Fuzzy Sets, Near Sets, and Rough Sets.
Sets in the Computational Intelligence Spectrum

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Abstract. This keynote talk considers how one might utilize fuzzy sets, near sets, and rough sets, taken separately or taken together in hybridizations in solving a variety of problems commonly faced in science and engineering. These technologies offer set theoretic approaches to solving problems such as classifying sensor output, image retrieval and image correspondence. Fuzzy sets result from the introduction of a membership function that generalizes the traditional characteristic function. The notion of a fuzzy set was introduced by L. Zadeh in 1965. Fifteen years later, rough sets were introduced by Z. Pawlak in 1981. A set is considered rough the boundary between its lower and upper approximation is non-empty. Of the three forms of sets, a near set is newest, introduced in 2007 by J.F. Peters in a perception-based approach to the study of perceptual objects. Perceptual systems provide stepping stones leading to nearness relations and properties of near sets. This work has been motivated by an interest in finding a solution to the problem of discovering perceptual granules that are, in some sense, near each other. Near set theory provides a description-based approach to observing, comparing and classifying perceptual granules. Near sets result from the introduction of a description-based approach to perceptual objects and a generalization of the traditional rough set approach to granulation that is independent of the notion of the boundary of a set approximation. Near set theory has strength by virtue of the strength it gains from rough set theory, starting with a $L_2$ norm-based perceptual indiscernibility relation, a new extension of the traditional indiscernibility equivalence relation. This keynote talk highlights a context for three forms of sets that are now part of the computational intelligence spectrum of tools useful in pattern recognition. By way of introduction to near sets, we consider various perceptual nearness relations that define partitions of sets of perceptual objects that are near each other. Every perceptual granule is represented by a set of perceptual objects that have their origin in the physical world. Objects that have the same appearance are considered perceptually near each other, i.e., objects with matching descriptions. Pixels, pixel windows, and segmentations of digital images are given by way of illustration of sample near sets. The contribution of this paper is an overview of the links between fuzzy sets, near sets and rough sets as well as the relation between these sets and the original notion of a set introduced by Cantor in 1883.

Keywords: Description, fuzzy set, near set, perceptual system, rough set.

Near To

How near to the bark of a tree are drifting snowflakes,
swirling gently round, down from winter skies?
How near to the ground are icicles,
slowly forming on window ledges?

—Fragment of a Philosophical Poem.
1 Introduction

This paper briefly presents the notion of a Cantor set. From the definition of a Cantor set, it is pointed out that fuzzy sets, near sets and rough sets are special forms of Cantor sets. In addition, this paper points to links between the three types of sets that are part of the computational intelligence spectrum. Probe functions in near set theory provide a link between fuzzy sets and near sets, since every fuzzy membership function is a particular form of probe function. Probe functions are real-valued functions introduced by M. Pavel in 1993 as part of a study of image registration and a topology of images [6]. Z. Pawlak originally thought of a rough set as a new form of fuzzy set [7]. It has been shown that every rough set is a near set (this is Theorem 4.8 in [11]) but not every near set is a rough set. For this reason, near sets are considered a generalization of rough sets. The contribution of this paper is an overview of the links between fuzzy sets, near sets and rough sets as well as the relation between these sets and the original notion of a set introduced by Cantor in 1883 [3].

2 Cantor Set

By a ‘manifold’ or ‘set’ I understand any multiplicity, which can be thought of as one, i.e., any aggregate [inbegriff] of determinate elements which, can be united into a whole by some law.

–Foundations of a General Theory of Manifolds,
–G. Cantor, 1883.

In this mature interpretation of the notion of a set, G. Cantor points to a property or law that determines elementhood in a set and “unites [the elements] into a whole” [3], elaborated in [2], and commented on in [4]. In 1851, Bolzano [1] writes that “an aggregate so conceived that is indifferent to the arrangement of its members I call a set”. At that time, the idea that a set could contain just one element or no elements (null set) was not contemplated. This is important in the current conception of a near set, since such a set must contain pairs of perceptual objects with similar descriptions and such a set is never null. That is, a set is a perceptual near set if, and only if it is never empty and it contains pairs of perceived objects that have descriptions that are within some tolerance of each other (see Def. 1).

3 Near Sets

The basic idea in the near set approach to object recognition is to compare object descriptions. Sets of objects $X, Y$ are considered near each other if the sets contain objects with at least partial matching descriptions.

–Near sets. General theory about nearness of objects,

In a more recent interpretation of the notion of a near set, the nearness of sets is considered in the context of a perceptual system [15]. The trivial case is excluded. That is, an element $x \in X$ is not considered near itself. In addition, the empty set is excluded from near sets, since the empty set is never something that we perceive, i.e., a set of perceived objects is never empty. In the case where one set $X$ is near another set $Y$, this leads to the realization that there is a third set containing pairs of elements $x, y \in X \times Y$ with similar descriptions. The key to an understanding of near sets is the
Table 1. Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Interpretation</th>
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<tr>
<td>$O, X$</td>
<td>Set of perceptual objects, $X, Y \subseteq O$,</td>
</tr>
<tr>
<td>$F, \mathcal{B}$</td>
<td>Sets of probe functions, $\mathcal{B} \subseteq F$,</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>$\varepsilon \in \mathbb{R}$ (reals) such that $\varepsilon \geq 0$,</td>
</tr>
<tr>
<td>$\phi_i(x)$</td>
<td>$i$th probe function $\phi_i : X \to \mathbb{R}$ representing feature of $x$,</td>
</tr>
<tr>
<td>$\phi(x)$</td>
<td>$(\phi_1(x), \phi_2(x), \ldots, \phi_l(x)), \phi_i \in F, x \in O$, description of $x$,</td>
</tr>
<tr>
<td>$\equiv_{\mathcal{B}, \varepsilon}$</td>
<td>${ (x, y) \in O \times O : | \phi(x) - \phi(y) |_2 \leq \varepsilon }$, tolerance relation</td>
</tr>
<tr>
<td>$X \equiv_{\mathcal{B}, \varepsilon} Y$</td>
<td>$X$ resembles (is near) $Y \iff X \equiv_{\mathcal{B}, \varepsilon} Y$.</td>
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notion of a description. Each perceived object is represented by a vector of feature values and each feature is represented by what is known as a probe function that maps an object to a real value. Since our main interest is in detecting similarities between seemingly quite disjoint sets such as subimages in an image or pairs of classes in coverings on a pair of images, a near set is defined in context of a tolerance space.

**Definition 1 Tolerance Near Sets** [12]
Let $\langle O, F \rangle$ be a perceptual system, where $O$ is a set of images (sets of points) and $\mathcal{B} \subseteq F$ be a set of probe functions representing image features. Let $\mathcal{B}$ contains probe functions used to measure features of subimages in $X, Y \subseteq O$. A set $X$ is perceptually near a $Y$ within the perceptual system $\langle O, F \rangle$ (i.e., $(X \equiv_{\mathcal{B}, \varepsilon} Y)$) iff there are $x \in X$ and $y \in Y$ and there is $\mathcal{B} \subseteq F$ such that $x \equiv_{\mathcal{B}, \varepsilon} y$.

In effect, a Cantor set is a near set contains pairs of objects that satisfy the nearness description property enunciated in Def. 1.

### 4 Fuzzy Set

A fuzzy set is a class of objects with a continuum of grades of membership.

--Fuzzy sets, Information and Control 8

...A fuzzy set is characterized by a membership function which assigns to each object its grade of membership (a number lying between 0 and 1) in the fuzzy set.

--A new view of system theory

The notion of a fuzzy set was introduced by L.A. Zadeh in 1965 [16]. In effect, a Cantor set is a fuzzy set if, and only if every element of the set has a grade of membership assigned to it by a specified membership function. Notice that a membership function $\phi : X \to [0, 1]$ is a special case of what is known as a probe function in near set theory. A fuzzy set $X$ is a near set relative to a set $Y$ if the grade of membership of the objects in sets $X, Y$ is assigned to each object by the same membership function $\phi$ and there is at least one pair of objects $x, y \in X \times Y$ such that $\| \phi(x) - \phi(y) \|_2 \leq \varepsilon$, i.e., the description of $x$ is similar to the description $y$ within some $\varepsilon$. 
5 Rough Set

A new approach to classification, based on information systems theory, given in this paper. This approach leads to a new formulation of the notion of fuzzy sets (called here the rough sets).

The axioms for such sets are given, which are the same as the axioms of topological closure and interior.

--Classification of objects by means of attributes.

--Z. Pawlak, 1981.

<table>
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<tr>
<td>∼_{\mathcal{B}}</td>
<td>{(x, y) \mid f(x) = f(y) \forall f \in \mathcal{B}}, indiscernibility relation, cf. [7].</td>
</tr>
<tr>
<td>x_{/\sim_{\mathcal{B}}}</td>
<td>x_{/\sim_{\mathcal{B}}} = {y \in X \mid y \sim_{\mathcal{B}} x}, elementary set (class).</td>
</tr>
<tr>
<td>O_{/\sim_{\mathcal{B}}}</td>
<td>O_{/\sim_{\mathcal{B}}} = {x_{/\sim_{\mathcal{B}}} \mid x \in O}, quotient set.</td>
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In effect, in its original conception, Z. Pawlak thought of a rough set as a new formulation of the notion of a fuzzy set [7]. In a rough set approach to classifying sets of objects X, one considers the size of the boundary region in the approximation of X. By contrast, in a near set approach to classification, one does not consider the boundary region of a set. In particular, assume that X is a non-empty set belonging to a universe U and that \(\mathcal{F}\) is a set of features defined either by total or partial functions. The lower approximation of X relative to \(\mathcal{B} \subseteq \mathcal{F}\) is denoted by \(\mathcal{B}_s(X)\) and the upper approximation of X is denoted by \(\mathcal{B}^+(X)\), where

\[
\mathcal{B}_s(X) = \bigcup_{x_{/\sim_{\mathcal{B}}} \subseteq X} x_{/\sim_{\mathcal{B}}},
\]

\[
\mathcal{B}^+(X) = \bigcup_{x_{/\sim_{\mathcal{B}}} \cap X \neq \emptyset} x_{/\sim_{\mathcal{B}}},
\]

The \(\mathcal{B}\)-boundary region of an approximation of a set X is denoted by \(Bnd_{\mathcal{B}}(X)\), where

\[
Bnd_{\mathcal{B}}(X) = \mathcal{B}^+(X) \setminus \mathcal{B}_s(X) = \{x \mid x \in \mathcal{B}^+(X) \text{ and } x \notin \mathcal{B}_s(X)\}.
\]

A set X is roughly classified whenever \(Bnd_{\mathcal{B}}(X)\) is not empty. In other words, X is a rough set whenever the boundary region \(Bnd_{\mathcal{B}}(X) \neq \emptyset\). In sum, a Cantor set is a rough set if, and only if its approximation boundary is non-empty. It should also be noted that rough sets differ from near sets, since near sets are defined without reference to an approximation boundary region. This means, for example, with near sets the image correspondence problem can be solved without resorting to set approximation.

6 Conclusion

From the beginning, the near set approach to perception has had direct links to rough sets in its approach to the perception of objects [7, 5] and the classification of objects [7, 10, 9, 8]. This is evident in the early work on nearness of objects and the extension of the approximation space model (see, e.g., [13, 14]). Unlike the focus on the approximation boundary of a set, the study of near sets
focuses on the discovery of affinities between perceptual granules such as digital images viewed as sets of points. In the context of near sets, the term affinity means close relationship relationship between perceptual granules (particularly images) based on common description. Affinities are discovered by comparing the descriptions of perceptual granules, e.g., descriptions of objects contained in classes found in coverings defined by the tolerance relation $\sim_{F,\varepsilon}$. In sum, fuzzy sets, near sets and rough sets are particular forms of Cantor sets. In addition, each of these sets in the computational intelligence spectrum offer very useful approaches in classifying objects.

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