Capacity of OFDM Systems in Nakagami-m Fading Channels: The Role of Channel Frequency Selectivity

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***Abstract*— In this paper, we analyze the capacity of orthogonal frequency division multiplexing (OFDM) systems with carrier frequency offset (CFO) in frequency-selective Nakagami-m fading channels. Previous work on this topic has not taken into account the frequency selectivity of the channel. In this work, we have explicitly attributed the effect of channel frequency selectivity, i.e. frequency domain correlations, in evaluating the OFDM system performance in the presence of CFO. A closed-form expression is derived of the probability density function (PDF) of the signal-to-interference-and-noise ratio (SINR) in terms of CFO and channel correlation vector. Capacity is evaluated using numerical integration. The frequency-flat fading scenario and the perfectly frequency-selective fading (uncorrelated subcarriers) scenario form the two extremes, i.e. bounds, of the achievable OFDM capacity in the presence of CFO in Nakagami-m fading channels.**

I. Introduction

Orthogonal frequency division multiplexing (OFDM) has recently become a key modulation technique for high data rate mobile wireless applications [1]. Computationally simple per subcarrier equalization in multipath frequency-selective fading channels makes OFDM more attractive compared to its single-carrier counterpart. The high spectral efficiency achieved through the use of orthogonal subcarriers is an additional advantage of OFDM and leads to efficient usage of the RF spectrum by multiple broadband systems.

A major drawback of OFDM is its relatively high sensitivity to the carrier frequency offset (CFO) errors, compared to a single carrier system [2]-[4]. The frequency offset error is caused by the misalignment in subcarrier frequencies at the receiver due to fluctuations in receiver RF oscillators or channel's Doppler frequency. This frequency offset can destroy the subcarrier orthogonality of the OFDM signal introducing inter-carrier-interference (ICI). The ICI results in severe degradation of the bit-error-rate (BER) performance of the OFDM systems. Although frequency offset correction techniques [3] can largely compensate for CFO, any residual error in frequency synchronization (CFO estimation error) contributes to the degradation of receiver performance.

In this paper we investigate the effect of the CFO error on the performance of OFDM systems in multipath frequency-selective Nakagami-m fading channels. In reported work in literature [4]-[6] on this topic, ICI noise is either ignored or it is assumed to be independent of the useful signal, which is only valid for highly frequency-selective channels, i.e. when the individual subcarrier responses of the OFDM system are totally uncorrelated (i.i.d.). Therefore, these results provide worse case capacity performance, i.e. lower-bound on the capacity in the presence of CFO. Some recent work [7] shows that in Rician channels with strong line-of-sight component the average OFDM capacity with CFO is only negligibly dependent on subcarrier correlations.

In contrast, in this paper, we provide an analytical technique of evaluating the capacity of an OFDM system with CFO in frequency-selective Nakagami-m fading channels. The exact capacity for a given frequency domain channel correlation function can be calculated using the proposed technique. Moreover, by using the frequency-flat fading (FFF) case and the perfectly frequency-selective fading (PFSF) case as extreme channel conditions, the dependency of achievable capacity on channel correlations in frequency domain is illustrated.

II. Interference due to Frequency offset

For an OFDM system, the post-FFT signal $y(k)$ at the kth subcarrier, where $0 \leq k \leq N-1$, can be given as [2]

$$y(k) = s_0 h(k) x(k) + \sum_{l=0, l\neq k}^{N-1} s_{l-k} h(l) x(l) + w(k) \quad (1)$$

where, $x(k)$ and $h(k)$ are the transmit data symbol and frequency response for the kth subcarrier, respectively. $w(k)$ is additive white Gaussian noise (AWGN). The sequence s_k (ICI coefficients) depends on the CFO and is given by [2]

$$s_k = \frac{\sin \pi(k+\epsilon)}{N \sin \frac{\pi}{N}(k+\epsilon)} \exp\left[j\pi \left(1 - \frac{1}{N}\right)(k+\epsilon)\right] \quad (2)$$

where, ϵ is the normalized frequency offset, which is the ratio between the CFO and the adjacent subcarrier spacing.

The decision variable $\hat{x}(k)$ after *per subcarrier equalization* can be formed as $\hat{x}(k) = \bar{s}_0 \bar{h}(k) y(k)$, where it is assumed that the effective channel $\tilde{h}(k) = s_0 h(k)$ is known, i.e. perfectly estimated. The complex conjugate of a complex number is denoted by $\bar{(.)}$. Thus, $\hat{x}(k)$ becomes

$$\hat{x}(k) = |s_0|^2 |h(k)|^2 x(k) + \bar{s}_0 \bar{h}(k) i(k) + \bar{s}_0 \bar{h}(k) w(k) \quad (3)$$

where, $i(k) = \sum_{l=0, l\neq k}^{N-1} s_{l-k} h(l) x(l)$ is the ICI interference term. In (3), the three terms show the signal, interference, and channel noise, respectively.

Note-1 Let u and v are two *correlated* zero-mean complex Gaussian R.V.s. The correlation coefficient between u and v are given by $r_{uv} = E\{u\bar{v}\}$. The conditional mean $\mu_{u|v}$ and the variance $\sigma^2_{u|v}$ of u given v are given by [8]

$$\mu_{u|v} = \frac{r_{uv}v}{\sigma_v^2} \quad \text{and} \quad \sigma^2_{u|v} = \sigma_u^2 - \frac{|r_{uv}|^2}{\sigma_v^2} \tag{4}$$

where σ_u^2 and σ_v^2 are the variances of u and v, respectively. Also, the conditional power of u given v is given by

$$\mathrm{E}\left\{|u|^2 \mid v\right\} = |\mu_{u|v}|^2 + \sigma^2_{u|v}. \tag{5}$$

The power associated with the three terms in (3) for a given channel realization $h(k)$ for the kth subcarrier can be calculated as follows. The signal power $P_{\mathrm{x}}(k)$ for the kth subcarrier becomes

$$P_{\mathrm{x}}(k) = |s_0|^4|h(k)|^4\mathrm{E}\left\{|x(k)|^2\right\} = |s_0|^4|h(k)|^4\sigma_{\mathrm{x}}^2 \tag{6}$$

where, $\sigma_{\mathrm{x}}^2 = \mathrm{E}\left\{|x(k)|^2\right\}$ is the average symbol power, and the notation $\mathrm{E}\{.\}$ depicts the expected value of a random variable. The interference power $P_{\mathrm{i}}(k)$ for a given $h(k)$ becomes

$$P_{\mathrm{i}}(k) = |s_0|^2|h(k)|^2\mathrm{E}\left\{|i(k)|^2 \mid h(k)\right\} \tag{7}$$

where, $\mathrm{E}\left\{|i(k)|^2 \mid h(k)\right\}$, the conditional power of $i(k)$ for a given $h(k)$ is given by ((8)), where $r_{lk} = \mathrm{E}\left\{h(l)\bar{h}(k)\right\}$ is the channel correlation coefficient between the lth and the kth ($k \neq l$) subcarriers. The constant average power of each subcarrier is given by $\sigma_h^2 = \mathrm{E}\left\{|h(l)|^2\right\}$. Assuming $\sigma_h^2 = 1$ (without loosing generality) and substituting (8) in (7) gives (9). Similarly, the channel noise power becomes $P_{\mathrm{w}} = |s_0|^2|h(k)|^2\sigma_{\mathrm{w}}^2$. Therefore, the *signal-to-interference-and-noise ratio* (SINR) becomes (10), where $\rho = \sigma_{\mathrm{x}}^2/\sigma_{\mathrm{w}}^2$ is the average channel SNR, and $0 \leq r_{lk} \leq 1$. Above (10) can be rewritten as

$$\gamma(k) = \frac{|s_0|^2|h(k)|^2}{\rho^{-1} + \lambda_1(\epsilon, \mathbf{r}) + |h(k)|^2\lambda_2(\epsilon, \mathbf{r})} \tag{11}$$

where,

$$\lambda_1(\epsilon, \mathbf{r}) = \sum_{l=0, l\neq k}^{N-1} |s_{l-k}|^2\left(1 - |r_{lk}|^2\right) \tag{12}$$

and

$$\lambda_2(\epsilon, \mathbf{r}) = \sum_{l=0, l\neq k}^{N-1} |s_{l-k}|^2|r_{lk}|^2. \tag{13}$$

The $N-1$ element vector $\mathbf{r}$ consists of the channel correlation coefficients $r_{l,k}$ for $l = 0, 1, \ldots, N-1$ except for $l = k$. For a Nakagami-m channel $|h(k)|^2$ is gamma distributed. Above (11) indicates that the distribution of $\gamma(k)$ is not only depends on the normalized CFO (ϵ) but also on the subcarrier correlation structure induced by the frequency selective channel given by $\mathbf{r}$. The main focus of this paper is to reveal the dependence of OFDM system performance, e.g. capacity, on $\mathbf{r}$ in the presence of CFO. In terms of frequency selectivity of the channel two extreme cases can be considered as special cases: (a) Frequency flat fading channel- this corresponds to a zero delay-spread channel, and (b) Perfectly frequency selective channel - this corresponds to a very high delay-spread channel.

A. Frequency Flat Fading (FFF)

For frequency flat fading subcarrier channel responses become perfectly correlated thus $r_{l,k} = 1, \forall k$, i.e. $\mathbf{r} = \mathbf{1}_{N-1}$. Thus $\lambda_1(\epsilon, \mathbf{r}) = 0$ and $\lambda_2(\epsilon, \mathbf{r}) = \sum_{l=0, l\neq k}^{N-1} |s_{l-k}|^2 = 1 - |s_0|^2$. Therefore, the SINR for the frequency flat fading becomes

$$\gamma(k) = \frac{|s_0|^2|h(k)|^2}{(1 - |s_0|^2)\,|h(k)|^2 + \rho^{-1}}. \tag{14}$$

B. Perfectly Frequency Selective Fading (PFSF)

For perfectly frequency selective channel, subcarrier channel responses become zero correlated thus $r_{l,k} = 0, \forall k$, i.e. $\mathbf{r} = \mathbf{0}_{N-1}$. Thus $\lambda_1(\epsilon, \mathbf{r}) = \sum_{l=0, l\neq k}^{N-1} |s_{l-k}|^2 = 1 - |s_0|^2$ and $\lambda_2(\epsilon, \mathbf{r}) = 0$. Therefore, the SINR for the frequency flat fading becomes

$$\gamma(k) = \frac{|s_0|^2|h(k)|^2}{1 - |s_0|^2 + \rho^{-1}}. \tag{15}$$

III. Density Function of SINR

In this section we derive a closed-form expression the density function (PDF) of SINR in Nakagami-m fading channels. The post-equalized SINR $\gamma(k)$ in (11) can be rewritten as

$$\gamma(k) = \frac{a|h(k)|^2}{b|h(k)|^2 + 1} \tag{16}$$

where,

$$a = \frac{|s_0|^2}{\rho^{-1} + \lambda_1(\epsilon, \mathbf{r})} \quad \text{and} \quad b = \frac{\lambda_2(\epsilon, \mathbf{r})}{\rho^{-1} + \lambda_1(\epsilon, \mathbf{r})}. \tag{17}$$

For a Nakagami-m fading channel, $|h(k)|$ follows the Nakagami distribution. Let $v = |h(k)|^2$ then v will be gamma distributed - thus for unit-power subcarriers, i.e. $E\{v\} = 1$, the PDF of v becomes

$$p_v(v) = \frac{m^m}{\Gamma(m)}v^{m-1}e^{-mv} \tag{18}$$

where, $\Gamma(.)$ is the gamma function and m is a parameter denoting the severity of channel fading, $m \geq 1/2$. The well-known Rayleigh fading corresponds to $m = 1$.

Note 2: The density $p_y(y)$ of the random variable $Y = f(X)$ (function of X) can be found in terms of the density $p_x(x)$ of the R. V. X as follows: To find $p_y(y)$ for a specific y solve the equation $y = f(x)$, denoting its real roots by x_1, x_2, $\ldots$, x_N. Then, $p_y(y)$ is given by [8]

$$p_y(y) = \frac{p_x(x_1)}{|f'(x_1)|} + \frac{p_x(x_2)}{|f'(x_2)|} + \ldots + \frac{p_x(x_N)}{|f'(x_N)|} \tag{19}$$

where $f'(x)$ is the derivative of $f(x)$.

Using (19), the PDF of $\gamma(k)$ can be calculated as follows:

$$\gamma = f(v) = \frac{av}{bv + 1} \quad \text{and} \quad f'(v) = \frac{a}{(bv+1)^2} \tag{20}$$

where, the subcarrier index k is dropped for simplicity. Also, $0 \leq \gamma \leq a/b$ as $v \geq 0$. $\gamma = f(v)$ has only one real root at $v_1 = \gamma/(a - b\gamma)$, and therefore using (19) the PDF of $\gamma(k)$ becomes

$$\begin{aligned} p_\gamma(\gamma) &= \frac{p_v(v_1)}{|f'(v_1)|} \\ &= \frac{m^m}{\Gamma(m)}\frac{a\gamma^{m-1}}{(a - b\gamma)^{m+1}}\exp\left(\frac{-m\gamma}{a - b\gamma}\right) \end{aligned} \tag{21}$$

$$\begin{aligned}\mathrm{E}\left\{|i(k)|^2 \mid h(k)\right\} &= \sum_{l=0,l\neq k}^{N-1} |s_{l-k}|^2 \left(|\mu_{h(l)|h(k)}|^2 + \sigma^2_{h(l)|h(k)}\right)\sigma_x^2 \\ &= \sum_{l=0,l\neq k}^{N-1} |s_{l-k}|^2 \left[\frac{|r_{lk}|^2|h(k)|^2}{\sigma_h^4} + \left(\sigma_h^2 - \frac{|r_{lk}|^2}{\sigma_h^2}\right)\right]\sigma_x^2 \end{aligned} \quad (8)$$

$$P_{\mathrm{i}}(k) = |s_0|^2|h(k)|^2\sigma_x^2 \sum_{l=0,l\neq k}^{N-1} |s_{l-k}|^2 \left[|r_{lk}|^2|h(k)|^2 + \left(1-|r_{lk}|^2\right)\right]. \quad (9)$$

$$\gamma(k) = \frac{P_{\mathrm{x}}(k)}{P_{\mathrm{w}}(k)+P_{\mathrm{i}}(k)} = \frac{\rho|s_0|^2|h(k)|^2}{1+\rho|h(k)|^2\sum_{l=0,l\neq k}^{N-1}|s_{l-k}|^2|r_{lk}|^2 + \rho\sum_{l=0,l\neq k}^{N-1}|s_{l-k}|^2\left(1-|r_{lk}|^2\right)} \quad (10)$$

where, $0 \le \gamma \le a/b$. It is important to note that $p_\gamma(\gamma) = 0$ for $\gamma > G$, where $G = a/b = |s_0|^2/\lambda_2(\epsilon, \mathbf{r})$. For a *frequency flat-fading channel* $a = \rho|s_0|^2$ and $b = \rho(1-|s_0|^2)$. For a *perfectly frequency selective channel* $a = |s_0|^2/(1-|s_0|^2+\rho^{-1})$ and $b = 0$.

IV. OFDM Capacity with CFO

In this section we derive an analytical expression for the ergodic capacity[1] of an OFDM system with CFO in a Nakagami-m fading channel with the effect of the frequency selectivity of the channel explicitly accounted for. Using the density function of SINR derived in (21) the ergodic capacity $C = C(\epsilon, \mathbf{r})$ can be given as

$$\begin{aligned} C &= \int_0^\infty \log_2(1+\gamma)p_\gamma(\gamma)d\gamma \\ &= I\int_0^G \frac{\gamma^{m-1}\log_2(1+\gamma)}{(a-b\gamma)^{m+1}}\exp\left(\frac{-m\gamma}{a-b\gamma}\right)d\gamma \end{aligned} \quad (22)$$

where,

$$I = I(m,a) = \frac{am^m}{\Gamma(m)}. \quad (23)$$

Note that the capacity C is function of both CFO and channel correlation structure as the parameters a and b are dependent on ϵ and $\mathbf{r}$. Capacity in a Rayleigh fading channel can be obtained by setting $m = 1$ in (22) to give

$$C = a\int_0^G \frac{\log_2(1+\gamma)}{(a-b\gamma)^2}\exp\left(\frac{-\gamma}{a-b\gamma}\right)d\gamma \quad (24)$$

For a perfectly frequency-selective channel $b = 0$ and $G = \infty$, thus the capacity can be calculated using

$$C = \frac{m^m}{a^m\Gamma(m)}\int_0^\infty \gamma^{m-1}\log_2(1+\gamma)\exp\left(\frac{-m\gamma}{a}\right)d\gamma. \quad (25)$$

V. Numerical Results

In this section we present numerical results obtained using the analytical expressions derived in Sections III and IV for the density function of the SINR and the ergodic capacity of an OFDM system with CFO.

[1]The capacity evaluated here is when the receiver uses conventional per subcarrier equalization, i.e. capacity limited by ICI noise and AWGN.

A. Density Function of SINR

Fig. 1 shows the density function (PDF) $p_\gamma(\gamma)$ of SINR for the Rayleigh fading case in frequency flat fading (FFF) when $\rho = 20$ dB. The PDF for CFO values of $\epsilon = 8\%, 10\%$, and 12% are shown separately. As can be seen from Fig. 1, the probability of lower SNR values increases with increasing values of CFO. Fig. 2 shows the density function (PDF) $p_\gamma(\gamma)$ of SINR for the Rayleigh fading case in perfectly frequency selective fading (PFSF) when $\rho = 10$ dB. Again, the PDF for CFO values of $\epsilon = 8\%, 10\%$, and 12% are shown separately. As can be seen from Fig. 2, the probability of lower SNR values (close to $\gamma = 0$) increases with increasing values of CFO. Also, comparison of Fig. 1 and Fig. 2 for the same CFO value reveals that PDF for PFSF is much worse than that for FFF case.

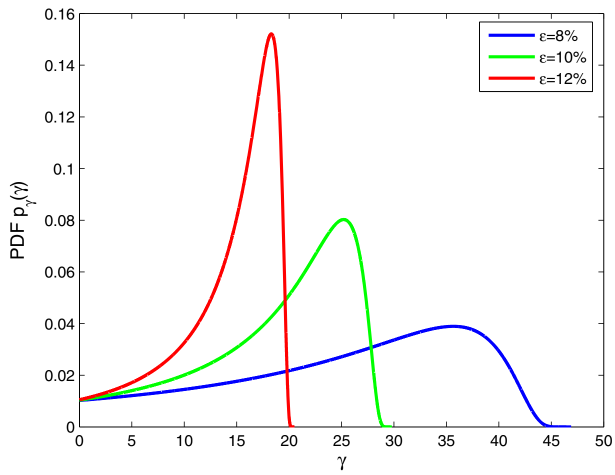

Fig. 1. The density function (PDF) $p_\gamma(\gamma)$ of SINR γ for $\epsilon = 8\%, 10\%$, and 12% and $m = 1$ (Rayleigh fading) in **frequency flat fading (FFF)** channel.

B. Capacity with CFO

Fig. 3, Fig. 4, and Fig. 5 show the average OFDM capacity versus average channel SNR curves numerically evaluated using the capacity expressions derived in Section IV, for

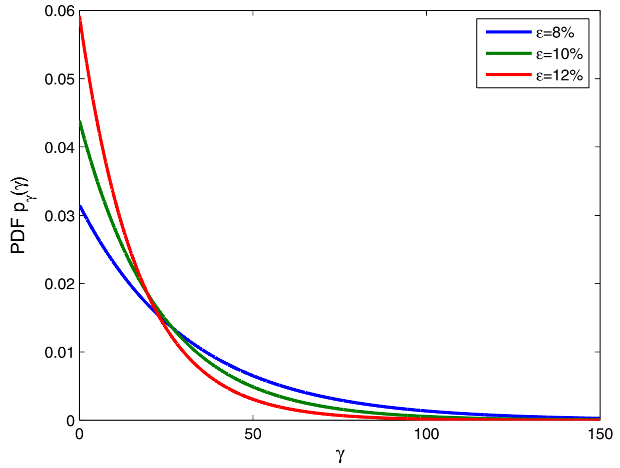

Fig. 2. The density function (PDF) $p_\gamma(\gamma)$ of SINR γ for $\epsilon = 8\%, 10\%$, and 12% and $m = 1$ (Rayleigh fading) in **perfectly frequency selective fading (PFSF)** channel.

$m = 0.5$, 1, and 2, respectively. Capacity for CFO values of $\epsilon = 5\%, 10\%$, and 20% and for the channel conditions of FFF and PFSF fading are shown separately. As can be seen from Fig. 3, Fig. 4, and Fig. 5, a significant capacity gap exists for the FFF and PFSF channel cases for the high SNR values. Note that at low SNR values the AWGN noise dominates over the ICI noise. Also, it is observable that the capacity gap is more for large CFO (ϵ) and deeper fading (low m).

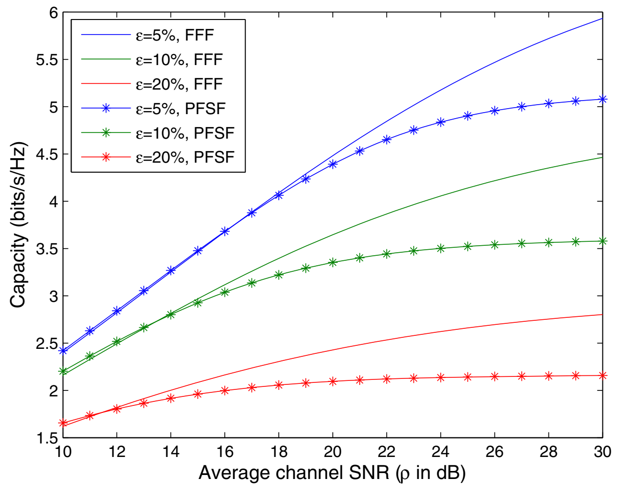

Fig. 3. Average OFDM Capacity versus average channel SNR for $m = 0.5$ **(deeper fading than Rayleigh)** and normalized CFO values of $\epsilon = 5\%, 10\%$, and 20%.

VI. Conclusions

In this paper, we demonstrated the role of channel frequency selectivity in determining the capacity of an OFDM system in the presence of CFO in Nakagami-m fading channels. A closed-form expression is derived for the probability density function of the signal-to-interference-and-noise ratio in terms of CFO and channel correlation vector. Numerical results show that the channel frequency selectivity plays an important role specially in the high SNR and deeper fading conditions.

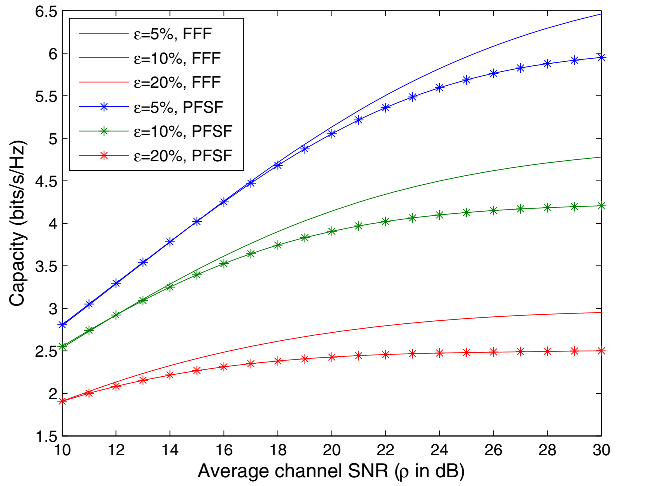

Fig. 4. Average OFDM Capacity versus average channel SNR for $m = 1$ **(Rayleigh fading)** and normalized CFO values of $\epsilon = 5\%, 10\%$, and 20%.

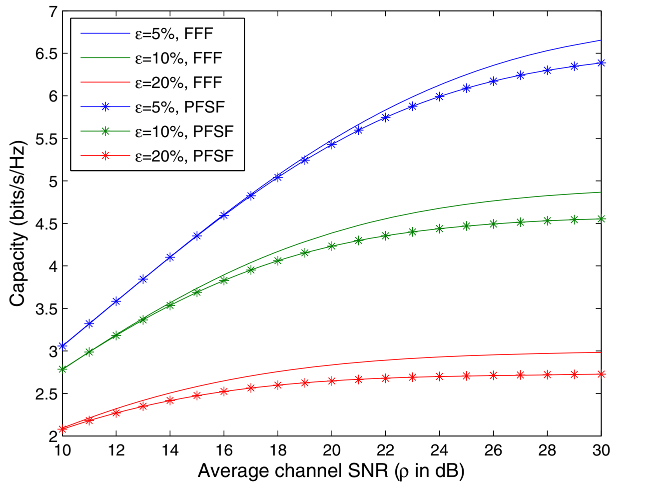

Fig. 5. Average OFDM Capacity versus average channel SNR for $m = 2$ **(shallower fading than Rayleigh)** and normalized CFO values of $\epsilon = 5\%, 10\%$, and 20%.

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