \sum_{n=1}^{\infty} \frac{2(T_o - T_\infty)(\sin\mu_n - \mu_n\cos\mu_n)}{(\mu_n - \sin\mu_n\cos\mu_n)} \frac{\sin(\frac{\mu_n}{r_o} r)}{\mu_n R} e^{-\kappa w^2 t} \quad (41)$$

Dividing (41) by θ_0 gives

$$\frac{\theta}{\theta_0} = \frac{T(r,t) - T_\infty}{T_0 - T_\infty} = \sum_{n=1}^{\infty} \frac{2(\sin\mu_n - \mu_n\cos\mu_n)}{(\mu_n - \sin\mu_n\cos\mu_n)} \frac{\sin(\mu_n R)}{\mu_n R} e^{-\kappa w^2 t} \tag{42}$$

The dimensionless average temperature is calculated according to formula

$$\frac{\bar{\theta}}{\theta_0} = \frac{\bar{T}(r,t) - T_\infty}{T_0 - T_\infty} = \frac{3}{r_o^3}\int_0^{r_o} \frac{\theta}{\theta_0} r^2 dr = 3\int_0^1 \frac{\theta}{\theta_0} R^2 dR \tag{43}$$

Substituting (42) into (43) and taking into account that

$$\int_0^1 R\sin(\mu_n R)dR = \frac{1}{\mu_n^2}(\sin\mu_n - \mu_n\cos\mu_n) \tag{44}$$

Equation (43) becomes

$$\frac{\bar{\theta}}{\theta_0} = \sum_{n=1}^{\infty} \frac{6}{\mu_n^3} \frac{(\sin\mu_n - \mu_n\cos\mu_n)^2}{\mu_n - \sin\mu_n\cos\mu_n} e^{-\mu_n^2 Fo} \tag{45}$$

In order to obtain the rate of temperature change, (41) is transformed to the following form:

$$T(r,t) = (T_0 - T_\infty)\sum_{n=1}^{\infty} \frac{2(\sin\mu_n - \mu_n\cos\mu_n)}{(\mu_n - \sin\mu_n\cos\mu_n)} \times \frac{\sin(\mu_n R)}{\mu_n R} e^{-\kappa w^2 t} + T_\infty \tag{46}$$

The differentiation of (46) with respect to time allows obtaining the formula for the rate of temperature changes in the sphere:

$$\frac{dT(r,t)}{dt} = \frac{\kappa(T_0 - T_\infty)}{r_z^2}\sum_{n=1}^{\infty} \frac{2\mu_n(\sin\mu_n - \mu_n\cos\mu_n)}{(\mu_n - \sin\mu_n\cos\mu_n)} \times \frac{\sin(\mu_n R)}{R} e^{-\kappa w^2 t} \tag{47}$$

Numerical Examples

The brief analysis of (46) and (47) allows concluding that the transient temperature profile and the rate of temperature changes in sphere strongly depend on the value of the heat transfer coefficient. To investigate how the different values of h influence the temperature distribution inside the sphere, three values of h are taken into consideration. The calculations are carried out using programs published in [1] for the following data: $r_o = 20$ mm, $\kappa = 10^{-5}\mathrm{m}^2/\mathrm{s}$, $k = 46\mathrm{W}/(\mathrm{m}\cdot\mathrm{K})$, $T_0 = 800\ °\mathrm{C}$, $T_\infty = 42\ °\mathrm{C}$.

The values of the heat transfer coefficient are listed in Table 2

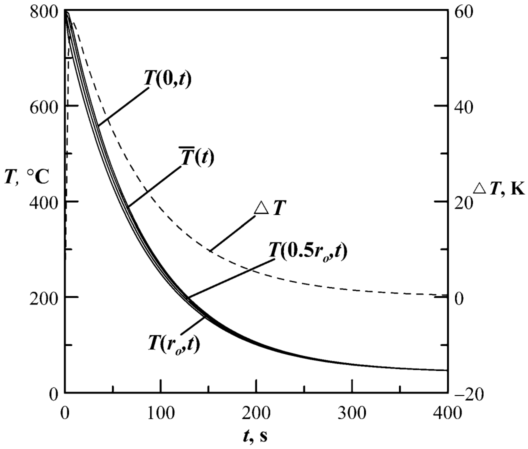
In Fig. 4, the transient temperature plots (see 46), obtained for the inner, outer, and mean radius, $r = 0.5r_o$, are presented. The calculations are carried out for $h = 400\ \mathrm{W}/(\mathrm{m}^2\cdot\mathrm{K})$.

Furthermore, the transient plots for the average temperature of the sphere wall are shown. The average temperature can be obtained by transforming (43) to the form presented below:

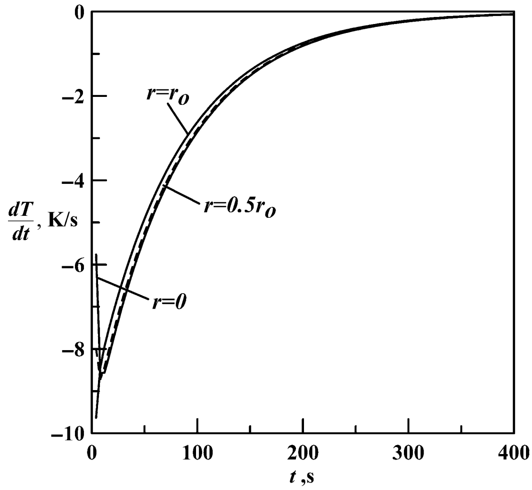
$$\bar{T}(r,t) = (T_0 - T_\infty) \times \sum_{n=1}^{\infty} \frac{6}{\mu_n^3} \frac{(\sin\mu_n - \mu_n\cos\mu_n)^2}{\mu_n - \sin\mu_n\cos\mu_n} e^{-\mu_n^2 Fo} + T_\infty \tag{48}$$

Transient Heat Conduction in Sphere, Table 2 Values of h for computational cases

Case no.	h $\mathrm{W}/(\mathrm{m}^2\cdot\mathrm{K})$
1	400
2	4,000
3	40,000



Transient Heat Conduction in Sphere, Fig. 4 The transient temperature plots for the center of sphere *T(0,t)*, outer surface of sphere *T(r_o, t)*, the mean radius of sphere *T(0.5r_o, t)*, average temperature $\bar{T}$, and temperature difference $\Delta T = T(0, t) - T(r_o, t)$, $h = 400 \mathrm{W}/(\mathrm{m}^2 \cdot \mathrm{K})$



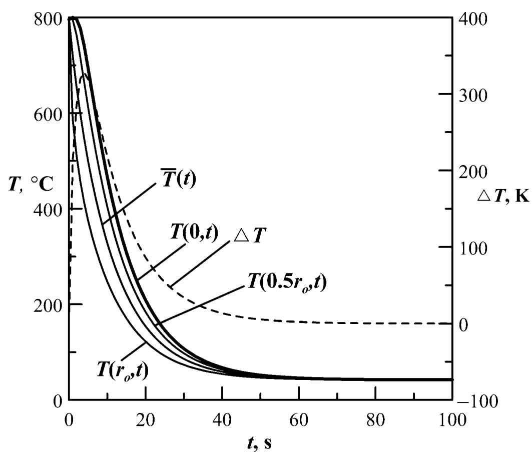
Transient Heat Conduction in Sphere, Fig. 5 The rate of temperature change at the outer surface of the sphere, on the mean radius, and at the center of the sphere, $h = 400\ \mathrm{W}/(\mathrm{m}^2 \cdot \mathrm{K})$

For low values of h, e.g., $h = 400\ \mathrm{W}/(\mathrm{m}^2 \cdot \mathrm{K})$, the temperature difference between the center of the sphere and its outer surface is low. Therefore the temperatures calculated at the center, at the outer surface of the sphere and the mean temperaure of the sphere have nearly the same values. The maximal value of the temperature difference ΔT does not exceed 60 K. With increasing time, the analyzed temperatures approach the temperature of surroundings.

In Fig. 5, the rates of temperature changes for the center, mean radius, and the outer surface of the sphere are presented.

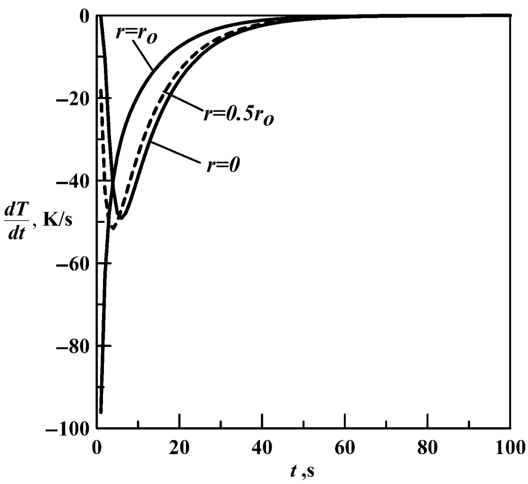
During the first 4 s of the heat transfer process, the rate of temperature changes is the largest at the outer surface and the lowest at the center of the sphere. This can be explained by the thermal inertia of the sphere's material. After the relatively short period of time, the rate of temperature changes is nearly equal for the all analyzed cases.

The temperature transients for $h = 4000\,\mathrm{W}/(\mathrm{m}^2 \cdot \mathrm{K})$ are presented in Fig. 6. It is possible to observe that temperature differences between the center and the outer surface of sphere are significantly larger than for the computational case no. 1, especially for the time period $t = 0$ to 40 s. The largest temperature difference occurs at $t = 10$ s and is equal to 310 °C. The steady state t_∞ is reached significantly faster in comparison to the first analyzed case.

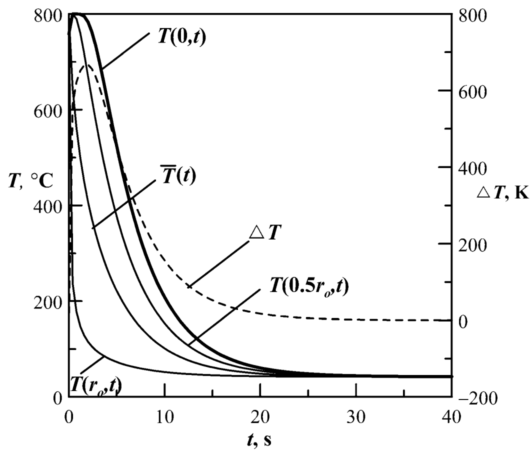


Transient Heat Conduction in Sphere, Fig. 6 The transient temperature plots for the center of sphere *T(0,t)*, outer surface of sphere *T(r_o, t)*, the mean radius of sphere *T(0.5r_o, t)*, average temperature $\bar{T}$, and temperature difference $\Delta \boldsymbol{T} = \boldsymbol{T}(\boldsymbol{0}, \boldsymbol{t}) - \boldsymbol{T}(\boldsymbol{r_o}, \boldsymbol{t})$, $h = 4{,}000\,\mathrm{W}/(\mathrm{m}^2 \cdot \mathrm{K})$

The rates of temperature changes for $h = 4000\,\mathrm{W}/(\mathrm{m}^2 \cdot \mathrm{K})$ are presented in Fig. 7. The largest values of $\frac{dT(r,t)}{dt}$ are obtained for $r = r_o$. Due to the thermal inertia of sphere's material, the lower values of $\frac{dT(r,t)}{dt}$ are obtained



Transient Heat Conduction in Sphere, Fig. 7 The rate of temperature change at the outer surface of the sphere, on the mean radius, and at the center of the sphere, $h = 4{,}000\ \mathrm{W/(m^2 \cdot K)}$



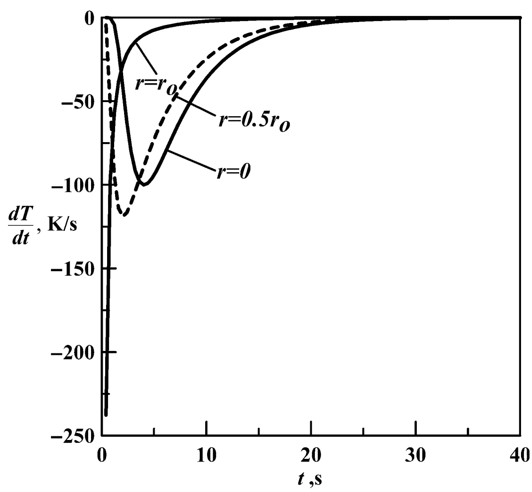
Transient Heat Conduction in Sphere, Fig. 8 The transient temperature plots for the center of sphere *T(0,t)*, outer surface of sphere $T(r_o, t)$, the mean radius of sphere $T(0.5r_o, t)$, average temperature $\bar{T}$, and temperature difference $\Delta T = T(0,t) - T(r_o,t)$, $h = 40{,}000\ \mathrm{W/(m^2 \cdot K)}$

for $r = 0.5r_o$ and the lowest for the center of sphere. Comparing Figs. 5 and 7 one can see, that the rates of temperature changes obtained for $h = 4000\ \mathrm{W/(m^2 \cdot K)}$ are significantly larger than if $h = 400\ \mathrm{W/(m^2 \cdot K)}$.

The temperature transients obtained for $h = 40000\ \mathrm{W/(m^2 \cdot K)}$ are presented in Fig. 8. For $h = 40000\ \mathrm{W/(m^2 \cdot K)}$ the temperature differences between the center of the sphere and the outer surface of the sphere are larger than for $h = 400\ \mathrm{W/(m^2 \cdot K)}$ and $h = 4000\ \mathrm{W/(m^2 \cdot K)}$. When $h = 40000\ \mathrm{W/(m^2 \cdot K)}$ the steady-state heat conduction occurs after 34 s, significantly faster than if $h = 400\ \mathrm{W/(m^2 \cdot K)}$ and $h = 4000\ \mathrm{W/(m^2 \cdot K)}$.

The rates of temperature change for $h = 40000\ \mathrm{W/(m^2 \cdot K)}$, presented in Fig. 9, are significantly larger than if $h = 400\ \mathrm{W/(m^2 \cdot K)}$ and $h = 4000\ \mathrm{W/(m^2 \cdot K)}$. For larger values of heat transfer coefficient h, the temperature of sphere rapidly reaches the temperature of medium.

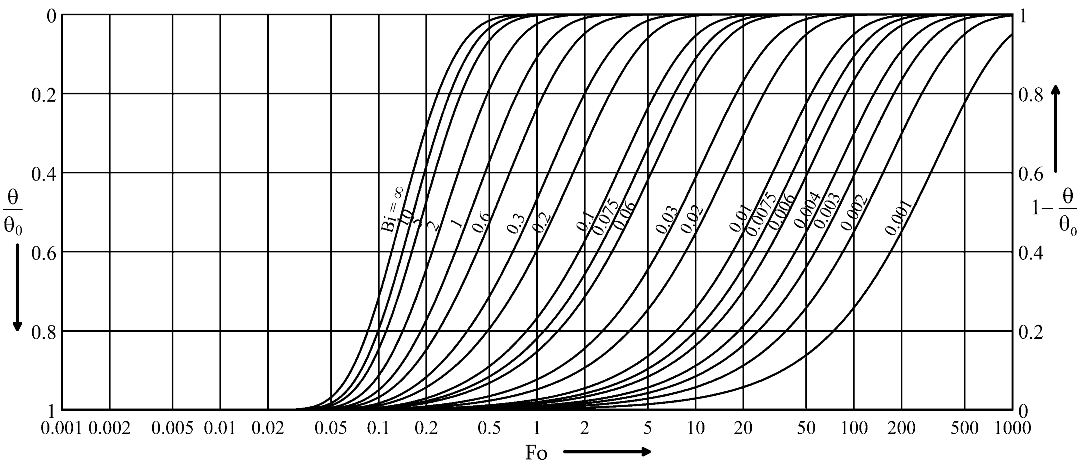
Comparing Figs. 4 and 5 it is possible to draw a conclusions that for low value of heat transfer coefficient the ratios of temperature changes at the center of the sphere, outer surface and at the mean radius of the sphere do not differ



Transient Heat Conduction in Sphere, Fig. 9 The rate of temperature change at the outer surface of the sphere, on the mean radius, and at the center of the sphere, $h = 40{,}000\ \mathrm{W/(m^2 \cdot K)}$

significantly. Therefore, the temperature of sphere material is close to uniform. On the other hand if the heat transfer coefficient h is large, then the rate of temperature changes for studied locations differs significantly. Consequently, the large temperature differences occur in the domain of sphere during the time.

T



Transient Heat Conduction in Sphere, Fig. 10 Dimensionless temperature at the center of the sphere $\frac{\theta(0,t)}{\theta_0} = \frac{T(0,t)-T_\infty}{T_0-T_\infty}$ as a function of Biot and Fourier numbers

To apply the the matematical models of transient heat conduction in sphere for engineering calculations, the specialized charts (see Fig. 10) were developed. These charts are developed for dimensionless parameters: Fourier number Fo, Biot number Bi and the dimensionless temperature Q/Q_0. If the values of Fo and Bi are known then it is possible to determine the value of dimensionless temperature Q/Q_0 at the outer surface and at the center of the sphere. Similary, the mean dimensionless temperaute can be calculated as a function of Fourier and Biot numbers.

On the basis of Fig. 10, the given values of Fourier and Biot numbers allow graphically determining the dimensionless temperature for the center of the sphere, which is defined as:

$$\frac{\theta(0,t)}{\theta_0} = \frac{T(0,t) - T_\infty}{T_0 - T_\infty} \tag{49}$$

Hence, aftere some manipulations of (49) the temperature at the center of the sphere is obtained

$$T(0,t) = \frac{\theta(0,t)}{\theta_0}(T_0 - T_\infty) + T_\infty \tag{50}$$

Similarly, the temperature of the sphere's outer surface is determined for the known Biot and Fourier numbers. The dimensionless temperature at the outer surface is given by the following formula:

$$\frac{\theta(r_o,t)}{\theta_0} = \frac{T(r_o,t) - T_\infty}{T_0 - T_\infty} \tag{51}$$

The values of $\frac{\theta(r_o,t)}{\theta_0}$ can be obtained from Fig. 11 for the known values of Biot and Fourier numbers.

The temperature at the outer surface of the sphere can be obtained by transforming (51) into the following form:

$$T(r_o,t) = \frac{\theta(r_o,t)}{\theta_0}(T_0 - T_\infty) + T_\infty \tag{52}$$

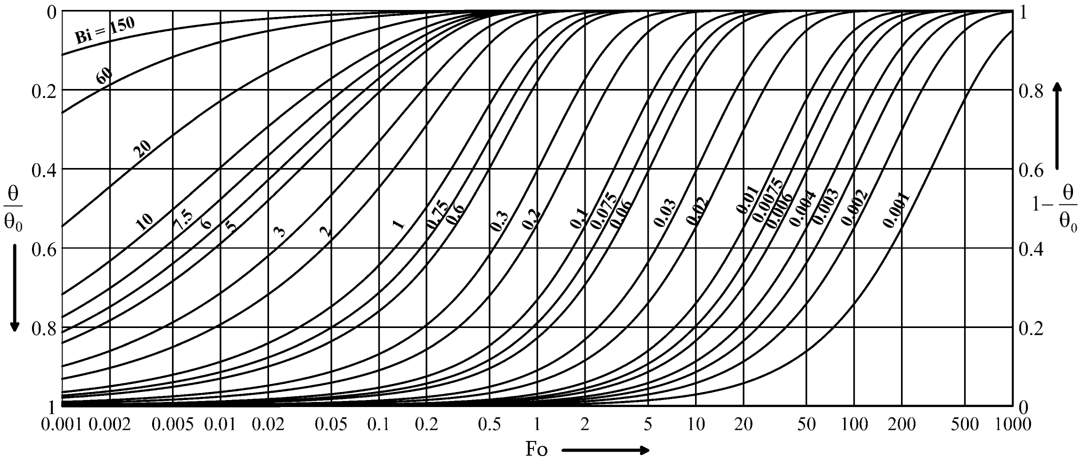
The same procedure may be applied in order to obtain the dimensionless average temperature of the sphere. According to (43), the dimensionless average temperature of the sphere becomes

$$\frac{\bar{\theta}}{\theta_0} = \frac{\bar{T}(r,t) - T_\infty}{T_0 - T_\infty} \tag{53}$$

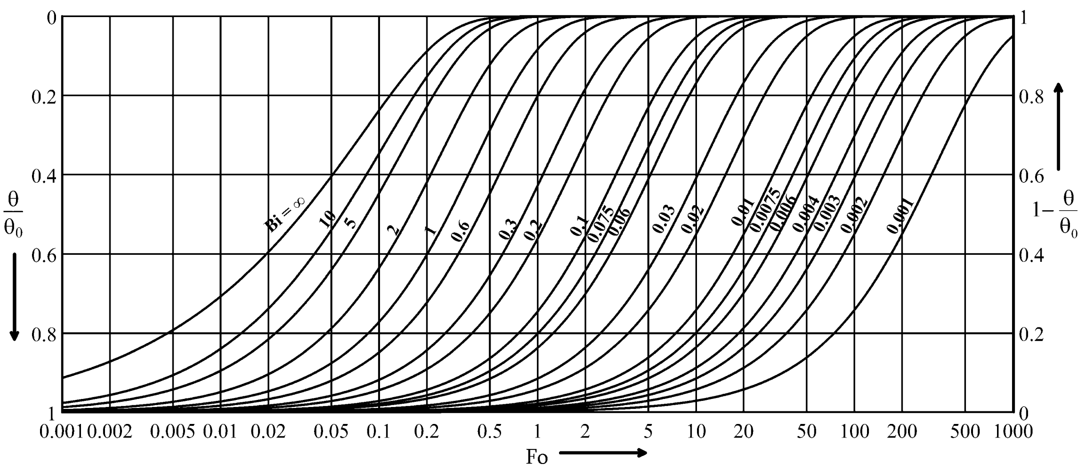
It may be obtained from Fig. 12 as a function of Biot and Fourier number.

The value of $\bar{T}(r_o,t)$ is calculated as follows:

$$\bar{T}(r_o,t) = \frac{\bar{\theta}(r,t)}{\theta_0}(T_0 - T_\infty) + T_\infty \tag{54}$$



Transient Heat Conduction in Sphere, Fig. 11 Dimensionless temperature at the outer sphere surface $\frac{\theta(r_o,t)}{\theta_0} = \frac{T(r_o,t)-T_\infty}{T_0-T_\infty}$ as a function of Biot and Fourier numbers



Transient Heat Conduction in Sphere, Fig. 12 Dimensionless average temperature of the sphere $\frac{\bar{\theta}(r,t)}{\theta_0} = \frac{\bar{T}(r,t)-T_\infty}{T_0-T_\infty}$ as a function of Biot and Fourier numbers

Summary

The transient heat conduction in sphere is described. The physical backgrounds of these phenomena are introduced. The analytical formulas, which describe the transient heat conduction in the sphere, are presented. The method of variables separation is used to obtain the temperature distribution as a function of time and position. The influence of the heat transfer coefficient from surroundings to the sphere is studied. Moreover, the charts that allow determining the value of temperature at the certain time point are presented. This method permits to calculate the temperature at the center of the sphere, the temperature at the outer surface of the sphere and the average temperature of the sphere at the specified time on the basis of dimensionless parameters – Biot Bi and Fourier *Fo* numbers.

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Transient Hertzian Contact Heat Transfer

► Heat Transfer During Impact

Transient Temperature and Thermal Stresses in Turbine Components

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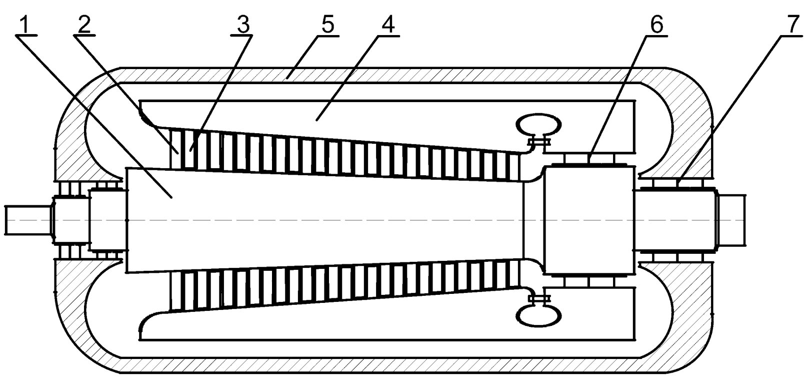
Overview

Steam turbines are used in thermal cycles that convert the chemical or the nuclear energy of a fuel into mechanical energy, which usually drives an electric power generator. The turbine is a thermal engine that converts the energy of flowing steam into mechanical energy. The turbines manufactured today are either drum-type turbines (so-called reaction turbines) or disc-type turbines (impulse turbines) [1]. Plots describing these turbines are shown in Figs. 1 and 2. The plots present the essential components of the turbines. The steam flows through the system of blades and expands, while its temperature decreases. The higher the temperature of the steam delivered to the turbine, the higher the efficiency of the conversion of the fuel chemical energy into mechanical energy. Furthermore, the temperature of the components that are exposed to the flowing steam increases. This process results in thermal stresses that occur in the components of the turbine.

Operation of Turbines

In modern turbines, the live steam pressure reaches 30 MPa, and the temperature exceeds 600 °C. Due to such large parameter values, the expansion process takes place in three or four sections of the turbine (Fig. 3). The steam flows from the outlet of the HP section to the boiler where the steam temperature is increased (usually to a level slightly above the temperature at the inlet to the HP section). Next, the steam flows into the IP section and subsequently into the LP section. Depending on the power output, a turbine may include one, two, or three LP sections.

The operation of steam turbines is performed in cycles [2]. The length of a single cycle depends on the application and varies from several hundreds of hours to several thousands of hours. A single cycle begins with a start-up process. In this period the parameters of the working fluid (the steam) delivered to the turbine are gradually increased (Fig. 4a). The rate of change of the pressure, the



Transient Temperature and Thermal Stresses in Turbine Components, Fig. 1 A reaction turbine: *1* rotor, *2* rotor blades, *3* stator blades, *4* inner casing, *5* outer casing, *6* seal on a balance piston, and *7* casing seal