Mathematical proofs in primary schools

James Russo shares his experiences of exploring proof with a group of 8- and 9- year students in an Australian primary school.

The first time I encountered the notion of mathematical proofs was in a mathematics class in my final two years at school. I found the process of attempting to prove a mathematical conjecture intriguing, mystifying and intimidating. I was confounded that mathematical proofs appeared to be a central aspect of what mathematicians do, yet they had not appeared in the curriculum until the end of my formal education

As a primary school teacher, I strive to reveal the power and elegance of mathematics to young students. This can be achieved through many different types of mathematical activity. I think a common thread running through such activities is that they provide opportunities for mathematical reasoning. Although I am a firm believer in using highly contextualised problem-solving tasks to engage students, I often wondered whether there might be a role for exploring mathematical proof in a primary school context. My feeling is that this might allow students to use the language of mathematics to reason and build an argument, whilst deepening their understanding of important mathematical ideas.

In this article I offer examples of student discussions, taken from audio recordings, when considering the properties of even and odd numbers, and multiplication. I conclude the article by putting forward three principles of task design that I think might be useful to other primary educators interested in developing problems of this type with their students.

Investigating and exploring properties of odd and even numbers: Proof 1.

> Zero is an even number. True or false? Prove it.

This was probably the most accessible of the activities. All students came up with at least one reason as to why zero must in fact be an even number. Many of these explanations focused around an examination of the pattern of even and odd numbers. For example:

> It is an even number because there is a pattern going even, odd, even, odd. Because 1 is odd, 0 must be even, because they're next to each other. (Diane, age 8)

> If we count by 2s, there is only even numbers, like 2, 4, 6, 8 ... The number before 2 would be zero, so zero must be even. (Jake, age 8)

> Because if zero was an odd number, it wouldn't make any sense. Because if we were counting all the odd numbers, and it went zero and then 1, 3, 5, 7, 9 such and such. It is not in a pattern and it doesn't make sense. (Kiara, age 9)

> Negative 1 is an odd number because the only number in it is 1 and it is an odd number… And the number 'actual 1' is an odd number. So, there would have to be an even number in between those two odd numbers to split the odd numbers. So, zero must be even. (Violet, age 9)

Several students attempted to argue by analogy, suggesting that all other numbers ending with zero are even, for example, 10, 20, 30, so zero itself must be even. As one student stated:

> I think zero is an even number because it goes 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. And it keeps going until 100. And if it keeps going until 100, then all of these (gesturing towards multiples of 10) are even. So all the numbers counting by 10 are even. (Jeremiah, age 8)

In some instances, this analogical argument was linked to the notion that even numbers can be halved. For example:

> If a number has zero at the end, in the units column, then you can halve this number. Like half of 30 is 15. If you can halve it, it must be an even number. So, zero must be even.(Jayesh, age 9)

> So, you had 21, you couldn't halve that, so it is an odd number. But say you had 20, it is even because you can halve it into 10 and 10. (Emmy, age 9)

> If you halve a number ending with zero, it is either going to end with a zero or a five after we halve it. And if we halve 0 it ends with 0, so 0 must be even. (Zalika, age 9)

One student explanation focussed on the notion that even numbers have "partners", whereas odd numbers do not. Although this reasoning seems potentially problematic for proving that zero is an even number, he attempted to emphasise that the lack of an "odd one out" proved that zero must be even.

> If you have three dots and you put it into two vertical lines, one side would have two and the other side would have one. That dot [gesturing towards the single dot] doesn't have a friend next to it, so it is odd. It's the odd one out. Just like in seven, because 2, 4, 6 … they all have partners. But there is one extra dot for seven and that one doesn't have a partner. With zero, there is no partners needed, so there is no odd one out. (Divit, age 8)

Indeed, this explanation was challenged by other students.

> Zero doesn't have a partner, so this doesn't make sense. You need a partner to be an even number? (Jayesh, age 9)

> What about negative numbers? They don't have partners? (Violet, age 8)

The group reached a near-consensus that given negative numbers are still odd or even, and that zero is still odd or even, the idea that even numbers have partners and odd numbers do not may have outgrown its usefulness.

Investigating and exploring properties of odd and even numbers: Proof 2.

> There are more even numbers altogether than odd numbers. True or false? Prove it.

Although intended to build on the previous discussion, this problem seemed more difficult for students to make meaningful progress with. However, part of the reason for me choosing to go in this direction was that, in addition to exploring the properties of odd and even numbers, this problem gave students an opportunity to grapple with the notion of infinity and what it represents. I was interested to see whether or not the introduction of ideas of infinity might be overwhelming for students, limiting their capacity to attempt to reason mathematically. For some students, this might have been the case. For example, after several minutes of reflection and discussion, these year 3 students concluded:

> Because the numbers go forever, there is no way of knowing. (Divit, age 8)

> It [infinity] goes up to like a quadrillion, billion, so you can't tell. (Jake, age 8)

Other students proposed that there are more even numbers because infinity itself was likely to be even. However, in students' minds, this appeared contingent on where you started counting from. Below is an exchange I had with two students who proposed that if we began counting at zero, there would be more even numbers, whereas if we began counting at one, there would be more odd numbers. The logic appears to be that beginning at one rather than zero shifts the entire set of possible numbers by one, meaning that instead of beginning and ending our counting on an even number, we will begin and end our counting on an odd number.

Jayesh: If the first number zero is even, then infinity should be even… so there is one more even number than odd number.

Teacher: Why do you think infinity might be an even number?

Zalika: Because the first number you started counting, you would probably end on what you started on, but higher. But if people started counting from one, that would mean there would be more odd numbers. It kind of depends on how you count them.

Teacher: So, it depends on where you start?

Zalika: Yep, yep.

Jayesh: So, if you start from an odd number, there will be one more odd number, but if you start from an even number, there will be one more even number.

Teacher: But if you start from one instead of zero Jayesh, how does the number you end on change?

Jayesh: Because infinity keeps on going, it goes odd, even, odd, even. But if you start from 1 instead of zero, it will go one back, and it will go one back from infinity from the even.

Zalika: So, if you start on zero you will have one more even number, and if you start on one, you will have one more odd number.

Teacher: So, changing where you begin shifts everything over? Is that what you are saying?

Jayesh Yeah, yeah.

Jayesh and Zalika were clearly genuinely engaged with the notion of infinity and how it might be manipulated to address this particular problem. However, the explanation that most closely bore semblance to a formal proof was provided by Violet and Emmy, who argued that there were more even numbers than odd numbers:

Teacher: So, why do you think there are more even numbers than odd numbers altogether?

Emmy: If we count up to 10 with the even numbers, it goes 0, 2, 4, 6, 8, 10 – there are six numbers. But if we count by 2s in the odd numbers,1, 3, 5, 7, 9 – there are only five.

Teacher: So, up to 10 there are more even numbers than odd numbers.

Emmy: And then if you count every group of 10 in the number alphabet …

Teacher: But, in the next set of 10, say up to 20, you wouldn't start on 10, you would start on 11. So, there wouldn't be more even numbers in this set. There would be five odd numbers and five even numbers.

Emmy: But that wouldn't matter because there would still be more even numbers, because there would be one more here [gesturing towards the zero to 10 set].

Teacher: So, can you summarise your argument?

Violet: If we count to 10 in the 2 times tables in the even numbers, it goes 0, 2, 4, 6, 8, 10. And there are six even numbers. But we count in the 2 times tables for the odd numbers, it goes 1, 3, 5, 7, 9. And then there are only five odd numbers. And if we do that to every (set of) 10 that there is, there will always be one more even number. Even though for the next group of 10 we start at 11, and then 21… For these groups, there's the same (amount of even and odd numbers). But it doesn't matter, because there is still always one more even number, because of the first group.

Although it is possible to offer an alternative proof which would contradict Emmy and Violet's conclusion, for example, ending our sets on odd numbers, like 9, 19, 29 and so on, and similar reasoning could be used to prove there are an equal number of odd and even numbers, it seems that these students have harnessed the language and logic of mathematics to form their argument.

Investigating and exploring the properties of multiplication: Proof 1.

> If I know all my other times-tables, I don't need to learn my 7 times-tables because I will know them anyway. True or false? Prove it.

This problem was intended to engage students in reflecting on the commutative property of multiplication. However, two students struggled to make progress with the problem, even after some teacher prompting. For example, after following the prompt to write out his 7 times-tables and look for patterns, one student concluded,

"It's false, because if you don't learn your 7's, you won't know much about your 7 times-tables." However, all other students in the group cited the commutative property of multiplication as the reason why learning the 7 times-tables independently was unnecessary.

> Because you know your 1s, 2s, 3s, 4s and other times-tables, you don't need to know your 7s. And it is the same with any other [times-table]. Because you know one 1 x 7, 2 x 7, 3 x 7. (Violet, age 9)

> Because if you learn like 1 x 7 and 2 x 7 it would be the same as 7 x 1 and 7 x 2. It is the same answer, even though it is the other way. (Jayesh, age 9)

> If you wrote out all your times-tables, and then just found the one's with 7s it would be the same. You swap it around and it would be the same thing. (Zalika, age 9)

> I was thinking it wasn't true, but then I got convinced because if you know all the others, you'll probably know your 7s. […] Because if you count by 2's all the way to 14 it would be 7 times, and 14 is 2 times 7. (Diane, age 8)

Towards the end of the discussion, several students realized that one multiplication fact would still remain missing, that is 7 × 7. The group decided that this fact would need to be learnt independently. I thought that this realisation would lead to a discussion about how the distributive property could be used as an alternative means of showing that learning the 7 times-tables independently is unnecessary (that is 7 groups of 7 is the same as 7 groups of 5 and 7 groups of 2), however, this was not the case. Instead, several students proceeded to discuss whether learning your 7 times-tables would be sufficient for knowing all other times-tables, that is, exploring whether the commutative property could be used to prove the inverse of the original problem. The group reached a consensus that this would in fact be impossible, as many multiplication facts would be "left out" such as, "5 times 6 and 6 times 5."

Investigating and exploring the properties of multiplication: Proof 2

> If I know all my times-tables up to 10 × 10, I don't need to learn my 12 times-tables because I can easily work them out. True or false? Prove it.

This problem was aimed at students developing an appreciation for the distributive property of multiplication, through them demonstrating that 12 times-tables are equivalent to either 'adding together' our 10 times-tables and our 2 times-tables, or doubling our 6 times-tables. Most students in the group did in fact use the distributive property to 'prove' that the statement was true. The most notable exchange was again with Violet and Emmy.

Emmy: We reckon it's true.

Teacher: Why?

Emmy: Say you have 12 × 5. You would add the 10s, so 10 × 5 = 50, and then you would add the 2s, so 2 × 5 = 10. And 50 and 10 equals 60.

Teacher: Wow, that seems like a fairly powerful idea. I wonder whether you could you use the same logic for the 13 times-tables?

Emmy: Yeah, say you had 13 × 7. You would do 10 × 7 which is 70 and 3 × 7 which is 21. And you could add it together, and it would be 91.

Teacher: Wow. And, if that is the case, I wouldn't need to know my 14s or my 15s either. I wouldn't need to know any of my times-tables up until my 20s.

Violet: But, 20s would be easy. Say we had to do 24 × 5. You would times the 5 by 10, and then you would do it again. And then you would add 4 more groups of 5. So it would be 120.

Task design principles for developing 'proof-type' problems.

Hopefully the above discussion has demonstrated that asking students to engage in problems requiring them to prove or falsify a particular conjecture is worth pursuing, even in a primary school context. I will conclude by proposing three task design principles which teachers interested in developing similar problems may wish to consider.

Principle 1: The problem should be worded as a statement, with an attached follow-up question "True or false? Prove it".

Putting forward the problem as a conjecture that needs to be proven or falsified is very different from conventional question-type problems we normally ask of students. I feel that presenting problems in this form both encourages students to first take a definite position. They have to decide whether they believe the statement to be true or false which gets them into the mindset of needing to find evidence to support their position. This shifts the focus of the lesson to the quality of students' mathematical reasoning and their capacity to adopt the language of mathematics to develop a persuasive argument.

Principle 2: The mathematical knowledge required to engage productively with the problem is accessible to most students beforehand.

It seemed to me that asking students to engage in these problems is cognitively demanding, particularly when students are unfamiliar with problems of this structure. Consequently, I feel that such problems are probably better introduced towards the end of a particular unit of work, as they offer a great opportunity for students to apply recently acquired mathematical knowledge and skills.

Principle 3: An important mathematical idea should lie at the heart of the problem.

I think that these types of activities are an excellent means of exposing students to important mathematical ideas. In some instances, it may be that such problems allow the discovery of such ideas, as appeared to be the case with Violet and Emmy realising the power of organising numbers into discrete sets whilst attempting to prove that there are more even numbers than odd numbers altogether. In other instances, it may be that such problems allow already known ideas to be brought to life and given new relevance, as was the case when the same two students demonstrated that applying the distributive property of multiplication meant that learning our 12 times-tables, and even our 24 times-tables, was redundant.

Finally, I would love to hear from other primary school teachers and teacher-educators who have developed similar problems which they have used successfully in their own classrooms or with pre-service teachers.

James Russo teaches at Belgrave South Primary School and Monash University, Melbourne, Australia. Email: james.russo@monash.edu.